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STUDY OF NONLINEAR WAVE PROCESSES IN PLASMAS USING THE FORMALISM OF A SPECIAL LORENTZ TRANSFORMATION FOR A SPACE-INDEPENDENT FRAME

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ABSTRACT. A study is made of nonlinear waves in plasmas using the formalism of a special Lorentz transformation for a space-independent frame, S'. This special transformation is used to transform the space-time dependent equations in a cold, relativistic, magnetized plasma to the S' frame. Then the transformed equations are employed to derive the expressions for the Lagrangian and the Hamiltonian in the S' frame. The Lagrangian and the Hamiltonian for a strong circularly polarized laser beam have also been obtained in the S' frame. The exact form of the nonlinear dispersion relation is derived for circularly polarized waves. Then the results for the frequency and the wave number shifts of these waves in a cold, magnetized relativistic plasma are obtained with some discussion on the nature of the frequency shifts. Finally, numerical results are presented for the radiation of Nd-glass laser in dense plasmas.

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1. INTRODUCTION.

Using a special Lorentz transformation (SLT), the space-time dependent variables of an electromagnetic wave in a plasma in the laboratory frame, S become space independent in a moving frame, S' when the latter frame moves with a relative velocity c^2/V, V is the group velocity of the wave. Winkles and Eldridge [1] first employed the SLT to obtain self-consistent solutions of the relativistic Vlasov-

Maxwell equations, and have shown that a pure transverse wave cannot exist, but the coupled longitudinal field does necessarily appear. They also obtained a nonlinear dispersion relation correct up to the squares of the field amplitude. Thus the Lorentz transformation (LT) is found to be very useful because of the fact that it can transform a system of nonlinear partial differential equations for a plasma into a set of ordinary differential equations.

Several authors including Clemmow [2-3], Chian and Clemmow [4], Kennal and Pellat [5], Shih [6], Decoster [7], Lee and Lerche [8-11], Clemmow and Harding [12] have used the special Lorentz transformation to study nonlinear problems in plasma dynamics. On the other hand, Akhiezer and Polovin [13], Wong [14], Wong and Lojko [15] have employed similar transformation relations for the investigation of nonlinear wave propagation in relativistic plasmas. Recently, Paul and Chakraborty [16-17] developed the theory of transformations of nonlinear plasma equations, and extended it to yield the nonlinear precessional rotation of an elliptically polarized, electromagnetic wave in an unmagnetized, cold, collisionless plasma in addition to the nonlinear shifts of wave parameters.

The main purpose of the present work is to develop the work of Winkles and Eldridge [1] for the transformation of field variables from a laboratory inertial frame, S to the space independent frame, S'. Such a study is very useful for the investigation of several nonlinear effects in plasmas. In particular, the study of self-action effects including self-focusing, self-steepening, self-phase modulation, self-precession is expected to be simpler by using the transformation relations for the S' frame for several reasons. First, some of the field variables become either constant or zero. For instance, the number density of electrons and the scalar potential become constant, and the oscillation of the magnetic field vanishes. Second, some nonlinear terms, which appear in the S frame, vanish in the S' frame. Consequently, the nonlinear terms are fewer in number in the S' frame than in the S frame.

Motivated by the above discussions, we shall use the special Lorentz transformation to transform the space-time dependent equations in a cold, relativistic, magnetized plasma to the S' frame. These transformed equations are then employed to derive the expressions for the Lagrangian and the Hamiltonian for a strong circularly polarized laser beam in the S' frame. The exact expression for the nonlinear dispersion relation is derived for circularly polarized waves. Then the results for the frequency and the wave number shifts of the waves in a cold, magnetized relativistic plasma are obtained. Some attention is given to the nature of the frequency shift for different intensities of the wave and the static magnetic field. Finally, some numerical results for the radiation of Nd-glass laser in dense plasmas are presented.

2. BASIC EQUATIONS AND ASSUMPTIONS.

We make the following assumptions:

(i) The plasma is cold, homogeneous and stationary, and is subject to a strong radiation with intensity less than 3×10^{22} watts/cm^2 resulting in electron velocity becoming relativistic. The ion motion is negligibly small in comparison with the electron motion.

(ii) The forces due to other sources including gravitational and ponderomotive forces are also negligible.

With these assumptions, the basic equations of plasma in the S-frame are given by

$$[D + (\underline{v}\cdot\underline{\nabla})]\, \underline{p} = -e\underline{E} - \frac{e}{c}\,(\underline{v} \times \underline{H}) \tag{2.1}$$

$$DN + \underline{\nabla}\cdot(N\underline{v}) = 0 \tag{2.2}$$

$$\underline{\nabla} \times \underline{E} = -\frac{1}{c}\, D\, \underline{H} \tag{2.3}$$

$$\underline{\nabla} \times \underline{H} = -\frac{1}{c}\, D\, \underline{E} - \frac{4\pi e}{c}(N\underline{v}) \tag{2.4}$$

$$\underline{\nabla}\cdot\underline{E} = -\,4\pi e\,(N-N_i) \tag{2.5}$$

$$\underline{\nabla}\cdot\underline{H} = 0 \tag{2.6}$$

where $\underline{p} = m_0\gamma\underline{v}$, $\gamma = (1 - \frac{v^2}{c^2})^{-\frac{1}{2}}$, $D \equiv \frac{\partial}{\partial t}$; (2.7abc)

and N_i and N are the number densities of ions and electrons, m_0 and $-e$ are the rest mass and charge of an electron, and other parameters have their usual meanings.

