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ON SOME RESULTS FOR λ-SPIRAL FUNCTIONS OF ORDER α

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ABSTRACT. The results of various kinds concerning λ-spiral functions of order α in the unit disk U are given in this paper. They represent mainly the generalizations of the previous results of the authors.

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1. INTRODUCTION.

Let A_n denote the class of functions of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \qquad (n \in N = \{1,2,3,\cdots\}) \tag{1.1}$$

which are regular in the unit disk $U = \{z: |z| < 1\}$.

For a function f(z) belonging to the class A_n we say that f(z) is λ-spiral of order α if and only if

$$\text{Re}\left\{e^{i\lambda}\frac{zf'(z)}{f(z)}\right\} > \alpha\cos\lambda \tag{1.2}$$

for some real λ ($|\lambda| < \pi/2$), for some α ($0 \leqq \alpha < 1$), and for all $z \in U$.

We denote by $S_n^\lambda(\alpha)$ the class of all such functions. In the case $n = 1$ we write A and $S^\lambda(\alpha)$ instead of A_1 and $S_1^\lambda(\alpha)$, respectively. The class $S^\lambda(\alpha)$ has been considered by Libera [1].

We note that $S^\lambda(0) = S^\lambda$, $S_n^0(\alpha) = S_n^*(\alpha)$, and $S^0(0) = S^*$, where S^λ, $S_n^*(\alpha)$, and S^* denote the classes of functions which are λ-spiral, starlike of order α and type (1.1), and starlike, respectively.

Let $f(z)$ and $g(z)$ be regular in the unit disk U. Then a function $f(z)$ is said to be subordinate to $g(z)$ if there exists a regular function $w(z)$ in the unit disk U satisfying $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $f(z) = g(w(z))$. For this relation the following symbol $f(z) \prec g(z)$ is used. In particular, if $g(z)$ is univalent in the unit disk U the subordination $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

2. RESULTS AND CONSEQUENCES.

First we give the following result due to Miller and Mocanu [2].

LEMMA 1. Let $\Phi(u,v)$ be a complex valued function,

$$\Phi: D \longrightarrow \mathbb{C}, \quad D \subset \mathbb{C} \times \mathbb{C} \quad (\mathbb{C} \text{ is the complex plane}),$$

and let $u = u_1 + iu_2$, $v = v_1 + iv_2$. Suppose that the function $\Phi(u,v)$ satisfies the following conditions:

(i) $\Phi(u,v)$ is continuous in D;

(ii) $(1,0) \in D$ and $\mathrm{Re}\{\Phi(1,0)\} > 0$;

(iii) $\mathrm{Re}\{\Phi(iu_2,v_1)\} \leqq 0$ for all $(iu_2,v_1) \in D$ and such that $v_1 \leqq -n(1 + u_2^2)/2$.

Let $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \cdots$ be regular in the unit disk U such that $(p(z), zp'(z)) \in D$ for all $z \in U$. If

$$\mathrm{Re}\{\Phi(p(z), zp'(z))\} > 0 \qquad (z \in U),$$

then

$$\mathrm{Re}\{p(z)\} > 0 \qquad (z \in U).$$

By using the above lemma, we prove

THEOREM 1. Let the function $f(z)$ defined by (1.1) be in the class $S_n^\lambda(\alpha)$ and let

$$0 < \beta \leqq \frac{n}{2(1-\alpha)\cos\lambda}. \tag{2.1}$$

Then we have

$$\mathrm{Re}\left\{\left(\frac{f(z)}{z}\right)^{\beta e^{i\lambda}}\right\} > \frac{n}{2\beta(1-\alpha)\cos\lambda + n} \qquad (z \in U).$$

