

GENERALIZED RESOLVENTS AND SPECTRUM FOR A CERTAIN CLASS OF PERTURBED SYMMETRIC OPERATORS

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Received 3 February 2004 and in revised form 20 May 2004

The generalized resolvents for a certain class of perturbed symmetric operators with equal and finite deficiency indices are investigated. Using the Weinstein-Aronszajn formula, we give a classification of the spectrum.

1. Introduction

The present paper is concerned with the study of spectral properties for a certain class of linear symmetric operator T , defined in the Hilbert space H of the form $T = A + B$, where A is a closed linear symmetric operator, with nondensely defined domain in general, $D(A) \subset H$, and B is a finite-rank operator of the form

$$Bf = \sum_{k=1}^n a_k(f, y_k) y_k, \quad (1.1)$$

where y_1, y_2, \dots, y_n is a linearly independent system in H , $a_1, a_2, \dots, a_n \in \mathbb{R}$. We remark that the operator T can be considered as a perturbation of the operator A by the finite-rank operator B .

The case when A is a first-order or second-order differential operator in the spaces $L^2(0, 2\pi)$, $L^2(0, \infty)$ or in the Hilbert space of vector-valued functions, and B is a one-dimensional perturbation ($n = 1$), has been studied by many authors (see, e.g., [9, 20, 24]).

In particular, certain integrodifferential equations of the above type occur in quantum mechanical scattering theory [8].

In this paper, the generalized resolvents of perturbed symmetric operator T with equal and finite deficiency indices are investigated. Using the Weinstein-Aronszajn formula (see, e.g., [18]), we give a classification of the spectrum. Finally, the obtained results are applied to the study of two classes of first-order and second-order differential operators.

We note that the spectral theory of perturbed symmetric and selfadjoint operators have been investigated using various methods by many authors [3, 4, 5, 6, 11, 12, 13, 14, 15, 16, 17, 21, 22].

2. Preliminaries

Let A be a closed symmetric operator with nondensely defined domain in a separable Hilbert space H with equal deficiency indices (m, m) , and $m < \infty$. We denote by $\rho(A)$ the resolvent set of the operator A , the resolvent operator $R_\lambda(A)$ of A is defined as $R_\lambda(A) = (A - \lambda I)^{-1}$. The complement of $\rho(A)$ in the complex plane is called the spectrum of A and denoted by $\sigma(A)$. There is a decomposition of the spectrum $\sigma(A)$ into three disjoint subsets, at least one of which is not empty [1, 2, 10]:

$$\sigma(A) = P\sigma(A) \cup C\sigma(A) \cup PC\sigma(A), \quad (2.1)$$

$P\sigma(A)$ is called the point spectrum, $C\sigma(A)$ the continuous spectrum, and $PC\sigma(A)$ the point-continuous spectrum. We denote the essential spectrum of the operator A by $\sigma_e(A) = C\sigma(A) \cup PC\sigma(A)$.

For arbitrary $\lambda \in \mathbb{C}$, we denote $P_\lambda = N_\lambda \cap (D(A) \oplus N_{\bar{\lambda}})$, where $N_\lambda = H \ominus (A - \lambda I)D(A)$ is the deficiency subspace of the operator A [1, 2].

It is known [23] that $P_\lambda = \{0\}$ if and only if $\overline{D(A)} = H$, and if $\overline{D(A)} \neq H$, then the subset

$$G_\lambda = \{[\varphi, \psi] \in N_\lambda \times N_{\bar{\lambda}} : \varphi - \psi \in D(A)\} \quad (2.2)$$

is a graph of the isometric operator X_λ with domain P_λ and values in $P_{\bar{\lambda}}$.

We denote by \mathfrak{I} the set of linear operators F defined from N_i to N_{-i} , such that $\|F\| \leq 1$. For each analytic operator-valued function $F(\lambda)$ in \mathbb{C}^+ , with $\mathbb{C}^+ = \{\lambda : \operatorname{Im} \lambda > 0\}$, and values in \mathfrak{I} , we introduce the set $\Omega_F(\infty)$ consisting of elements $h \in N_i$ such that

$$\lim_{\lambda \rightarrow \infty, \lambda \in \mathbb{C}_\varepsilon^+} |\lambda| [\|h\| - \|F(\lambda)h\|] < \infty, \quad (2.3)$$

where $\mathbb{C}_\varepsilon^+ = \{\lambda \in \mathbb{C}^+ : \varepsilon < \arg \lambda < \pi - \varepsilon\}$, $0 < \varepsilon < \pi/2$.