3. SPACE-INDEPENDENT FRAME AND TRANSFORMATION OF QUANTITIES TO IT.

The Lorentz transformation from the S-frame to the S'-frame moving with a relative velocity V_0 parallel to the z-axis is given by

$$t = (t' + V_0 z'/c^2)\gamma_0,\ z = (z' + V_0 t')\gamma_0,\ x = x'\ ,\ y = y'\ , \tag{3.1}$$

where $\gamma = \frac{1}{(1-\beta_0^2)^{\frac{1}{2}}}$, $\beta_0 = \frac{V_0}{c}$ (3.2ab)

We follow Decoster [7] to obtain

$$\beta_0 = \tanh \Psi_0 \tag{3.3}$$

and find from the relations (3.1) that

$$\left.\begin{aligned} \gamma_0 = \cosh \Psi_0,\ \beta_0\gamma_0 = \sinh \Psi_0\ , \\ t = t' \cosh \Psi_0 + \frac{z'}{c} \sinh \Psi_0\ , \\ z = z' \cosh \Psi_0 + ct' \sinh \Psi_0\ , \\ x = x',\ y = y'\ . \end{aligned}\right\} \tag{3.4abc}$$

It is obvious that Ψ_0 is the hyperbolic angle for the S'-system relative to the S-system. The reverse transformation from S' to S is obtained by changing the sign of V_0 or Ψ_0, that is,

$$\left.\begin{aligned} t' = t \cosh \Psi_0 - \frac{z}{c} \sinh \Psi_0\ , \\ z' = z \cosh \Psi_0 - ct \sinh \Psi_0 \\ x' = x,\ y' = y \end{aligned}\right\} \tag{3.5abc}$$

These relations show that the wave phase

$$\omega t-kz = (\omega \cosh \Psi_0 - kc \sinh \Psi_0)t'-(k \cosh \Psi_0 - \frac{\omega}{c} \sinh \Psi_0)z' , \qquad (3.6)$$

where k and ω are the constant wave number and wave frequency respectively of an electromagnetic wave.

Following Winkles and Eldridge [1] we consider the transformation from S to S' at the velocity $V_0 = kc^2/\omega$ to obtain the phase velocity

$$V = \omega/k = c^2/V_0 \qquad (3.7)$$

Then (3.6) reduces to

$$\omega t - kz = \omega (1-\beta_0)^{\frac{1}{2}} \cdot t' \qquad (3.8)$$

This transformation enables us to change the field variables from the space-time dependent S frame to the space-independent S'-frame of primed variables. It is to be noted here that for transverse waves, the phase velocity $V(= \omega|k) > c$ and so $V_0 < c$. Therefore the velocity of S' relative to S is not unphysical.

Replacing t' by T we can write

$$\left.\begin{aligned} \omega' T &= \omega t - kz \\ v_z &= \dot{z} = V - \frac{V}{\gamma_0}\frac{\partial T}{\partial t} \end{aligned}\right\} \qquad (3.9ab)$$

$$\left.\begin{aligned} \omega' &= \omega (1-\beta_0)^{\frac{1}{2}} = \omega (1-\frac{c^2}{V^2})^{\frac{1}{2}} = \omega/\gamma_0 \\ \gamma_0^2 &= 1/(1- \frac{c^2}{V^2}) = 1/(1- \frac{V_0^2}{c^2}) \end{aligned}\right\} \qquad (3.10ab)$$

$$\left.\begin{aligned} \omega t - kz &= \omega (1-\frac{c^2}{V^2})^{\frac{1}{2}} \cdot T \\ \frac{\partial}{\partial t} &= \gamma_0 \frac{\partial}{\partial T} , \quad \frac{\partial}{\partial z} = - \frac{\gamma_0}{V}\frac{\partial}{\partial T} \end{aligned}\right\} \qquad (3.11ab)$$

Akhiezer et al. [18] considered a linear transformation rule

$$\omega T = \omega t - kz, \quad V = \omega/k, \qquad (3.12ab)$$

to solve the plasma equations (2.1) to (2.6) neglecting collision effect for some nonlinear problems. Boyd and Sanderson [19] considered the special value, $V = c$ for some investigations.

The transformation of some field variables from the S-frame to the S'-frame is given by

$$\left\{\begin{aligned} v_x &= v_x'/\gamma_0(1 + \beta_0 v_z'/c) = (cv_x')/(c \cosh \Psi_0 + v_z' \sinh \Psi_0) , \\ v_y &= v_y'/\gamma_0(1+\beta_0 v_z'/c) = (cv_y')/(c \cosh \Psi_0 + v_z' \sinh \Psi_0) , \\ v_z &= (v_z' + \beta_0 c)/(1+\beta_0 v_z'/c) = c(v_z' \cosh \Psi_0 + c \sinh \Psi_0)/(c \cosh \Psi_0 + v_z' \sinh \Psi_0) \end{aligned}\right. \qquad (3.13abc)$$

$$\left.\begin{aligned} E_x &= \gamma_0(E_x' + \beta_0 H_y') = E_x' \cosh \Psi_0 + H_y' \sinh \Psi_0 , \\ E_y &= \gamma_0(E_y' - \beta_0 H_x') = E_y' \cosh \Psi_0 - H_x' \sinh \Psi_0 , \\ E_z &= E_z' \end{aligned}\right\} \qquad (3.14abc)$$

$$\left.\begin{aligned} H_x &= \gamma_0(H_x' - \beta_0 E_y') = (H_x' \cosh \Psi_0 - E_y' \sinh \Psi_0) \\ H_y &= \gamma_0(H_y' + \beta_0 E_x') = H_y' \cosh \Psi_0 + E_x' \sinh \Psi_0 , \\ H_z &= H_z' \end{aligned}\right\} \qquad (3.15abc)$$