PROOF. If we put

$$B = \frac{n}{2\beta(1-\alpha)\cos\lambda + n} \tag{2.2}$$

and

$$\left(\frac{f(z)}{z}\right)^{\beta e^{i\lambda}} = (1-B)p(z) + B, \tag{2.3}$$

where β satisfies (2.1) then $p(z)$ is regular in the unit disk U and $p(z) = 1 + p_n z^n + p_{n+1}z^{n+1} + \cdots$. From (2.3) after taking the logarithmical differentiation we have that

$$\beta e^{i\lambda}\frac{zf'(z)}{f(z)} - \beta e^{i\lambda} = (1-B)\cdot\frac{zp'(z)}{(1-B)p(z)+B}, \tag{2.4}$$

and from there

$$e^{i\lambda}\frac{zf'(z)}{f(z)} - \alpha\cos\lambda = e^{i\lambda} - \alpha\cos\lambda + \frac{(1-B)zp'(z)}{\beta\{(1-B)p(z)+B\}} \tag{2.5}$$

Since $f(z) \in S_n^\lambda(\alpha)$ then from (2.5) we get

$$\operatorname{Re}\left\{e^{i\lambda} - \alpha\cos\lambda + \frac{(1-B)zp'(z)}{\beta\{(1-B)p(z)+B\}}\right\} > 0 \qquad (z \in U). \tag{2.6}$$

Let consider the function $\Phi(u,v)$ defined by

$$\Phi(u,v) = e^{i\lambda} - \alpha\cos\lambda + \frac{(1-B)v}{\beta\{(1-B)u+B\}}$$

(it is noted $u = p(z)$ and $v = zp'(z)$). Then $\Phi(u,v)$ is continuous in $D = (\mathbb{C} - \{-B/(1-B)\}) \times \mathbb{C}$. Also, $(1,0) \in D$ and $\operatorname{Re}\{\Phi(1,0)\} = (1-\alpha)\cos\lambda > 0$. Furthermore, for all $(iu_2, v_1) \in D$ such that $v_1 \leqq -n(1+u_2^2)/2$ we have

$$\begin{aligned}
\operatorname{Re}\{\Phi(iu_2,v_1)\} &= (1-\alpha)\cos\lambda + \operatorname{Re}\left\{\frac{(1-B)v_1}{\beta\{(1-B)iu_2+B\}}\right\} \\
&= (1-\alpha)\cos\lambda + \frac{B(1-B)v_1}{\beta\{(1-B)^2u_2^2+B^2\}} \\
&\leqq (1-\alpha)\cos\lambda - \frac{nB(1-B)(1+u_2^2)}{2\beta\{(1-B)^2u_2^2+B^2\}} \\
&= \frac{(1-B)\{4\beta^2(1-\alpha)^2\cos^2\lambda - n^2\}u_2^2}{2\beta\{(1-B)^2u_2^2+B^2\}} \\
&\leqq 0
\end{aligned}$$

because $0 < B < 1$ and $2\beta(1-\alpha)\cos\lambda - n \leqq 0$. Therefore, the function $\Phi(u,v)$ satisfies the conditions in Lemma 1. This proves that $\mathrm{Re}\{p(z)\} > 0$ for $z \in U$, that is, that from (2.3),

$$\mathrm{Re}\left\{\left(\frac{f(z)}{z}\right)^{\beta e^{i\lambda}}\right\} > B \qquad (z \in U),$$

which is equivalent to the statement of Theorem 1.

Taking $\alpha = 0$ and $\beta = n/2\cos\lambda$ in Theorem 1, we have

COROLLARY 1. Let the function $f(z)$ defined by (1.1) be in the class $S_n^\lambda(0)$. Then

$$\mathrm{Re}\left\{\left(\frac{f(z)}{z}\right)^{(n/2\cos\lambda)e^{i\lambda}}\right\} > \frac{1}{2} \qquad (z \in U).$$

REMARK 1. If $\lambda = 0$ in Theorem 1, then we have the former result given by the authors in [3]. If $\lambda = 0$ in Corollary 1, then we have the earlier result given by Golusin [4].

THEOREM 2. Let β be a fixed real number, $0 \leqq \beta < 1$, and let the function $f(z)$ be in the class $S^\lambda(\alpha)$. Let

$$g(z) = z\left(\frac{f(z)}{z}\right)^{\gamma} \frac{1}{(1-z)^{2\mu e^{-i\lambda}\cos\lambda}} \qquad (z \in U), \tag{2.7}$$

where $0 < \gamma \leqq (1-\beta)/(1-\alpha)$ and $\mu = 1 - \beta - \gamma(1-\alpha)$. Then the function $g(z)$ is in the class $S^\lambda(\beta)$.

