# QUASI-DEFINITENESS OF GENERALIZED UVAROV TRANSFORMS OF MOMENT FUNCTIONALS

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*Received 11 March 2001*

When $\sigma$ is a quasi-definite moment functional with the monic orthogonal polynomial system $\{P_n(x)\}_{n=0}^{\infty}$, we consider a point masses perturbation $\tau$ of $\sigma$ given by $\tau := \sigma + \lambda\sum_{l=1}^{m}\sum_{k=0}^{m_l}((-1)^k u_{lk}/k!)\delta^{(k)}(x-c_l)$, where $\lambda$, $u_{lk}$, and $c_l$ are constants with $c_i \neq c_j$ for $i \neq j$. That is, $\tau$ is a generalized Uvarov transform of $\sigma$ satisfying $A(x)\tau = A(x)\sigma$, where $A(x) = \prod_{l=1}^{m}(x-c_l)^{m_l+1}$. We find necessary and sufficient conditions for $\tau$ to be quasi-definite. We also discuss various properties of monic orthogonal polynomial system $\{R_n(x)\}_{n=0}^{\infty}$ relative to $\tau$ including two examples.

## 1. Introduction

In the study of Padé approximation (see [5, 10, 21]) of Stieltjes type meromorphic functions

$$\int_a^b \frac{d\mu(x)}{z-x} + \sum_{l=1}^{m}\sum_{k=0}^{m_l} C_{lk}\frac{k!}{(z-c_l)^{k+1}}, \tag{1.1}$$

where $-\infty \leq a < b \leq \infty$, $C_{lk}$ are constants, and $d\mu(x)$ is a positive Stieltjes measure, the denominators $R_n(x)$ of the main diagonal sequence of Padé approximants satisfy the orthogonality

$$\int_a^b R_n(x)\pi(x)d\mu(x) + \sum_{l=1}^{m}\sum_{k=0}^{m_l} C_{lk}\partial^k\left[\pi R_n\right](c_l) = 0, \quad \pi \in \mathbb{P}_{n-1}, \tag{1.2}$$


Journal of Applied Mathematics 1:2 (2001) 69–90
2000 Mathematics Subject Classification: 33C45
URL: http://jam.hindawi.com/volume-1/S1110757X01000225.html

where $\mathbb{P}_n$ is the space of polynomials of degree $\le n$ with $\mathbb{P}_{-1} = \{0\}$. That is, $R_n(x)$ $(n \ge 0)$ are orthogonal with respect to the measure

$$d\mu + \sum_{l=1}^{m} \sum_{k=0}^{m_l} (-1)^k C_{lk} \delta^{(k)}\big(x - c_l\big), \tag{1.3}$$

which is a point masses perturbation of $d\mu(x)$. Orthogonality to a positive or signed measure perturbed by one or two point masses arises naturally also in orthogonal polynomial eigenfunctions of higher order $(\ge 4)$ ordinary differential equations (see [14, 15, 16, 19]), which generalize Bochner's classification of classical orthogonal polynomials (see [6, 18]). On the other hand, many authors have studied various aspects of orthogonal polynomials with respect to various point masses perturbations of positive-definite (see [1, 2, 8, 14, 27, 28]) and quasi-definite (see [3, 4, 9, 11, 12, 20, 23]) moment functionals. In this work, we consider the most general such situation. That is, we consider a moment functional $\tau$ given by

$$\tau := \sigma + \lambda \sum_{l=1}^{m} \sum_{k=0}^{m_l} \frac{(-1)^k u_{lk}}{k!} \delta^{(k)}\big(x - c_l\big), \tag{1.4}$$

where $\sigma$ is a given quasi-definite moment functional, $\lambda$, $u_{lk}$, and $c_l$ are complex numbers with $u_{l,m_l} \neq 0$ and $c_i \neq c_j$ for $i \neq j$, that is, $\tau$ is obtained from $\sigma$ by adding a distribution with finite support. We give necessary and sufficient conditions for $\tau$ to be quasi-definite. When $\tau$ is also quasi-definite, we discuss various properties of orthogonal polynomials $\{R_n(x)\}_{n=0}^{\infty}$ relative to $\tau$ in connection with orthogonal polynomials $\{P_n(x)\}_{n=0}^{\infty}$ relative to $\sigma$. These generalize many previous works in [4, 9, 11, 12, 20, 23].