It is known [27] that $\Omega_F(\infty)$ is a vector space and for each $h \in \Omega_F(\infty)$,

$$\lim_{\lambda \rightarrow \infty, \lambda \in \mathbb{C}_\varepsilon^+} F(\lambda)h = F_0(\infty)h \quad (2.4)$$

exists in the sense of the strong topology, and $F_0(\infty)$ is an isometric operator.

According to the theory of Štraus [28], the generalized resolvents of A are given by the formula

$$R_\lambda(A) = R_\lambda = (A_{F(\lambda)} - \lambda I)^{-1}, \quad R_{\bar{\lambda}} = R_\lambda^*, \quad \lambda \in \mathbb{C}^+, \quad (2.5)$$

where $A_{F(\lambda)}$ is an extension of A which is determined by the function $F(\lambda)$, whose values are operators from the deficiency subspace N_i to the deficiency subspace N_{-i} such that $\|F(\lambda)\| \leq 1$ and $F(\lambda)$ satisfy the condition

$$F_0(\infty)\psi = X_i\psi, \quad \text{for } \psi = 0 \text{ only}, \quad (2.6)$$

then $A_{F(\lambda)}$ is a restriction on H of a selfadjoint operator defined in a certain extended Hilbert space and is called quasiselfadjoint extension of the operator A [28] defined on $D(A_{F(\lambda)}) = D(A) + (F(\lambda) - I)N_i$ by

$$A_{F(\lambda)}(f + F(\lambda)\varphi - \varphi) = Af + iF(\lambda)\varphi + i\varphi, \quad f \in D(A), \varphi \in N_i. \quad (2.7)$$

For selfadjoint extensions with exit in the space in which acts the considered operators, see, for example, [12, 21] and the references therein.

We denote by \mathfrak{N} the set of analytic operator functions $F(\lambda)$ in \mathbb{C}^+ with values in \mathfrak{I} satisfying the condition (2.6).

Remark 2.1. To each selfadjoint extension of the operator A corresponds a certain constant operator function $F(\lambda) = V$, where V is an isometric operator defined from N_i over N_{-i} satisfying the condition $V\psi = X_i\psi$ for $\psi = 0$ only, and reciprocally.

We denote by \mathring{A} a selfadjoint extension of A and we introduce the operator

$$\mathring{U}_{\lambda\lambda_0} = (\mathring{A} - \lambda_0 I)(\mathring{A} - \lambda I)^{-1}, \quad \text{Im } \lambda > 0. \quad (2.8)$$

We note that (see [19])

$$\mathring{U}_{\lambda\lambda_0} N_{\lambda_0}^- = N_{\bar{\lambda}}, \quad (\text{Im } \lambda)(\text{Im } \lambda_0) \neq 0. \quad (2.9)$$

We denote by

$$\varphi_i^{(1)}, \varphi_i^{(2)}, \dots, \varphi_i^{(m)} \quad (2.10)$$

a basis of N_{-i} . From (2.9), $\varphi_{\bar{\lambda}}^{(k)} = \mathring{U}_{\lambda i} \varphi_i^{(k)}$, $k = 1, 2, \dots, m$ form a basis for $N_{\bar{\lambda}}$. In particular, the vectors

$$\varphi_{-i}^{(k)} = \mathring{U} \varphi_i^{(k)}, \quad k = 1, 2, \dots, m, \quad (2.11)$$

where $\mathring{U} = \mathring{U}_{-ii}$ is the Cayley transform [1, 2] of \mathring{A} , form an orthogonal basis of N_i .

To get a convenient formula of the generalized resolvents of A , we will need the following notation:

$$\Phi_{\lambda\mu} = (\lambda - \bar{\mu})[(\varphi_{\lambda}^{(k)}, \varphi_{\mu}^{(s)})]_{k,s=1}^m, \quad C(\lambda) = \Phi_{\lambda i}^{-1} \Phi_{\lambda(-i)}, \quad (2.12)$$

where E is the identity matrix of order m , $\Omega(\lambda)$ is an analytic matrix function in \mathbb{C}^+ corresponding, in the bases (2.10) and (2.11), to the operator function $F(\lambda) \in \mathfrak{N}$ and $\varphi_{\lambda} = (\varphi_{\lambda}^{(1)}, \dots, \varphi_{\lambda}^{(m)})^t$, $(f, \varphi_{\bar{\lambda}})^t = ((f, \varphi_{\bar{\lambda}}^{(1)}), \dots, (f, \varphi_{\bar{\lambda}}^{(m)}))$, t denotes the transpose, and (φ_{λ}, g) is defined analogously.