PROOF. Let β $(0 \leqq \beta < 1)$ be a given real number. Then from (2.7) by using the logarithmic differentiation and some simple transformations, we have that

$$e^{i\lambda}\frac{zg'(z)}{g(z)} - \beta\cos\lambda$$

$$= \gamma\left\{e^{i\lambda}\frac{zf'(z)}{f(z)} + \alpha\cos\lambda\right\} + (1-\gamma)e^{i\lambda} - \left\{\beta - \alpha\gamma - \frac{2\mu z}{1-z}\right\}\cos\lambda$$

$$= \gamma\left\{e^{i\lambda}\frac{zf'(z)}{f(z)} - \alpha\cos\lambda\right\} + \mu\cos\lambda\frac{1+z}{1-z} - i(1-\gamma)\sin\lambda. \tag{2.8}$$

From (10) we have

$$\begin{aligned}&\mathrm{Re}\left\{e^{i\lambda}\frac{zg'(z)}{g(z)} - \beta\cos\lambda\right\}\\ &= \mathrm{Re}\left\{e^{i\lambda}\frac{zf'(z)}{f(z)} - \alpha\cos\lambda\right\} + \mu\cos\lambda\left\{\frac{1+z}{1-z}\right\}\\ &> 0 \qquad (z \in U), \qquad (2.9)\end{aligned}$$

because of the suppositions for γ, $f(z)$ and μ given in the statement of Theorem 2. Thus we complete the proof of Theorem 2.

COROLLARY 2. Let $0 \leqq \beta \leqq \alpha$, and let $f(z) \in S^{\lambda}(\alpha)$. Then the function $g(z)$ defined by

$$g(z) = \frac{f(z)}{(1-z)^{2(\alpha-\beta)e^{-i\lambda}\cos\lambda}} \qquad (2.10)$$

belongs to the class $S^{\lambda}(\beta)$.

PROOF. Since $0 \leqq \beta \leqq \alpha$, it follows that $1 \leqq (1-\beta)/(1-\alpha)$ and we may choose $\gamma = 1$ in Theorem 2. Also, we have $\mu = \alpha - \beta$.

REMARK 2. In particular, for $\lambda = 0$ in Corollary 2 we have that if $f(z) \in S^{*}(\alpha)$ and $0 \leqq \beta \leqq \alpha$, then the function

$$g(z) = \frac{f(z)}{(1-z)^{2(\alpha-\beta)}}$$

belongs to the class $S^{*}(\beta)$.

This is the earlier result given by the authors [5].

Letting $\lambda = 0$ in Theorem 2, we have

COROLLARY 3. Let $f(z) \in S^{*}(\alpha)$ and let $0 \leqq \beta < 1$. Then the function $g(z)$ defined by

$$g(z) = z\left(\frac{f(z)}{z}\right)^{\gamma}\frac{1}{(1-z)^{2\mu}}$$

belongs to the class $S^*(\beta)$, where $0 < \gamma \leqq (1 - \beta)/(1 - \alpha)$ and $\mu = 1 - \beta - \gamma(1 - \alpha)$.

In order to derive the following theorem, we shall apply the next result given by Robertson [6].

LEMMA 2. Let the function $f(z)$ belonging to A be univalent in the unit disk U. For $0 \leqq t \leqq 1$, let $F(z,t)$ be regular in the unit disk U with $F(z,0) \equiv f(z)$ and $F(0,t) \equiv 0$. Let p be a positive real number for which

$$F(z) = \lim_{t \to +0} \frac{F(z,t) - F(z,0)}{zt^p}$$

exists. Further, let $F(z,t)$ be subordinate to $f(z)$ in the unit disk U for $0 \leqq t \leqq 1$. Then

$$\operatorname{Re}\left\{\frac{F(z)}{f'(z)}\right\} \leqq 0 \qquad (z \in U).$$

If, in addition, $F(z)$ is also regular in the unit disk U and $\operatorname{Re}\{F(0)\} \neq 0$, then

$$\operatorname{Re}\left\{\frac{f'(z)}{F(z)}\right\} < 0 \qquad (z \in U). \tag{2.11}$$

