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A VARIANT OF A FIXED POINT THEOREM OF BROWDER-FAN AND REICH

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ABSTRACT. Let S be a convex, weakly compact subset of a locally convex Hausdorff space (E, τ) and $f: S \to E$ be a continuous multifunction from its weak topology ω to τ. Let p be a continuous seminorm on (E, τ) and for subsets A, B, of E, let $p(A, B) = \inf\{p(x - y): x \in A, y \in B\}$. In this paper, sufficient conditions are developed for the existence of an $x \in S$ satisfying $p(x, fx) = p(fx, S)$. The result is then used to prove several fixed point theorems.

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1. INTRODUCTION

Let (E, τ) be a locally convex Hausdorff topological vector space with topology τ and $E^* = (E, \tau)^*$ be its topological dual. Let $\omega = \omega(E, E^*)$ be the weak topology of E. Let P and Q denote the family of continuous semi-norms generating the topologies τ and ω respectively. For sets A and B of E and a $p \in P$, let $p(A, B) = \inf\{p(x - y): x \in A, y \in B\}$. In this paper, we prove the following result.

THEOREM 1. Let S be a nonempty convex, ω-compact subset of E and $f: (S, \omega) \to (E, \tau)$ be a continuous multifunction such that $f(x)$ is convex and ω-compact for each $x \in S$. Then for each $p \in P$ there exists a $x \in S$ satisfying

$$p(x, fx) = p(fx, S). \tag{1.1}$$

Futher if $p(x, fx) > 0$ then $x \in \partial(S, \omega) \cap \partial(S, \tau)$ where ∂ denotes the boundary.

It may be remarked the since $\omega \subseteq \tau$, f in Theorem 1 is also a continuous multifunction from $(S, \omega) \to (E, \omega)$. Consequently it follows by Reich (Lemma 1.6 [1]) that each $q \in Q$ satisfies (1.1) for some $x \in S$. However, since $Q \subseteq P$, the lemma in [1] is not applicable for arbitrary $p \in P$. In fact, Theorem 1 contains the above lemma [1] (see Corollary 2) and it provides a generalization of a well-known result of Ky Fan [2] for single valued mappings.

2. PRELIMINARY RESULTS.

Recall that if X, Y are topological spaces then a multifunction $f: X \to Y$ ($fx \neq \emptyset$ for each x) is upper (lower) semicontinuous iff for each closed (open) subset A of Y, $f^{-1}(A) = \{x \in X: f(x) \cap A \neq \emptyset\}$ is a closed (open) subset of X. It follows by definition that f is l.s.c. iff $fx \cap U \neq \emptyset$ for some open set U of Y and x in X then $fz \cap U \neq \emptyset$ for each z in some neighborhood V of X. Further, it is well-known (i) that if f is u.s.c. and a net $x_\alpha \to x$ in X and $y_\alpha \to y$ in Y with $y_\alpha \in fx_\alpha$ then $y \in fx$; (ii) if X is compact and f is u.s.c. with compact values then fX is compact. A multifunction which is both u.s.c. and l.s.c. is called continuous.

We prove two lemmas that simplify the proof of Theorem 1. Throughout, let E be as stated in the beginning and S a nonempty subset of E.

LEMMA 1. Let A, B be ω-compact sets of E and $p \in P$. Then $p(A, B) = p(x, B) = p(x - y)$ for some $x \in A$, $y \in B$.

PROOF. Choose sequences $\{x_n\} \subseteq A$, $\{y_n\} \subseteq B$ such that $p(x_n - y_n) \downarrow p(A, B)$. We may assume that $x_n \to x$ weakly for some $x \in A$ and $y_n \to y$ weakly for some $y \in B$. By Hahn Banach Theorem (see [3], Cor. 2, p. 29) there exists a $x^* \in E^*$ with $x^*(x - y) = p(x - y)$ and $|x^*(u)| \leq p(u)$ for each $u \in E$. Consequently, since $x_n - y_n \to x - y$ weakly,

$$p(x, B) \leq p(x - y) = x^*(x - y) = \lim|x^*(x_n - y_n)| \leq \underline{\lim}\, p(x_n - y_n) = p(A, B) \leq p(x, B)$$

LEMMA 2. Let S be ω-compact subset of E and $f: (S, \omega) \to (E, \tau)$ be a l.s.c. multifunction with weakly compact values. If a net $x_\alpha \to x$ weakly in S, then for each $p \in P$ and $\varepsilon > 0$, $p(fx_\alpha, S) \leq p(fx, S) + \varepsilon$ eventually.

PROOF. It follows by Lemma 1 that there is a $y \in fx$ with $p(fx, S) = p(y, S)$. Let $U = \{x \in E: p(x - y) < \varepsilon\}$. Then U is τ-open and $y \in fx \cap U$. Hence by l.s.c., $fx_\alpha \cap U \neq \emptyset$ eventually. For such α, let $y_\alpha \in fx_\alpha \cap U$. Then eventually,

$$p(fx_\alpha, S) \leq p(y_\alpha, S) \leq p(y_\alpha - y) + p(y, S) \leq p(fx, S) + \varepsilon.$$