In what follows, we denote by Φ the set of matrices $\Omega(\lambda)$, $\lambda \in \mathbb{C}^+$, associated in the bases (2.10) and (2.11) to the operator functions $F(\lambda) \in \mathfrak{K}$.

According to the notation used in [7], the generalized resolvents of A are given by

$$\begin{aligned} R_\lambda(A)f &= R_\lambda f = \mathring{R}_\lambda f + (f, \varphi_{\bar{\lambda}})^t [E - \Omega(\lambda)] [C(\lambda)\Omega(\lambda) - E]^{-1} \Phi_{\lambda i}^{-1} \varphi_\lambda, \\ R_{\bar{\lambda}} &= R_\lambda^*, \quad \lambda \in \mathbb{C}^+, \end{aligned} \quad (2.13)$$

where \mathring{R}_λ is the resolvent of \mathring{A} and $\Omega(\lambda) \in \Phi$.

Remark 2.2. The formula (2.13) defines a resolvent of a selfadjoint extension of A if and only if $\Omega(\lambda)$ is a unitary constant matrix.

3. Resolvent and spectrum of a symmetric perturbed operator

Let $T = A + B$ be defined on $D(T) = D(A)$, where A is a linear closed symmetric operator in H and B is a finite-rank operator.

LEMMA 3.1. *For $\lambda \in \rho(A) \cap \rho(T)$, the resolvent $R_\lambda(T)$ of the operator T is given by*

$$R_\lambda(T) = R_\lambda(A) - R_\lambda(A)[I + BR_\lambda(A)]^{-1}BR_\lambda(A). \quad (3.1)$$

Proof. For $\lambda \in \rho(A) \cap \rho(T)$, the operator

$$R_\lambda(A)[I + BR_\lambda(A)]^{-1} = R_\lambda(T) \quad (3.2)$$

exists and is bounded. Then, we get

$$\begin{aligned} (T - \lambda I)[R_\lambda(A) - R_\lambda(A)(I + BR_\lambda(A))^{-1}BR_\lambda(A)] \\ = (A - \lambda I + B)[R_\lambda(A) - R_\lambda(A)(I + BR_\lambda(A))^{-1}BR_\lambda(A)] \\ = I + BR_\lambda(A) - (I + BR_\lambda(A))(I + BR_\lambda(A))^{-1}BR_\lambda(A) = I \end{aligned} \quad (3.3)$$

as required. \square

Remark 3.2. If $\|BR_\lambda(A)\| < 1$, then from (3.1), we obtain

$$R_\lambda(T) = R_\lambda(A)(I + BR_\lambda(A))^{-1} = R_\lambda(A) \sum_{k=0}^{\infty} (-1)^k [BR_\lambda(A)]^k. \quad (3.4)$$

Now, the aim is to give a convenient expression of $(I + BR_\lambda(A))^{-1}$ in a more specific case.

So, we study in detail the case when B is a finite-rank operator. Then,

$$Bf = \sum_{k=1}^n a_k(f, y_k) y_k, \quad f \in H, \quad (3.5)$$

where $a_1, a_2, \dots, a_n \in \mathbb{R}$; $\{y_1, y_2, \dots, y_n\}$ is a linearly independent system in H . If we put

$$(I + BR_\lambda(A))^{-1}BR_\lambda(A)f = y, \quad (3.6)$$

we have

$$y = BR_\lambda(A)f - BR_\lambda(A)y, \quad (3.7)$$

then, $y \in \text{Im } B$, so that

$$y = \sum_{k=1}^n c_k y_k. \quad (3.8)$$

From (3.7) and (3.8), we get

$$\sum_{k=1}^n c_k y_k = BR_\lambda(A)f - \sum_{k=1}^n c_k BR_\lambda(A)y_k, \quad (3.9)$$

with

$$c_k + a_k \sum_{j=1}^n c_j (R_\lambda(A)y_j, y_k) = a_k (R_\lambda(A)f, y_k). \quad (3.10)$$