The transformation of mass is

$$m = m'\gamma_0(1 + \beta_0 v_z'/c) \tag{3.16}$$

as may be seen from Hughes and Young [20, p18-19] where m_0 (but not m) is the rest mass. The momentum components are transformed into the form

$$\left.\begin{aligned} P_x &= mv_x = m'\gamma_0(1+\beta_0 v_z'/c)\frac{c}{(c\cosh\Psi_0 + v_z'\sinh\Psi_0)} = m'v_x' = P_x', \\ P_y &= mv_y = m'v_y' = P_y', \\ P_z &= mv_z = m'\gamma_0(1+\beta_0 v_z'/c)\frac{c(v_z'\cosh\Psi_0 + c\sinh\Psi_0)}{(c\cosh\Psi_0 + v_z'\sinh\Psi_0)} \\ &= P_z'\gamma_0 + m'c\beta_0\gamma_0 = \gamma_0(P_z' + m_0 V_0\gamma') \end{aligned}\right\} \tag{3.17abc}$$

Defining now the momentum like quantities q and q' as

$$q = (m_0^2 c^2 + p^2)^{\frac{1}{2}}, \quad q' = (m_0^2 c^2 + p'^2)^{\frac{1}{2}}, \tag{3.18ab}$$

$$q'^2 = m_0^2 c^2 + \frac{m_0^2 V'^2}{(1-V'^2/c^2)} = \frac{m_0^2 c^2}{(1-V'^2/c^2)} = m'^2 c^2 \tag{3.19}$$

where V' is the velocity of the S'-frame relative to the rest system S_0. So we have

$$P_z = P_z'\cosh\Psi_0 + q'\sinh\Psi_0, \tag{3.20}$$

$$\begin{aligned} q &= mc = m'\gamma_0 c(1+\beta_0 v_z'/c) = m'\gamma_0(c + \beta_0 v_z') \\ &= \gamma_0(q' + P_z'\beta_0) = q'\cosh\Psi_0 + P_z'\sinh\Psi_0 \end{aligned} \tag{3.21}$$

If N, N', N_0 are the symbols for the number density in the three systems S, S', S_0 respectively, then again following Hughes and Young [20] we get

$$N' = N_0/(1-V'^2/c^2)^{\frac{1}{2}} \tag{3.22}$$

so we can write

$$m_0 N' = m_0 N_0/(1-V'^2/c^2)^{\frac{1}{2}} = m' N_0 \tag{3.23}$$

Similarly, we can also write

$$m_0 N = m N_0 \tag{3.24}$$

$$\frac{N'}{N} = \frac{m'}{m} = \frac{m'c}{m c} = \frac{q'}{q} \tag{3.25}$$

Since $\underline{A}$ and ϕ form a four vector ($\underline{A}$ is vector potential and ϕ is the scalar potential), and we can write

$$\left.\begin{aligned} A_z &= \gamma_0(A_z'+\beta_0\phi') = A_z'\cosh\Psi_0 + \phi'\sinh\Psi_0, \quad \underline{A}_\perp = \underline{A}_\perp' \\ \phi &= \gamma_0(\beta_0 A_z' + \phi') = \phi'\cosh\Psi_0 + A_z'\sinh\Psi_0 \end{aligned}\right\} \tag{3.26ab}$$

The reverse transformation is obtained by replacing Ψ_0 by $-\Psi_0$.

Equation (3.21) gives

$$N_0 = N'\,\text{sech}\,\Psi' = \text{constant} \tag{3.27}$$

and (3.25) can be written as

$$N = N'q/q' = N_0\, q\cosh\Psi'/\{q\cosh(\Psi'-\Psi) - P_z\sinh(\Psi'-\Psi)\} \tag{3.28}$$

Here Ψ is the hyperbolic angle for the S-system relative to the S_0 frame and Ψ' is the same for the S' system. Therefore, in the S-system, the number density is not constant, and the electron and ion densities are not necessarily equal.

4. TRANSFORMATION OF FIELD EQUATIONS TO THE SPACE INDEPENDENT FRAME

Using the transformation relations of section 3, we obtain from (2.1) for the x-component

$$\gamma_0(1-\frac{v_z'}{V})\frac{\partial P_x'}{\partial T} = \{\gamma_0 V - V_0 + (\gamma_0-1)v_z'\}(\frac{\partial P_x'}{\partial T})/V(1+\beta_0 v_z'/c)$$

$$= -e\gamma_0(E_x'+\beta_0 H_y') + (e/c)(v_z'+V_0)\gamma_0(H_y'+\beta_0 E_x')/(1+\beta_0 v_z'/c) - ev_y'H_z'/\gamma_0 c(1+\beta_0 v_z'/c)$$

$$= -\frac{\beta_0 c\, E_x'\{\gamma_0 V - V_0+(\gamma_0-1)v_z'\}}{c(1+\beta_0 v_z'/c)} - \frac{eH_y'\{V_0(\gamma_0-1)+(\beta_0^2\gamma_0-1)v_z'\}}{c(1+\beta_0 v_z'/c)} - \frac{e\, v_y' H_z'}{\gamma_0 c(1+\beta_0 v_z'/c)} \quad (4.1)$$

This equation can be simplified to the form

$$\frac{\partial P_x'}{\partial T} = -eE_x' - \frac{eH_y' V}{c(\gamma_0 V - V_0)}\left[V_0(\gamma_0-1) - \frac{Vv_z'}{\gamma_0^3\{\gamma_0 V - V_0+(\gamma_0-1)v_z'\}}\right] - \frac{eV\, v_y' H_z'}{\gamma_0 c\{\gamma_0 V - V_0+(\gamma_0-1)v_z'\}} \quad (4.2)$$

Similarly from (2.1) the equation for P_y' becomes

$$\frac{\partial P_y'}{\partial T} = -eE_y' + \frac{eH_x' V}{c(\gamma_0 V - V_0)}\left[V_0(\gamma_0-1) - \frac{Vv_z'}{\gamma_0^3\{\gamma_0 V - V_0+(\gamma_0-1)v_z'\}}\right] + \frac{eV\, v_x' H_z'}{\gamma_0 c\{\gamma_0 V - V_0+(\gamma_0-1)v_z'\}} \quad (4.3)$$