Applying the above lemma, we prove

THEOREM 3. Let the function $f(z)$ be in the class A, and let the function $g(z)$ defined by

$$g(z) = \frac{1}{1 - \alpha\cos\lambda}\left\{f(z) - \alpha\cos\lambda\int_0^z \frac{f(s)}{s}\,ds\right\}$$

$$= z + \cdots \tag{2.12}$$

be univalent in the unit disk U, where λ is a real number with $|\lambda| < \pi/2$ and $0 \leqq \alpha < 1$. If the function $G(z,t)$ defined by

$$G(z,t) = \frac{1}{1 - \alpha\cos\lambda}\left\{(1 - te^{-i\lambda})f(z) - \alpha\cos\lambda(1 - t^2)\int_0^z \frac{f(s)}{s}\,ds\right\} \tag{2.13}$$

is subordinate to $g(z)$, that is, $G(z,t) \prec g(z)$, in the unit disk U for fixed α and λ, and for each t $(0 \leqq t \leqq 1)$, then $f(z)$ is in the class $S^{\lambda}(\alpha)$.

PROOF. It is easy to show that $G(z,0) \equiv g(z)$ and $G(0,t) \equiv 0$. We choose $p = 1$ and $F(z,t)$ to be the function $G(z,t)$ defined by (2.13) in Lemma 2. Then we have

$$G(z) = \lim_{t \to +0} \frac{G(z,t) - G(z,0)}{zt} = \lim_{t \to +0} \frac{\partial G(z,t)/\partial t}{} = \frac{-e^{-i\lambda} f(z)}{(1 - \alpha \cos \lambda) z}. \tag{2.14}$$

From (2.14) we know that $G(z)$ is regular in the unit disk U and

$$\mathrm{Re}\{G(0)\} = \frac{-\cos\lambda}{1 - \alpha\cos\lambda} \neq 0.$$

Since

$$g'(z) = \frac{1}{1 - \alpha\cos\lambda}\left\{f'(z) - \alpha\cos\lambda \frac{f(z)}{z}\right\},$$

it follows from (2.11) in Lemma 2 that

$$\mathrm{Re}\left\{\frac{g'(z)}{G(z)}\right\} < 0 \qquad (z \in U),$$

which is equivalent to

$$\mathrm{Re}\left\{e^{i\lambda}\frac{zf'(z)}{f(z)} - \alpha\cos\lambda\right\} > 0 \qquad (z \in U).$$

Consequently, we prove that $f(z)$ is in the class $S^{\lambda}(\alpha)$.

If we put $\alpha = 0$ in Theorem 3, then we have

COROLLARY 4. Let the function $f(z)$ belonging to the class A be univalent in the unit disk U such that

$$(1 - te^{-i\lambda})f(z) \prec f(z) \qquad (z \in U),$$

where λ is real such that $|\lambda| < \pi/2$ and $0 \leqq t \leqq 1$. Then $f(z)$ is in the class $S^{\lambda}(0)$. This is the former result due to Robertson [6].

For $\lambda = 0$ in Theorem 3, we have the following result for starlike functions of order α.

COROLLARY 5. Let the function f(z) be in the class A and let the function g(z) defined by

$$g(z) = \frac{1}{1-\alpha}\left\{ f(z) - \alpha\int_0^z \frac{f(s)}{s}\,ds \right\} \qquad (0 \leqq \alpha < 1)$$

be univalent in the unit disk U. If

$$G(z,t) = \frac{1}{1-\alpha}\left\{(1-t)f(z) - \alpha(1-t^2)\int_0^z \frac{f(s)}{s}\,ds \right\} \prec g(z)$$

in the unit disk U, then f(z) belongs to the class $S^*(\alpha)$. This is the previous result given by Obradović [7].

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