The determinant $\Delta(\lambda)$ of the system (3.10) is given by

$$\Delta(\lambda) = \det \{ [\delta_{kj} + a_k (R_\lambda(A)y_j, y_k)]_{k,j=1}^n \}, \quad (3.11)$$

where δ_{kj} is the Kronecker symbol. If we suppose that $\Delta(\lambda) \neq 0$, the solution of (3.10) is given by

$$c_k = c_k(\lambda; f) = \frac{(f, \Delta_k(\lambda))}{\Delta(\lambda)}, \quad k = 1, 2, \dots, n, \quad (3.12)$$

where $\Delta_k(\lambda)$ is the determinant obtained from $\overline{\Delta(\lambda)}$ by replacing the k th column by $[a_j R_\lambda(A)y_j]_{j=1}^n$. So, from (3.1), we have

$$R_\lambda(T)f = R_\lambda(A)f - \sum_{k=1}^n \frac{(f, \Delta_k(\lambda))}{\Delta(\lambda)} R_\lambda(A)y_k. \quad (3.13)$$

This completes the proof of the following theorem.

THEOREM 3.3. *Let $\lambda \in \rho(A)$ such that $\Delta(\lambda) \neq 0$. Then, $\lambda \in \rho(T)$ and the resolvent of the operator T is given by (3.13).*

Remark 3.4. From (3.13), we note that the resolvent $R_\lambda(T)$ is a perturbation of $R_\lambda(A)$ by a finite-rank operator.

Remark 3.5. For the particular case $n = 1$ and $a_1 = 1$, the formula (3.13) was established in [9].

Remark 3.6. If $\lambda \in \rho(A)$ such that $\Delta(\lambda) = 0$, then λ is an eigenvalue of the operator T .

Proof. We can show that there exists an element

$$\psi = \sum_{k=1}^n \alpha_k y_k \quad (3.14)$$

such that $R_\lambda(A)\psi$ is an eigenvector of the operator T , corresponding to the eigenvalue λ . Consequently, we have

$$a_k \sum_{j=1}^n \alpha_j (R_\lambda(A)y_j, y_k) + \alpha_k = 0, \quad k = \overline{1, n}. \quad (3.15)$$

Since the determinant of this system $\Delta(\lambda) = 0$, it admits a nontrivial solution, which gives the desired result. \square

THEOREM 3.7. *Let μ be a fixed complex number. Then, the following holds.*

- (a) *If $\mu \in \rho(A)$ and $\Delta(\mu) \neq 0$, then $\mu \in \rho(T)$.*
- (b) *If $\mu \in \rho(A)$ and $\Delta(\mu) = 0$, then $\mu \in P\sigma(T)$ and the multiplicity of μ as an eigenvalue of T is equal to the order of the zero of $\Delta(\lambda)$ at μ .*
- (c) *If $\mu \in P\sigma(A)$ and μ of multiplicity $k > 0$ and if μ is a pole of $\Delta(\lambda)$ of multiplicity p ($k \geq p$), then*
 - (1) *for $k > p$, it holds that $\mu \in P\sigma(T)$ of multiplicity $(k - p)$,*
 - (2) *for $k = p$, it holds that $\mu \in \rho(T)$.*
- (d) *If $\mu \in P\sigma(A)$ is neither a zero, nor a pole of $\Delta(\lambda)$, then $\mu \in P\sigma(T)$.*
- (e) *If $\mu \in P\sigma(A)$ of multiplicity k and μ is a root of the function $\Delta(\lambda)$ of order p , then $\mu \in P\sigma(T)$ of order $(k + p)$.*
- (f) *The essential spectra $\sigma_e(A)$ and $\sigma_e(T)$, respectively of the operators A and T , coincide.*

Proof. It is sufficient to evaluate the function

$$C(\lambda) = \det \{I + BR_\lambda(A)\}. \quad (3.16)$$

To this end, let $y \in \text{Im } B$. Then,

$$BR_\lambda(A)y = \sum_{k=1}^n a_k (y, R_\lambda^*(A)y_k) y_k, \quad (3.17)$$

it is clear that $C(\lambda) = \Delta(\lambda)$, and the function $\Delta(\lambda)$ is meromorphic in $\rho(A) \cup P\sigma(A)$. From the formula of Weinstein and Aronszajn [18], we have

$$\overline{\vartheta}(\lambda; T) = \overline{\vartheta}(\lambda; A) + \vartheta(\lambda; \Delta), \quad (3.18)$$