Now, the equation for P_z' obtained from (2.1) is

$$\frac{\partial P_z'}{\partial T} + m_0 V_0\frac{\partial \gamma'}{\partial T} = -\frac{eV\{E_z' + \beta_0(\underline{\beta}'\cdot\underline{E}') + (\underline{\beta}'\times\underline{H}')_z\}}{\gamma_0\{\gamma_0 V - V_0 + (\gamma_0-1)v_z'\}}, \quad (4.4)$$

where

$$\frac{\partial\gamma'}{\partial T} = -\frac{1}{2}(1-\frac{v'^2}{c^2})^{-3/2}\cdot(-\frac{2}{c^2})(\underline{v}'\cdot\frac{\partial \underline{v}'}{\partial T}) = \gamma'^3(\underline{\beta}'\cdot\frac{\partial\underline{\beta}'}{\partial T}) \quad (4.5)$$

Since $\beta = \underline{P}/(P^2 + m_0^2c^2)^{\frac{1}{2}}$, and $P^2 + m_0^2c^2 = m_0^2c^2\gamma^2$, we obtain

$$\frac{\partial\underline{\beta}}{\partial T} = \frac{1}{m_0 c\gamma}\frac{\partial\underline{P}}{\partial T} - \frac{\underline{P}(\underline{P}\cdot\partial\underline{P}/\partial T)}{m_0^3c^2\gamma^3}. \quad (4.6)$$

Therefore,

$$(\underline{\beta}'\cdot\frac{\partial\underline{\beta}'}{\partial T}) = \frac{(\underline{P}'\cdot\partial\underline{P}'/\partial T)}{(m_0c\gamma')^2} - \frac{P'^2(\underline{P}'\cdot\partial\underline{P}'/\partial T)}{(m_0c\gamma')^4} = \frac{1}{m_0^2c^2\gamma'^4}(\underline{P}'\cdot\frac{\partial\underline{P}'}{\partial T}) \quad (4.7)$$

Hence

$$\frac{\partial P_z'}{\partial T} + \frac{V_0}{m_0c^2\gamma'}(\underline{P}'\cdot\frac{\partial\underline{P}'}{\partial T}) = \frac{eV\{E_z' + \beta_0(\underline{\beta}'\cdot\underline{E}') + (\underline{\beta}'\times\underline{H}')_z\}}{\gamma_0\{\gamma_0 V - V_0 + (\gamma_0-1)v_z'\}} \quad (4.8)$$

Multiplying (4.2) by P_x', (4.3) by P_y' and (4.8) by P_z' and then adding, we get after simplification

$$(\underline{P}' \cdot \frac{\partial \underline{P}'}{\partial T}) = -\frac{eV(P_x'E_x' + P_y'E_y')\{\gamma_0(\gamma_0 V - V_0) + v_z'(\gamma_0^2 - \gamma_0 + 1)\}}{\gamma_0(V + v_z')\{\gamma_0 V - V_0 + (\gamma_0 - 1)v_z'\}}$$
$$- \frac{eVP_z'E_z'}{\gamma_0\{\gamma_0 V - V_0 + (\gamma_0 - 1)v_z'\}}$$
$$- \frac{eV^2(\underline{P}'x\underline{H}')_z\{v_z'(V - \gamma_0^2 V_0 + \gamma_0^4 V) + V\gamma_0^3(\gamma_0 V - V_0)\}}{\gamma_0^3 c(V + v_z')(\gamma_0 V - V_0)\{\gamma_0 V - V_0 + (\gamma_0 - 1)v_z'\}}$$
$$- \frac{eV(\underline{P}'x\underline{v}')_z H_z'}{c\gamma_0\{\gamma_0 V - V_0 + (\gamma_0 - 1)v_z'\}} \tag{4.9}$$

Therefore, the equation for P_z' becomes

$$\frac{\partial P_z'}{\partial T} = -\frac{eVE_z'}{\gamma_0\{\gamma_0 V - V_0 + (\gamma_0 - 1)\, v_z'\}}$$
$$+ \frac{e(P_x'E_x' + P_y'E_y')\{V(\gamma_0^2 V - \gamma_0 V_0 - V_0) + v_z(V(\gamma_0^2 - \gamma_0 + 1) - V_0)\}}{m_0\gamma_0\gamma' V_0(V - v_z')\{\gamma_0 V - V_0 + (\gamma_0 - 1)v_z'\}}$$
$$+ \frac{eVP_z'E_z'}{m_0\gamma_0^3\gamma' V_0\{\gamma_0 V - V_0 + (\gamma_0 - 1)v_z'\}}$$
$$+ \frac{e(\underline{P}'x\underline{H}')_z V^2\{v_z'(V + V\gamma_0^6 - V_0\gamma_0^5) + \gamma_0^4(\gamma_0 V_0 - V_0)^2\}}{c\, m_0\gamma_0^5\gamma' V_0(V + v_z')(\gamma_0 V - V_0)\{\gamma_0 V - V_0 + (\gamma_0 - 1)v_z'\}}$$
$$+ \frac{eVV_0(\underline{P}'x\underline{v}')_z H_z'}{m_0 c^3\gamma_0\gamma'\{\gamma_0 V - V_0 + (\gamma_0 - 1)v_z'\}} \tag{4.10}$$

The equation of continuity (2.2) assumes the form

$$\gamma_0 \frac{\partial}{\partial T}(\frac{N'\gamma}{\gamma'}) - \frac{\gamma_0}{V}\frac{\partial}{\partial T}\{\frac{N'\gamma}{\gamma'} \cdot \frac{c(\beta_z' + \beta_0)}{(1 + \beta_0\beta_z')}\} = 0 \tag{4.11}$$

