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ON ALPHA-CLOSE-TO-CONVEX FUNCTIONS OF ORDER BETA

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ABSTRACT Let $M_\beta(\alpha)$ $[\alpha \geq 0$ and $\beta \geq 0]$ denote the class of all functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

analytic in the unit disc U with $f'(z)f(z)/z \neq 0$ and which satisfy for $z=re^{i\theta}\in U$ the condition

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} d\theta > -\beta\pi$$

for all $\theta_2 > \theta_1$. In this note we show that each $f\in M_\beta(\alpha)$ is close-to-star of order β when $0 < \beta \leq \alpha$.

KEY WORDS AND PHRASES. Close-to-star functions, close-to-convex functions, α-convex functions.

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I. INTRODUCTION.

A shall denote the class of all functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

analytic in the unit disc $U = \{z:|z|<1\}$ and S shall denote the subclass of functions in A which are univalent in U.

Let $\alpha\geq 0$, $\beta\geq 0$, and let $f \in A$ with $f'(z)f(z)/z \neq 0$ in U, and let

$$J(\alpha, f) = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1+\frac{zf''(z)}{f''(z)}\right). \tag{1.1}$$

If for $z=re^{i\theta}\in U$

$$\int_{\theta_1}^{\theta_2} \left\{\mathrm{Re}\, J(\alpha, f)\right\} d\theta > -\beta\pi \tag{1.2}$$

whenever $0 \leq \theta_1 < \theta_2 < 2\pi$, then f is said to be an α-close-to-convex function of order β or $f \in M_\beta(\alpha)$. The class $M_\beta(\alpha)$ was introduced by Nasr[1].

It was shown [1] that $M_\beta(\alpha) \subset S$ if and only if $0 \le \beta \le \alpha$.

Note that $M_\beta(1) = K_\beta$, is the class of close-to-convex functions of order β introduced by Reade [2] and studied by Goodman [3] for $\beta \ge 1$ and by Pommerenke [4] for $0 \le \beta \le 1$, and $M_\beta(0) = R_\beta$ is the class of close-to-star functions of order β introduced by Goodman [3]. Moreover $M_0(\alpha) = M(\alpha)$ is the class of α-convex functions introduced by Mocanu [5], and $M_{\gamma/\alpha}(\frac{1}{\alpha}) = \beta_\gamma(\alpha)$, $\alpha > 0$, is the class of Bazilevič functions of order γ introduced by Nasr [6].

In this note we continue the investigation of α-close-to-convex functions of order β studied in [1].

2. RESULTS

In this section we show that each $f \in M_\beta(\alpha)$ is close-to-star of order β when $0 < \beta \le \alpha$. For $\alpha \ge 1$ we show each $f \in M_\beta(\alpha)$ is close-to-convex of order β when $0 < \beta \le \alpha$ and if $f \in M_\beta(\alpha)$, then $f \in M_\beta(\gamma)$ when $0 < \beta \le \gamma \le \alpha$.

We assume, unless otherwise stated, that θ is a real number, that $0 < r < 1$ and that $z=re^{i\theta}$. Also that $0 < \beta \le \alpha$.

We shall need the following result.

LEMMA 1: If $f \in M_\beta(\gamma)$, then the function h given by

$$h(z) = f(z) \cdot (zf'(z)/f(z))^\alpha \tag{2.1}$$

belongs to R_β. (The powers taken are the principal values).

PROOF: Let $f \in M_\beta(\alpha)$. If we choose the branch of $(zf'(z)/f(z))^\alpha$ which is equal to 1 when $z = 0$, a simple calculation shows that the function h defined by (2.1) belongs to R_β.

THEOREM 1: If $f \in M_\beta(\alpha)$ then $f \in R_\beta$.

PROOF: Since $f \in M_\beta(\alpha)$, it follows from Lemma 1 that

$$\int_{\theta_1}^{\theta_2} \text{Re}\left\{ \frac{\frac{d}{dx}\, h(z)^{1/\alpha}}{\frac{d}{dz}\, f(z)^{1/\alpha}} \right\} d\theta = \int_{\theta_1}^{\theta_2} \text{Re}\left\{ \frac{zh'(z)}{h(z)} \right\} d\theta > -\pi\beta. \tag{2.2}$$

In (2.2) we choose the branches for $f(z)^{1/\alpha}$ and $h(z)^{1/\alpha}$ for which

$$h(z)^{1/\alpha}/f(z)^{1/\alpha} = (h(z)/f(z))^{1/\alpha} \tag{2.3}$$

with value 1 for $z=0$. If we use (2.1), (2.2) and (2.3) it is easy to prove that

$$\int_{\theta_1}^{\theta_2} \text{Re}\left\{h(z)/f(z)^{\ 1/\alpha}\right\} d\theta > -\pi\beta. \tag{2.4}$$