where

$$\begin{aligned}\bar{\vartheta}(\lambda; A) &= \begin{cases} 0 & \text{if } \lambda \in \rho(A), \\ k & \text{if } \lambda \in P\sigma(A) \text{ and of multiplicity } k, \\ +\infty & \text{otherwise,} \end{cases} \\ \vartheta(\lambda; \Delta) &= \begin{cases} k & \text{if } \lambda \text{ is a zero of } \Delta(\lambda) \text{ of order } k, \\ -k & \text{if } \lambda \text{ is a pole of } \Delta(\lambda) \text{ of order } k, \\ 0 & \text{for other } \lambda \in \Omega, \end{cases}\end{aligned}\quad (3.19)$$

which gives the desired result. \square

4. Generalized resolvents

Now, we suppose that A is a symmetric operator with deficiency indices (m, m) , $m < \infty$.

LEMMA 4.1. *Let $\lambda \in \mathbb{C}$ such that $\text{Im } \lambda > 0$ and $\varphi_\lambda(A) \in N_{\bar{\lambda}}(A)$. Then, the element $\varphi_\lambda(T)$, defined by the formula*

$$\varphi_\lambda(T) = D(\lambda)\varphi_\lambda(A) = \varphi_\lambda(A) - \sum_{k=1}^n \frac{(\varphi_\lambda(A), \mathring{g}_k(\lambda))}{\mathring{\Delta}(\lambda)} R_\lambda(\mathring{A})y_k, \quad (4.1)$$

is an element of the deficiency subspace $N_{\bar{\lambda}}(T)$, where

$$D(\lambda) = I - R_\lambda(\mathring{A})[I + BR_\lambda(\mathring{A})]^{-1}B = I - R_\lambda(\mathring{T})B, \quad \mathring{g}_k(\lambda) = (\mathring{A} - \bar{\lambda}I)\mathring{\Delta}_k(\lambda), \quad (4.2)$$

$\mathring{\Delta}(\lambda)$ and $\mathring{\Delta}_k(\lambda)$ are defined similarly as $\Delta(\lambda)$ and $\Delta_k(\lambda)$ in the formula (3.13) by putting the operator \mathring{A} instead of the operator A .

Proof. Since the operators \mathring{A} and $\mathring{T} = \mathring{A} + B$ are selfadjoint and λ is nonreal, then $\lambda \in \rho(\mathring{A}) \cap \rho(\mathring{T})$. In addition, from Theorem 3.3 we have $\mathring{\Delta}(\lambda) \neq 0$. Furthermore, for each $f \in D(A) = D(T)$, we have

$$\begin{aligned}([\mathring{T} - \bar{\lambda}I]f, D(\lambda)\varphi_\lambda(A)) &= (D^*(\lambda)[\mathring{T} - \bar{\lambda}I]f, \varphi_\lambda(A)) \\ &= ([I - BR_{\bar{\lambda}}(\mathring{T})](\mathring{T} - \bar{\lambda}I)f, \varphi_\lambda(A)) \\ &= ((\mathring{A} - \bar{\lambda}I)f, \varphi_\lambda(A)) \\ &= 0,\end{aligned}\quad (4.3)$$

and the equality

$$\varphi_\lambda(T) = \varphi_\lambda(A) - \sum_{k=1}^n \frac{(\varphi_\lambda(A), \mathring{g}_k(\lambda))}{\mathring{\Delta}(\lambda)} R_\lambda(\mathring{A})y_k \quad (4.4)$$

results from (3.13). \square

Remark 4.2. We note that if $\varphi_\lambda(A) \neq 0$, then $\varphi_\lambda(T) \neq 0$.

Proof. If we suppose the contrary, we obtain $R_\lambda(\mathring{T})B\varphi_\lambda(A) = \varphi_\lambda(A)$, which gives $\mathring{A}\varphi_\lambda(A) = \lambda\varphi_\lambda(A)$. This leads to a contradiction, since a selfadjoint operator can not have nonreal eigenvalues. \square

Remark 4.3. If $D(A)$ is dense in H , then $\varphi_\lambda(A)$ and $\varphi_\lambda(T)$ are, respectively, eigenfunctions of the operators A^* and T^* , corresponding to the eigenvalues $\bar{\lambda}$.