It can be written as

$$\gamma_0 \frac{\partial}{\partial T}\ [\frac{N'(1 - \beta_0^2)}{\beta_0}] = 0. \tag{4.12}$$

And so, $N' = N_0$, where N_0 is a constant. (4.13)

Equation for H_x', obtained from (2.3), is given by

$$-\frac{1}{V}\frac{\partial}{\partial T}(E_y' - \beta_0 H_x') = \frac{1}{c}\frac{\partial}{\partial T}(H_x - \beta_0 E_y') \tag{4.14}$$

It can be written as

$$(\frac{1}{\beta_0} - \beta_0)\frac{\partial H_x'}{\partial T} = 0 \tag{4.15}$$

So, $$H_x' = \text{constant} \tag{4.16}$$

And the equation for H_y', obtained from equation (2.3), is given by

$$\frac{\partial}{\partial T}[E_x'(1 - \frac{\beta_0 V}{c}) + H_y'(\beta_0 - \frac{V}{c})] = 0 \tag{4.17}$$

Hence, $$H_y' = \text{constant}\ . \tag{4.18}$$

The x-component of equation (2.4) is

$$-\frac{\gamma_0^2}{V}\frac{\partial}{\partial T}(H_y' + \beta_0 E_x') = -\frac{\gamma_0^2}{c}\frac{\partial}{\partial T}(E_x' + \beta_0 H_y') + \frac{4\pi e N_0 v_x'}{c\gamma_0(1+\beta_0\beta_z')} \ . \tag{4.19}$$

Or

$$\frac{\partial E_x'}{\partial T} = \frac{4\pi e N_0 v_x'}{\gamma_0(1+\beta_0\beta_z')} \ . \tag{4.20}$$

Similarly, the y- and z-components of equation (2.4) become

$$\frac{\partial E_y'}{\partial T} = \frac{4\pi e N_0 v_y'}{\gamma_0(1+\beta_0\beta_z')} \ . \tag{4.21}$$

and

$$\frac{\partial E_z'}{\partial T} = \frac{4\pi e N_0 (v_z' + V_0)}{\gamma_0(1+\beta_0\beta_z')} \ . \tag{4.22}$$

We shall now expand $1/(1+\beta_0\beta_z')$ in powers of β_z'. Since $v_z = 0$ when $v_z' = -v_0$, we put $v_z = -V_0 + \delta v_z'$ to avoid a physical impossibility. Therefore,

$$\frac{1}{\gamma_0(1+\beta_0\beta_z')} = \gamma_0^2[1 - \gamma_0^2(\frac{\delta v_z}{V}) + \gamma_0^4(\frac{\delta v_z'}{V})^2 - \ldots] \tag{4.23}$$

So, the velocity components become

$$v_x = v_x'\gamma_0[1 - \gamma_0^2(\frac{\delta v_z'}{V}) + \gamma_0^4(\frac{\delta v_z'}{V})^2 - \ldots] \tag{4.24}$$

$$v_y = v_y'\gamma_0[1 - \gamma_0^2(\frac{\delta v_z'}{V}) + \gamma_0^4(\frac{\delta v_z'}{V})^2 - \ldots] \tag{4.25}$$

$$v_z = \gamma_0^2\delta v_z'[1 - \gamma_0^2(\frac{\delta v_z'}{V}) + \gamma_0^4(\frac{\delta v_z'}{V})^2 - \ldots] \tag{4.26}$$

where v_x' and v_y' are the first order velocity components in the directions parallel to OX and OY respectively, $\delta v_z'$ is the second order velocity component along -OZ in the S'-frame.

The continuity equation (2.2) gives

$$N = N_0 + \delta N' = N_0/(1 - \gamma_0^2\delta v_z'/V) \tag{4.27}$$

and so

$$\delta N' = N_0\gamma_0^2\delta v_z'/V \ , \tag{4.28}$$

where N_0 is a constant, and higher powers of $\delta v_z'$ are neglected.

Therefore, equations (4.20)-(4.22) for the electric field components become

$$\frac{\partial E_x'}{\partial T} = \frac{4\pi e}{\gamma_0} \cdot \frac{(N_0 + \delta N')v_x'}{(1+\beta_0\beta_z')} = \frac{\gamma_0 m_0 \omega_e^2}{e} v_x' \ , \tag{4.29}$$

$$\frac{\partial E_y'}{\partial T} = \frac{\gamma_0 m_0 \omega_p^2}{e} v_y' \ , \tag{4.30}$$

$$\frac{\partial E_z'}{\partial T} = \frac{\gamma_0 m_0 \omega_p^2}{e} \delta v_z' \ , \tag{4.31}$$

where $\omega_p = (4\pi N_0 e^2/m_0)^{\frac{1}{2}}$, ω_p being the electron plasma frequency. Again from the transformation relations (3.14abc) and (3.15abc), neglecting the transverse magnetic field components in the S'-frame, we obtain from the transformation relations

$$E_x = \gamma_0 E_x', \ E_y = \gamma_0 E_y', \ E_z = E_z' \tag{4.32abc}$$

$$H_x = -\gamma_0\beta_0 E_y', \ H_y = -\gamma_0\beta_0 E_x', \ H_z = H_z' = H_0 \tag{4.33abc}$$