In fact, since f is univalent, we can let $w = f(z)$, $z = z(w) = f^{-1}(w)$ and $w = \rho e^{i\phi}$ to obtain

$$\frac{h(z)^{1/\alpha}}{f(z)^{1/\alpha}} = \frac{1}{w^{1/\alpha}} \int_0^w \frac{d}{dw}\left[h(z(w))\right]^{1/\alpha} dw$$

$$= \frac{1}{\rho^{1/\alpha}} \int_0^\rho \frac{d\,[\,h(z(\rho e^{i\phi}))]^{1/\alpha}}{d\,(\rho e^{i\phi})^{1/\alpha}}\, 1/\alpha\ \rho^{1/\alpha - 1}\, d\rho$$

$$= \frac{1}{\rho^{1/\alpha}} \int_0^\rho \frac{d\,[\,h(z)]^{1/\alpha}}{d\,[f(z)]^{1/\alpha}}\, 1/\alpha\ \rho^{1/\alpha - 1}\, d\rho .$$

Hence

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left\{\frac{h(z)^{1/\alpha}}{f(z)^{1/\alpha}}\right\} d\theta =$$

$$\frac{1}{\rho^{1/\alpha}} \int_0^\rho \int_{\theta_1}^{\theta_2} \left[\mathrm{Re}\left\{\frac{d[h(z)]^{1/\alpha}}{d[f(z)]^{1/\alpha}}\right\} d\theta\right] 1/\alpha\ \rho^{1/\alpha\ -1}\, d\rho .$$

The result now follows from (2.1) and (2.4).

COROLLARY 1: If $f \in M_\beta(\alpha)$, $\alpha \geq 1$, then $f \in K_\beta$.

PROOF: Let $f \in M_\beta(\alpha)$, $\alpha \geq 1$, then

$$\alpha \int_{\theta_1}^{\theta_2} \mathrm{Re}\left\{(1+ \frac{zf''(z)}{f'(z)})\right\} d\theta > (\alpha-1) \int_{\theta_1}^{\theta_2} \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} d\theta - \pi\beta .$$

Now from THEOREM 1, we have $\int_{\theta_1}^{\theta_2} \mathrm{Re}\left\{zf'(z)/f(z)\right\} d\theta > -\pi\beta$,
and therefore

$$\alpha \int_{\theta_1}^{\theta_2} \mathrm{Re}\left\{(1+ \frac{zf''(z)}{f'(z)})\right\} d\theta > -(\alpha-1)\pi\beta - \pi\beta = -\alpha\pi\beta$$

and the proof of Corollary 1 is complete.

COROLLARY 2: If $f \in M_\beta(\alpha)$, $0 < \beta \leq \gamma \leq \alpha$, then $f \in M_\beta(\gamma)$.

PROOF:

By THEOREM 1, $f \in R_\beta$. Suppose there exists a γ, $0 < \beta \leq \gamma \leq \alpha$, such that $f \in M_\beta(\gamma)$. Then there is $\tau \in U$ for which

$$\int_{\theta_1}^{\theta_2} \left\{\mathrm{Re}\left[\frac{\tau f''(\tau)}{f'(\tau)} + 1 - \frac{\tau f'(\tau)}{f(\tau)}\right]\right\} d\theta$$

$$< -\frac{\pi\beta}{\gamma} - \frac{1}{\gamma} \int_{\theta_1}^{\theta_2} \left\{\mathrm{Re}\left[\frac{\tau f'(\tau)}{f(\tau)}\right]\right\} d\theta. \tag{2.5}$$

However, for $f \in M_\beta(\alpha)$,

$$0 < \pi\beta + \int_{\theta_1}^{\theta_2} \left\{\mathrm{Re}\left[\frac{\tau f'(\tau)}{f(\tau)}\right]\right\} d\theta + \alpha \int_{\theta_1}^{\theta_2} \left\{\mathrm{Re}\left[\frac{\tau f''(\tau)}{f'(\tau)} + 1 - \frac{\tau f'(\tau)}{f(\tau)}\right]\right\} d\theta. \tag{2.6}$$

Substituting (2.5) into (2.6), we obtain

$$0 < (1 - \frac{\alpha}{\gamma}) [\beta\pi + \int_{\theta_1}^{\theta_2} \left\{ \text{Re} \left[\frac{\tau f'(\tau)}{f(\tau)} \right] \right\} d\theta].$$

But $(1 - \frac{\alpha}{\gamma})<0$ implies $\int_{\theta_1}^{\theta_2} \left\{ \text{Re} \left[\frac{\tau f'(\tau)}{f(\tau)} \right] \right\} d\theta < -\pi\beta$, which contradicts the assumption that $f \varepsilon R_\beta$. Thus $f \varepsilon M_\beta(\gamma)$.

REMARK:

For $\beta = 0$ we obtain results due to Miller, Mocanu and Reade [7].

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