Let $\varphi_i^{(k)}(T) = D(i)\varphi_\lambda^{(k)}(A)$, $k = 1, 2, \dots, m$, defined by the formula (4.1). If $\varphi_i^{(1)}(A), \varphi_i^{(2)}(A), \dots, \varphi_i^{(m)}(A)$ is a basis of the deficiency subspace $N_i(A)$ of the operator A , then $\varphi_i^{(1)}(T), \varphi_i^{(2)}(T), \dots, \varphi_i^{(m)}(T)$ is a basis of the deficiency subspace $N_i(T)$ of the operator T . Putting

$$\begin{aligned} \mathring{U}_{\lambda\lambda_0}(\mathring{T}) &= (\mathring{T} - \lambda_0 I)R_\lambda(\mathring{T}), & \varphi_\lambda^{(k)}(T) &= \mathring{U}_{\lambda i}(\mathring{T})\varphi_i^{(k)}(T), & k &= 1, 2, \dots, m, \\ \varphi_\lambda(T) &= (\varphi_\lambda^{(1)}(T), \dots, \varphi_\lambda^{(m)}(T))^t, & \Phi_{\lambda\mu}(T) &= (\lambda - \bar{\mu})[(\varphi_\lambda^{(k)}(T), \varphi_\mu^{(j)}(T))]_{k,j=1}^m, \end{aligned} \quad (4.5)$$

$C(\lambda) = \Phi_{\lambda i}^{-1}(T)\Phi_{\lambda(-i)}(T)$ denotes the characteristic matrix of the operator T , and $\omega(\lambda)$ the corresponding matrix of order $m \times m$, in the bases $\varphi_i^{(1)}(T), \varphi_i^{(2)}(T), \dots, \varphi_i^{(m)}(T)$ and $\varphi_{-i}^{(1)}(T), \varphi_{-i}^{(2)}(T), \dots, \varphi_{-i}^{(m)}(T)$.

THEOREM 4.4. *The set of all generalized resolvents of the operator T is given by*

$$R_\lambda(T)f = R_\lambda(\mathring{T})f + (f, \varphi_{\bar{\lambda}}(T))^t [E - \omega(\lambda)][C(\lambda)\omega(\lambda) - E]^{-1}\Phi_{\lambda i}^{-1}(T)\varphi_\lambda(T), \quad \forall f \in H, \quad (4.6)$$

where

$$R_\lambda(\mathring{T})f = R_\lambda(\mathring{A})f - \sum_{k=1}^n \frac{(f, \mathring{\Delta}_k(\lambda))}{\mathring{\Delta}(\lambda)} R_\lambda(\mathring{A})y_k. \quad (4.7)$$

Proof. The proof results from Lemma 4.1 and formula (2.13). \square

We denote, respectively, by A_ω and T_ω the quasiselfadjoint extensions of operators A and T corresponding to the operator function $F(\lambda) \in \mathfrak{I}$, defined by the matrix $\omega(\lambda)$.

Remark 4.5. To selfadjoint extensions of these operators correspond the constant unitary matrices $\omega = [\omega_{ij}]$.

THEOREM 4.6. *Suppose that $y_1, y_2, \dots, y_n \in \text{Im } A$, μ is an eigenvalue of the quasiselfadjoint extension A_ω of the operator A , $\mu \in P\sigma(A_\omega)$. If $\mu \in \rho(\mathring{A})$ and $\mathring{\Delta}(\mu) \neq 0$, then μ is an eigenvalue of the operator $T_\omega = A_\omega + B$ and the corresponding eigenfunction $\varphi_\mu(T_\omega)$ is given by*

$$\varphi_\mu(T_\omega) = D(\mu)\varphi_\mu(A_\omega) = \varphi_\mu(A_\omega) - \sum_{k=1}^n \frac{(\varphi_\mu(A_\omega), \mathring{g}_k(\mu))}{\mathring{\Delta}(\mu)} R_\mu(\mathring{A})y_k, \quad (4.8)$$

where $\varphi_\mu(A_\omega)$ is the eigenfunction of the operator A_ω , corresponding to the eigenvalue μ .