5. THE LAGRANGIAN AND HAMILTONIAN IN THE SPACE-INDEPENDENT FRAME

Following Landau and Lifshitz [21], the expressions for the Lagrangian and Hamiltonian in plasmas in the laboratory frame are given by

$$\mathcal{L} = -N_0[m_0c^2(1 - \frac{v^2}{c^2})^{\frac{1}{2}} + \frac{e}{c}(\underline{A}\cdot\underline{v}) + e\phi] + \frac{E^2 - H^2}{8\pi} \tag{5.1}$$

$$H' = N_0\underline{v}[\frac{m_0\underline{v}}{(1 - \frac{v^2}{c^2})^{\frac{1}{2}}} - \frac{e}{c}\underline{A}] - \mathcal{L} \tag{5.2}$$

where the vector and scalar potentials $\underline{A}$ and ϕ respectively are defined by

$$\underline{H} = \text{curl}\,\underline{A} \tag{5.3}$$

$$\underline{E} = -\frac{1}{c}\frac{\partial\underline{A}}{\partial t} + \underline{\nabla}\phi \tag{5.4}$$

$$\text{div.}\,\underline{A} = -\frac{1}{c}\frac{\partial\phi}{\partial t}. \tag{5.5}$$

The Lorentz gauge condition (5.5), when transformed to the space-independent frame, gives

$$-\frac{\gamma_0^2}{V}\frac{\partial}{\partial T}(A_z' + \beta_0\phi') = -\frac{\gamma_0^2}{c}\frac{\partial}{\partial T}(\phi' + \beta_0 A_z') \tag{5.6}$$

Since the terms containing A_z' cancel from both sides, and since a constant potential is neglected here, it turns out that

$$\phi' = 0. \tag{5.7}$$

Therefore, in the S'-frame the Lagrangian and the Hamiltonian are found to be

$$\begin{aligned}\mathcal{L}' = &-N_0[m_0c^2\{1 - \frac{\gamma_0^2}{c^2}(v_+v_- + \gamma_0^2\delta v_z'^2)(1 - \frac{\gamma_0^2}{V}\delta v_z' + \frac{\gamma_0^4}{V^2}\delta v_z'^2 - \ldots)^2\}^{\frac{1}{2}} + \\ &+e\gamma_0(1 - \frac{\gamma_0^2}{V}\delta v_z' + \frac{\gamma_0^4}{V^2}\delta v_z'^2 - \ldots)\{\tfrac{1}{2}(A_+v_- + A_-v_+) + \gamma_0^2 A_z'\delta v_z'\} + \\ &+e\gamma_0\beta_0 A_z'] + \frac{\gamma_0^2}{8\pi}(1 - \beta_0^2)(E_+E_-) + E_z'^2\end{aligned} \tag{5.8}$$

$$\begin{aligned}H' = &N_0[m_0c^2\{1 - \frac{\gamma_0^2}{c^2}(v_+v_- + \gamma_0^2\delta v_z'^2)(1 - \frac{\gamma_0^2}{V}\delta v_z' + \frac{\gamma_0^4}{V^2}\delta v_z'^2 - \ldots)^2\}^{-\frac{1}{2}} + e\gamma_0\beta_0 A_z'] \\ &- \frac{\gamma_0^2}{8\pi}(1 - \beta_0^2)(E_+E_-) - E_z'^2\end{aligned} \tag{5.9}$$

where $A_\pm = A_x' \pm iA_y'$, $E_\pm = E_x' \pm iE_y'$, $v_\pm = v_x' \pm iv_y'$.

For a circularly polarized laser beam, we put

$$E_x' = a\cos\theta,\; E_y' = a\sin\theta,\; E_z' = 0 \tag{5.10abc}$$

where a is the constant amplitude of the wave, and $\theta = \omega_\pm' T$, the subscripts + and - signs indicate the left and right circular polarized components respectively. In this case, $E_\pm = ae^{\pm i\theta}$, and the solution of the plasma field equations (3.1abcd)-(3.9ab) is exact.

For a circularly polarized wave (5.8) and (5.9) reduce to

$$\mathcal{L}' = -N_0[m_0c^2(1 - \frac{e^2a^2\omega_\pm'^2}{\omega_p^4 m_0^2 c^2})^{\frac{1}{2}} + \frac{e^2a^2}{m_0^2\omega_p^2}] + \frac{\gamma_0^2 a^2}{8\pi}(1 - \frac{c^2}{V^2}) \tag{5.11}$$

$$H' = N_0[m_0c^2(1 - \frac{e^2a^2\omega_\pm'^2}{\omega_p^4m_0^2c^2})^{-\frac{1}{2}}] - \frac{\gamma_0^2 a^2}{8\pi}(1 - \frac{c^2}{V^2}) \tag{5.12}$$

The results for the Lagrangian and Hamiltonian in the laboratory inertial frame are obtained simply by changing $\omega_\pm'$ to $\omega_\pm/\gamma_0$ in (5.11) and (5.12), where $\gamma_0^2 = \omega_\pm^2/(\omega_\pm^2 - k_\pm^2c^2)$.

Following the method of averaged Lagrangian developed by Whitham [22-25] and extended by Dysthe [26], Das and Sihi [27-29] and others, the Lagrangian derived in this section can be used for finding the nonlinear effects including the shifts of wave parameter in the space-independent frame and then in the laboratory frame with the help of the transformation relations in Section 3.

6. NONLINEAR DISPERSION RELATION

For a purely transverse circularly polarized wave,

$$E_\pm = a\exp(\pm i\theta) \quad \text{and} \quad E_z' = 0. \tag{6.1}$$

So, we have

$$P_z' = 0,\ v_z' = -V_0 + \delta v_z' = -V_0 \tag{6.2}$$

From (4.1), (4.3), (2.7abc), (3.13abc), and (3.17abc), we obtain

$$\dot{P}_\pm' = -eE_\pm' \pm \frac{ieVv_\pm' H_0}{\gamma_0^3 c(V - V_0)} \tag{6.3}$$

$$\dot{E}_\pm' = \frac{\gamma_0 m_0 \omega_p^2}{e} v_\pm' \tag{6.4}$$

$$P_\pm' = m_0 v_\pm'/[1 - \frac{V_0^2}{c^2} + \frac{(v_+' \cdot v_-')}{c^2}]^{\frac{1}{2}} \tag{6.5}$$

where $P_\pm' = P_x' \pm iP_y'$, $E_\pm' = E_x' \pm iE_y'$, $v_\pm' = v_x' \pm iv_y'$, and a dot denotes the derivative with respect to time.