3. MAIN RESULTS.

PROOF OF THEOREM 1. Let $p \in P$. Define a multifunction $g: (S, \omega) \to (S, \omega)$ by

$$g(x) = \{y \in S: p(y, fx) = p(fx, S)\}.$$

Then by Lemma 1, $g(x) \neq \emptyset$ and is clearly convex. Further, since S is τ-closed and for any $y, z \in g(x)$, the triangular inequality implies

$$|p(y, fx) - p(z, fx)| \leq p(y - z).$$

It follows $g(x)$ is τ-closed convex and hence a ω-compact subset of S. We show that g is u.s.c. Let C be a weakly closed (hence weakly compact) aubset of S. We show that $x \in g^{-1}(C)$, that is $g(x) \cap C \neq \emptyset$. Choose for each α, $y_\alpha \in gx_\alpha \cap C$. We may assume that $y_\alpha \to y$ weakly for some $y \in C$. Also since $p(y_\alpha, fx_\alpha) = p(fx_\alpha, S)$, there exists $z_\alpha \in fx_\alpha$ with $p(y_\alpha - z_\alpha) = p(fx_\alpha, S)$. Further $f: (S, \omega) \to (E, \omega)$ being u.s.c., it follows that fS is weakly compact and hence we may assume that $z_\alpha \to z$ weakly for some $z \in fx$. Thus $y_\alpha - z_\alpha \to y - z$ weakly. Choose as before a $x^* \in E^*$ such that $x^*(y - z) = p(y - z)$ and $|x^*(u)| \leq p(u)$ for each $u \in E$. Let $\varepsilon > 0$. Choose $\alpha_0 \in \Delta$ such that $p(fx_\alpha, S) \leq p(fx, S) + \varepsilon$ for $\alpha \geq \alpha_0$. Consequently, for $\alpha \geq \alpha_0$ $|x^*(y_\alpha - z_\alpha)| \leq p(fx_\alpha, S) \leq p(fx, S) + \varepsilon$ and hence

$$p(y, fx) \leq p(y - z) = \lim|x^*(y_\alpha - z_\alpha)| \leq p(fx, S) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary and $p(fx, S) \leq p(y, fx)$, we have $p(y, fx) = p(fx, S)$ that is $y \in g(x) \cap C$. Thus g is u.s.c. Hence by Glicksberg [4] there exists a $x \in S$ with $x \in g(x)$. This implies $p(x, fx) = p(fx, S)$.

Now, suppose $p(x, fx) > 0$. Then $fx \cap S = \emptyset$. Choose by Lemma 1, a $y \in fx$ satisfying $p(x - y) = p(x, fx)$. Now, if $x \in \text{int}(S, \omega)$ or $\text{int}(S, \tau)$, then since S being weakly closed and convex, there is a $z \in (x, y) \cap S$ with $0 < p(fx, S) \leq p(y - z) < p(x - y) = p(fx, S)$, a contradiction. This proves the result.

As a consequence of Theorem 1, we have

COROLLARY 1. Let S be a convex and weakly compact set in E and $f: (S, \omega) \to (E, \tau)$ be a continuous multifunction with convex and ω-compact values. Then either f has a fixed point or there exists a $p \in P$ and $x \in S$ satisfying $0 < p(x, fx) = p(fx, S)$.

PROOF. For each $p \in P$, let $x_p \in S$ satisfying (1). If $p(x_p, fx_p) = 0$ for each $p \in P$, then using the implication that $f: (S,\omega) \to (S,\omega)$ is continuous, it follows that $A_p = \{x \in S: p(x, fx) = 0\}$ is nonempty, weakly compact and the family $\{A_p: p \in P\}$ has finite intersection property. Consequently, there exists $x \in S$ with $p(x, fx) = 0$ for each $p \in P$. Now, if $x \notin fx$, then since $x - fx$ is τ-closed and convex and $0 \notin x - fx$, there exists (see [3], Cor. 1, p. 30) a $x^* \in E^*$ such that $0 \notin \{x^*(x - y): y \in fx\}$. Let $p = |x^*|$. Then $p \in P$ and $p(x, fx) \neq 0$, a contradiction.

The following corollaries result from Theorem 1.

COROLLARY 2. (Reich [1]). Let S be a compact and convex in (E, τ) and $f: (S, \tau) \to (E, \tau)$ be a continuous mutifunction with convex and compact values. Then either f has a fixed point or there exists a $p \in P$ and $x \in S$ satisfying $0 < p(x, fx) = p(fx, S)$.

COROLLARY 3. (Waters [5]). Let S be a compact and convex subset of (E, τ) and $f: (S, \tau) \to (E, \tau)$ be a continuous multifunction with convex and weakly compact values. Then for each $p \in P$, there exists a $x \in S$ satisfying (1.1).

PROOF. It suffices to show that the hypotheses in Corollary 2 and Corollary 3 imply that $f: (S, \omega) \to (E, \tau)$ is a continuous multifunction. Let A be τ-closed in E. Then $f^{-1}(A)$ is τ-compact subset of S. Since S is weakly closed, it follows that $f^{-1}(A)$ is weakly closed. Thus f is u.s.c. Similarly if A is τ-open set in E then $S \setminus f^{-1}(A) = f^{-1}(E \setminus A)$ is ω-closed and hence $f^{-1}(A)$ is ω-open. Thus f is l.s.c.

In the setting of semi-reflexive locally convex spaces, we have

COROLLARY 4. Let S be a closed, bounded and convex subset of a semi-reflexive locally convex space E. If $f: (S, \omega) \to (E, \tau)$ is continuous multifunction with closed, bounded and convex values then for each $p \in P$, there exists $x \in S$ satisfying (1.1).

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