Proof. Since $y_1, y_2, \dots, y_n \in \text{Im } A$, then $B\varphi_\mu(A) \in \text{Im } A$. We also have

$$\varphi_\mu(T_\omega) = D(\mu)\varphi_\mu(A_\omega) = \varphi_\mu(A_\omega) - R_\mu(\overset{\circ}{T})B\varphi_\mu(A) = \varphi_\mu(A_\omega) - \psi_\mu, \quad (4.9)$$

where

$$\psi_\mu = R_\mu(\overset{\circ}{T})B\varphi_\mu(A) \in D(A). \quad (4.10)$$

Then,

$$\begin{aligned} T_\omega\varphi_\mu(T_\omega) &= T_\omega(\varphi_\mu(A_\omega) - \psi_\mu) \\ &= (A_\omega + B)\varphi_\mu(A_\omega) - T_\omega R_\mu(\overset{\circ}{T})B\varphi_\mu(A) \\ &= \mu\varphi_\mu(A_\omega) + B\varphi_\mu(A_\omega) - B\varphi_\mu(A_\omega) + \mu R_\mu(\overset{\circ}{T})B\varphi_\mu(A) \\ &= \mu\varphi_\mu(T_\omega). \end{aligned} \quad (4.11)$$

□

5. Applications

5.1. Perturbed first-order differential operator. Consider in $L^2(0, 2\pi)$ the operator $T = A + B$, where A is defined by $Ay = iy'$ with domain $D(A) = H_0^1(0, 2\pi)$ and B is given by

$$(By)(x) = \sum_{k=1}^n a_k(y, y_k)y_k(x), \quad (5.1)$$

where $y_1, y_2, \dots, y_n \in L^2(0, 2\pi)$ and $a_k \in \mathbb{R}$, for all $k = \overline{1, n}$. From [1, 2], the operator A is regular symmetric of deficiency indices $(1, 1)$ and each selfadjoint extension of A has a discrete spectrum.

THEOREM 5.1. *The generalized resolvent $R_\lambda(T_\theta)$ of T , corresponding to the function $\omega(\lambda) = \theta(\lambda)$, is an integral operator with kernel*

$$K(x, t) = \left[1_{[x, 2\pi]}(x) + \frac{1}{\theta(\lambda)e^{2\pi\lambda i} + 1} \right] e^{i\lambda(t-x)} + \sum_{k=1}^n \theta_k(\lambda, x)\phi_k(\lambda, t), \quad (5.2)$$

where $1_{[x, 2\pi]}(x)$ is the characteristic function of the interval $[x, 2\pi]$,

$$\phi_k(\lambda, t) = (\Delta_k^\theta(\lambda))(t), \quad \theta_k(\lambda, x) = \frac{(R_\lambda(A_\theta)y_k)(x)}{\Delta^\theta(\lambda)}, \quad (5.3)$$

where $R_\lambda(A_\theta)$, associated to the function $\theta(\lambda)$, is given by

$$(R_\lambda(A_\theta)y)(x) = \int_0^x y(t)e^{i\lambda(t-x)}dt - \frac{1}{\theta(\lambda)e^{2\pi\lambda i} + 1} \int_0^{2\pi} y(t)e^{\lambda i(t-x)}dt \quad (5.4)$$

with

$$\Delta^\theta(\lambda) = \{\delta_{k_j} + a_k(R_\lambda(A_\theta)y_j, y_k)\}, \quad (5.5)$$

and Δ_k^θ is the determinant obtained from $\overline{\Delta^\theta(\lambda)}$ replacing the k th column by $[a_k R_{\overline{\lambda}}(A_\theta)y_k]_1^n$.

Proof. The proof results from [26] and Theorem 3.3. \square

COROLLARY 5.2. *Let T_θ be a selfadjoint extension of T corresponding to the function θ , $|\theta| = 1$.*

(1) *The spectrum of T_θ is simple if and only if the roots of $\Delta^\theta(\lambda)$ are simple and for $k = 0, \pm 1, \pm 2, \dots$, $\Delta^\theta(1/2 + k - \varphi_0/2\pi) \neq 0$, where $\{1/2 + k - \varphi_0/2\pi\}$ is the spectrum of A_θ , and $\varphi_0 = \arg \theta$.*

(2) *$\sigma(T_\theta) = P\sigma(T_\theta) = E_1 \cup E_2$, where E_1 is the set of points of $\sigma(A_\theta) = \{1/2 + k - \varphi_0/2\pi, k = 0, \pm 1, \pm 2, \dots\}$ in which $\Delta^\theta(\lambda)$ is analytic, E_2 is the set of roots of $\Delta^\theta(\lambda)$.*