Using (6.1) in (6.3) to (6.5), the nonlinear dispersion relation for the left and right circular polarization components in the space-independent frame can be obtained

$$\frac{\omega_\pm'^2}{\left[1 - \frac{e^2a^2\omega_\pm'^2}{\omega_p^4m_0^2c^2}\right]^{\frac{1}{2}}} - \omega_p^2 \pm \frac{V\Omega_0\omega_\pm'}{\gamma_0^3(V - V_0)} = 0\ , \tag{6.6}$$

where $\Omega_0 = eH_0/m_0c$. Replacing $\omega_\pm'$ by $\omega_\pm/\gamma_0$, the dispersion relation (6.6) in the laboratory frame becomes

$$\frac{\omega_\pm^2 - k_\pm^2c^2}{\left[1 - \frac{e^2a^2(\omega_\pm^2 - k_\pm^2c^2)}{\omega_p^4m_0^2c^2}\right]^{\frac{1}{2}}} - \omega_p^2 \pm \Omega_0(\omega_\pm^2 - k_\pm^2c^2)^{\frac{1}{2}} = 0\ . \tag{6.7}$$

In the absence of magnetic field, (6.7) becomes

$$\omega^2 - k^2c^2 = \omega_p^2(1 + \alpha_0^2)^{-\frac{1}{2}} \tag{6.8}$$

where α_0 is the dimensionless amplitude parameter $ea/m_0\omega c$. It is to be mentioned that (6.8) is identical to the results of Paul and Chakraborty [16-17]. Expanding the right-hand side of (6.8) in power of α_0^2 and retaining only the first two terms of binomial expansion we obtain the results of Arons and Max [30] if their symbols are adopted.

7. FREQUENCY AND WAVE NUMBER SHIFT OF AN ELECTROMAGNETIC WAVE

Nonlinearity in plasmas produces many interesting results in the medium and on the waves. Electromagnetic wave would have shifts in the wave parameters (wave number and frequency) as a result of nonlinear interactions in plasmas [31-41]. We shall derive here the expressions for the shifts of frequency and wave number of a circularly polarized wave propagating through a cold, magnetized relativistic plasma.

(a) Frequency Shifts:

For the temporal evolution problem, we can write

$$k_{\pm} = k \quad \text{and} \quad \omega_{\pm} = \omega_{\pm}^{0} + \delta\omega_{\pm} \tag{7.1}$$

where $\delta\omega_{\pm}$ are the increments of frequencies of the left and right circular polarization components of the wave. Therefore, neglecting higher powers of $\delta\omega_{\pm}$, we find that frequency shifts of the two polarization components are

$$\delta\omega_{\pm} = -\frac{\omega_1^2\left[1 + \dfrac{\alpha^2\omega_1^2}{2\omega_p^2} \mp \dfrac{\Omega_0}{\omega_1} - \dfrac{\omega_p^2}{\omega_1^2}\right]}{2\omega_{\pm}^{0}\left[1 + \dfrac{\alpha^2\omega_1^2}{\omega_p^2} \mp \dfrac{\Omega_0}{\omega_1}\right]} \tag{7.2}$$

where $\omega_1^2 = \omega_{\pm}^{0^2} - k^2c^2$. (7.3)

(b) Wave number shifts:

In the spatial evolution problem, we write

$$\omega_{\pm} = \omega, \; k_{\pm} = k_{\pm}^{0} + \delta k_{\pm} \tag{7.4}$$

where $\delta k_{\pm}$ are the increments of wave numbers of the left and right circular polarization components of the wave. Therefore, neglecting higher powers of $\delta k_{\pm}$; the expressions for the wave number shift of the two polarization components can be derived as

$$\delta k_{\pm} = -\frac{\omega_2^2\left[1 + \dfrac{\alpha^2\omega_2^2}{\omega_p^2} \mp \dfrac{\Omega_0}{\omega_2} - \dfrac{\omega_p^2}{\omega_2^2}\right]}{2k_{\pm}^{0}c^2\left[1 + \dfrac{\alpha^2\omega_2^2}{\omega_p^2} \mp \dfrac{\Omega_0}{\omega_2}\right]} \tag{7.5}$$

where $\omega_2^2 = \omega^2 - k_{\pm}^{0^2}c^2$. (7.6)

8. DISCUSSIONS AND CONCLUSIONS

From (7.2) and (7.5) it is observed that the intensity of the wave and static magnetic field have important roles to create the shifts of the wave parameters. It is seen that (i) when $\omega > kc$ and $\omega \simeq \Omega_0$, the LCP components of the wave do have a frequency shift when its intensity $\alpha \neq \sqrt{2}\,(\omega_p^2/\omega^2)$, but the frequency shifts of the RCP components exist only when α is greater or less than $\sqrt{2}(\omega_p/\omega)\times(\omega_p^2/\omega^2 - 2)^{\frac{1}{2}}$, and (ii) when $\omega > kc$ and $\omega \simeq \omega_p$, the frequency shift of LCP components become nonzero when $\alpha \neq (\Omega_0/\omega)^{\frac{1}{2}}$, but the RCP components always have frequency shifts for all possible values of α. The nature of the frequency shifts of RCP and LCP components of the electromagnetic wave can be well understood from Table 1 and Figures 1-3.