Proof. The proof results from (5.4), Theorem 3.7, and Lemma 4.1. \square

5.2. Perturbed second-order differential operator. Consider in $L^2(0, \infty)$ the operator $T = A + B$, where A is defined by

$$Ay = -y'' + x^2y \quad (5.6)$$

with domain $D(A)$ consisting of all variables y which satisfy

- (i) $y \in L^2(0, \infty)$,
- (ii) y' is absolutely continuous on all compact subintervals of $[0, \infty[$,
- (iii) $Ay \in L^2(0, \infty)$,
- (IV) $y(0) = y(\infty) = \lim_{x \rightarrow \infty} y(x) = 0$, $y'(0) = y'(\infty) = 0$,

and B is given by

$$(By)(x) = \sum_{k=1}^n a_k(y, y_k)y_k(x), \quad (5.7)$$

where $y_1, y_2, \dots, y_n \in L^2(0, 2\pi)$ and $a_k \in \mathbb{R}$, for all $k = \overline{1, n}$.

From [1, 2], the operator A is symmetric of deficiency indices $(1, 1)$. Let u_1, u_2 be two solutions of (5.6), satisfying the initial conditions

$$\begin{aligned} u_1(0, \lambda) &= 1, & u_1'(x, \lambda)|_{x=0} &= 0, \\ u_2(0, \lambda) &= 0, & u_2'(x, \lambda)|_{x=0} &= -1. \end{aligned} \quad (5.8)$$

There exists a function $m(\lambda)$ [29] analytic in $\mathbb{C} \setminus \mathbb{R}$ such that

$$\psi(x, \lambda) = u_2(x, \lambda) + m(\lambda)u_1(x, \lambda) \in L^2(0, \infty). \quad (5.9)$$

THEOREM 5.3. *The generalized resolvents $R_\lambda(T_\theta)$ of the operator T are defined by*

$$R_\lambda(T_\theta)y = R_\lambda(A_\theta)y - \sum_{k=1}^n \frac{(y, \Delta_k^\theta(\lambda))}{\Delta^\theta(\lambda)} R_\lambda(A_\theta)y_k, \quad \text{Im } \lambda > 0, \quad (5.10)$$

where

$$R_\lambda(A_\theta)y = \psi(x, \lambda) \int_0^x y(s)u_1(s, \lambda)ds + u_1(x, \lambda) \int_x^\infty y(s)\psi(s, \lambda)ds \\ - \frac{\psi(x, \lambda)}{\theta(\lambda) + m(\lambda)} \int_0^\infty y(s)\psi(s, \lambda)ds, \quad (5.11)$$

$$\Delta^\theta(\lambda) = \det\{\sigma_{j_k} + a_k(R_\lambda(A_\theta)y_j, y_k)\}, \quad \lambda \in \mathbb{C}^+, \quad (5.12)$$

with $\theta(\lambda)$ an arbitrary function analytic in \mathbb{C}^+ and such that $\text{Im}\theta(\lambda) \geq 0$ or $\theta(\lambda)$ is an infinite constant.

Proof. First, we show that for $\lambda \in \mathbb{C}^+$, $\Delta^\theta(\lambda) \neq 0$ (then, $\Delta^\theta \neq 0$). We know (see [1, 2]) that for each quasiselfadjoint extension of a symmetric operator, \mathbb{C}^+ is contained in the set of regular points of this operator. Then, if $\lambda \in \mathbb{C}^+$, we have $\lambda \in \rho(A_\theta)$ and $\lambda \in \rho(T_\theta)$. If we suppose that $\lambda \in \mathbb{C}^+$ and $\Delta^\theta(\lambda) = 0$, from Theorem 3.7, we obtain $\lambda \in P\sigma(T_\theta)$, which is a contradiction. The formula (5.11) results from [25]. Using Theorem 3.3, we end the proof. \square

COROLLARY 5.4. *Let T_θ be a selfadjoint extension associated to $\theta \in \overline{\mathbb{R}}$, let $\lambda_1, \lambda_2, \dots$ be the roots of $\Delta^\theta(\lambda)$ in $\rho(A_\theta)$ and let z_1, z_2, \dots be the poles of $\Delta^\theta(\lambda)$. Then,*

$$P\sigma(T_\theta) = (P\sigma(A_\theta) \setminus \{z_i\}_1^\infty) \cup \{\lambda_j\}_1^\infty. \quad (5.13)$$

Proof. The proof results from (b) and (c) of Theorem 3.7. \square

Acknowledgment

The author is grateful to the editor and the anonymous referees for their valuable comments and helpful suggestions which have much improved the presentation of the paper.

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