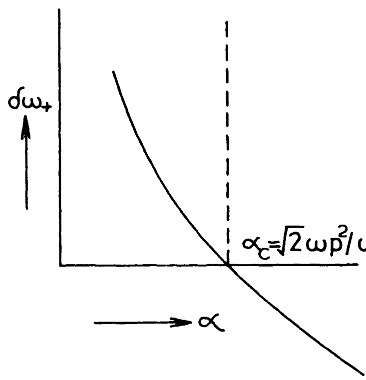

Fig. 1 Frequency shift of the LCP component V.S. Intensity of the wave (when $\omega > kc$ and $\omega \simeq \Omega_0$).

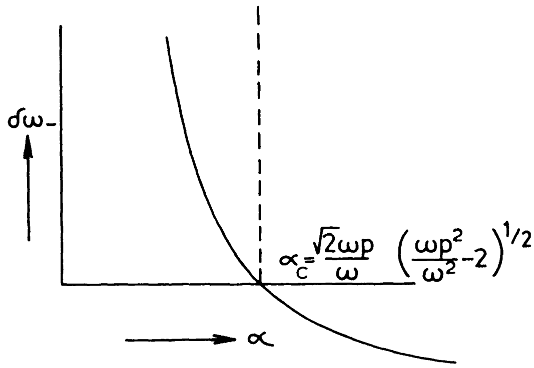

Fig. 2 Frequency shift of the RCP component V.S. Intensity of the wave (when $\omega > kc$ and $\omega \simeq \Omega_0$).

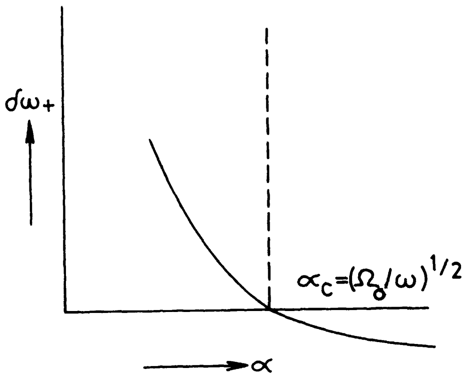

Fig. 3 Frequency shift of the LCP component V.S. Intensity of the wave (when $\omega > kc$ and $\omega \simeq \omega_p$).

9. NUMERICAL ESTIMATION

For the radiations of Nd-glass laser having wave length = 1.06 μm, frequency = 1.78×10^{15} c/s, Power = 10^{16} watts/cm^2 (which is less than the threshold power to generate self-action effects) passing through a dense plasma ($N_0 = 10^{20}$/c.c.), the frequency shift of LCP components = 1.01×10^{15} c/s and that of RCP components = -8.6×10^{14}c/s when $\omega \simeq \Omega_0$, i.e., $Ho \simeq 10^8$ gauss.

Under the situation $\omega \simeq \omega_p$, the frequency shift of the LCP and RCP components become 4.42×10^{12} c/s and 4.31×10^{12} c/s respectively when $Ho \simeq 10^5$ gauss. For magnetic field $\simeq 10^6$ gauss, the frequency shift of the LCP and RCP components become 3.48×10^{12} c/s and 5.22×10^{12} c/s respectively.

Table 1: Frequency shifts of electromagnetic waves having different amplitudes

Physical Conditions	The dimensional amplitude of the electromagnetic wave	Frequency shifts	Conclusion
(1) $\omega >> kc$ and $\omega \simeq \Omega_0$	$\alpha = \sqrt{2} \cdot \omega_p^2/\omega^2$	$\delta\omega_+ = 0$	Frequency shift of LCP component remains unchanged
	$\alpha > \sqrt{2} \cdot \omega_p^2/\omega^2$	$\delta\omega_+ = -ve$	Frequency shift of LCP component occurs towards red
	$\alpha < \sqrt{2} \cdot \omega_p^2/\omega^2$	$\delta\omega_+ = +ve$	Frequency shift of LCP component occurs towards blue
	$\alpha = \sqrt{2}(\omega_p/\omega)(\frac{\omega_p^2}{\omega^2} - 2)^{\frac{1}{2}}$	$\delta\omega_- = 0$	Frequency shift of RCP component remains unchanged
	$\alpha > \sqrt{2}(\omega_p/\omega)(\frac{\omega_p^2}{\omega^2} - 2)^{\frac{1}{2}}$	$\delta\omega_- = -ve$	Frequency shift of RCP component occurs towards red
	$\alpha < \sqrt{2}(\omega_p/\omega)(\frac{\omega_p^2}{\omega^2} - 2)^{\frac{1}{2}}$	$\delta\omega_- = +ve$	Frequency shift of RCP component occurs towards blue
(2) $\omega >> kc$ and $\omega \simeq \omega_0$	$\alpha = (\Omega_0/\omega)^{\frac{1}{2}}$	$\delta\omega_+ = 0$	Frequency shift of LCP component remains unchanged
	$\alpha > (\Omega_0/\omega)^{\frac{1}{2}}$	$\delta\omega_+ = -ve$	Frequency shift of LCP component is towards red
	$\alpha < (\Omega_0/\omega)^{\frac{1}{2}}$	$\delta\omega_+ = +ve$	Frequency shift of LCP component occurs towards blue
	α = any possible value	$\delta\omega_- = -ve$	Frequency shift of RCP component occurs always towards red

10. CLOSING REMARKS

Transformation of field variables from the laboratory frame to the space-independent frame is applicable only for a single electromagnetic wave because the condition required for this transformation, i.e., $V_0 = kc^2/\omega = c^2/V$ is not satisfied for two or more waves interacting in plasma.

To derive the results for the shifts of wave parameters with the help of special Lorentz transformation, field variables should be transformed to the space-independent frame at the beginning, and then one should proceed with the calculation. If equations are solved in the laboratory frame before transforming them to the space independent frame and transformations are used only in the subsequent steps, correct results would not appear in the space-independent frame.

11. ACKNOWLEDGEMENT

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