## 2. Necessary and sufficient conditions

For any integer $n \ge 0$, let $\mathbb{P}_n$ be the space of polynomials of degree $\le n$ and $\mathbb{P} = \bigcup_{n \ge 0} \mathbb{P}_n$. For any $\pi(x)$ in $\mathbb{P}$, let $\deg(\pi)$ be the degree of $\pi(x)$ with the convention that $\deg(0) = -1$. For the moment functionals $\sigma$, $\tau$ (i.e., linear functionals on $\mathbb{P}$) (see [7]), $c$ in $\mathbb{C}$, and a polynomial $\phi(x) = \sum_{k=0}^{n} a_k x^k$, let

$$\begin{aligned} &\langle \sigma', \pi \rangle := -\langle \sigma, \pi' \rangle; \qquad \langle \phi\sigma, \pi \rangle := \langle \sigma, \phi\pi \rangle; \\ &\big\langle (x-c)^{-1}\sigma, \pi \big\rangle := \big\langle \sigma, \theta_c \pi \big\rangle; \qquad \big(\theta_c \pi\big)(x) := \frac{\pi(x) - \pi(c)}{x - c}; \\ &(\sigma\phi)(x) := \sum_{k=0}^{n} \left( \sum_{j=k}^{n} a_j \sigma_{jk} \right) x^k; \qquad \langle \sigma\tau, \pi \rangle = \langle \sigma, \tau\pi \rangle, \quad \pi \in \mathbb{P}. \end{aligned} \tag{2.1}$$

We also let

$$F(\sigma)(z) := \sum_{n=0}^{\infty} \frac{\sigma_n}{z^{n+1}} \tag{2.2}$$

be the (formal) Stieltjes function of $\sigma$, where $\sigma_n := \langle \sigma, x^n \rangle$ $(n \geq 0)$ are the moments of $\sigma$. Following Zhedanov [29], for any polynomials $A(z)$, $B(z)$, $C(z)$, $D(z)$ with no common zero and $|C| + |D| \not\equiv 0$, let

$$S(A,B,C,D)F(\sigma)(z) := \frac{AF(\sigma)+B}{CF(\sigma)+D}. \tag{2.3}$$

If $S(A,B,C,D)F(\sigma) = F(\tau)$ for some moment functional $\tau$, then we call $\tau$ a rational (resp., linear) spectral transform of $\sigma$ (resp., when $C(z) = 0$). Then $S(A,B,C,D)F(\sigma) = F(\tau)$ if and only if

$$\begin{gathered} xA(x)\sigma = C(x)(\sigma\tau) + xD(x)\tau, \\ \langle \sigma, A \rangle + x\big(\sigma\theta_0 A\big)(x) + xB(x) = (\sigma\tau)\big(\theta_0 C\big)(x) + \langle \tau, D \rangle + x\big(\tau\theta_0 D\big)(x). \end{gathered} \tag{2.4}$$

In particular, for any $c$ and $\beta$ in $\mathbb{C}$, let

$$U(c,\beta)F(\sigma) := \frac{(z-c)F(\sigma)+\beta}{z-c} \tag{2.5}$$

be the Uvarov transform (see [28, 29]) of $F(\sigma)$. Then for any $\{c_i\}_{i=1}^k$ and $\{\beta_i\}_{i=1}^k$ in $\mathbb{C}$,

$$F(\tau) := U\big(c_k,\beta_k\big)\cdots U\big(c_1,\beta_1\big)F(\sigma) = \frac{A(z)F(\sigma)+B(z)}{A(z)}, \tag{2.6}$$

where $A(z) = \prod_{i=1}^k (z-c_i)$, $B(z) = \sum_{i=1}^k \beta_i \sum_{\substack{j=1 \\ j\neq i}}^k (z-c_j)$, and by (2.4)

$$A(x)\tau = A(x)\sigma. \tag{2.7}$$

In this case, we say that $\tau$ is a generalized Uvarov transform of $\sigma$. Conversely, if (2.7) holds for some polynomial $A(x)$ $(\not\equiv 0)$, then

$$F(\tau) = \frac{A(z)F(\sigma) + \big(\tau\theta_0 A\big)(z) - \big(\sigma\theta_0 A\big)(z)}{A(z)} \tag{2.8}$$

and $F(\tau)$ is obtained from $F(\sigma)$ by $\deg(A)$ successive Uvarov transforms (see [29]), that is, $\tau$ is a generalized Uvarov transform of $\sigma$.

In the following, we always assume that $\tau$ is a moment functional given by (1.4), where $\sigma$ is a quasi-definite moment functional. Let $\{P_n(x)\}_{n=0}^{\infty}$ be the monic orthogonal polynomial system (MOPS) relative to $\sigma$ satisfying the three term recurrence relation

$$P_{n+1}(x) = \big(x - b_n\big)P_n(x) - c_n P_{n-1}(x), \quad n \geq 0, \ \big(P_{-1}(x) = 0\big). \tag{2.9}$$

Since (1.4) implies (2.7) with $A(x) = \prod_{l=1}^{m}(x - c_l)^{m_l+1}$, $\tau$ is a generalized Uvarov transform of $\sigma$. Then our main concern is to find conditions under which a generalized Uvarov transform $\tau$, given by (1.4), of $\sigma$ is also quasi-definite. In other words, we are to solve the division problem (2.7) of the moment functionals.

Let

$$K_n(x,y) := \sum_{j=0}^{n} \frac{P_j(x)P_j(y)}{\langle \sigma, P_j^2 \rangle}, \quad n \geq 0 \tag{2.10}$$

be the $n$th kernel polynomial for $\{P_n(x)\}_{n=0}^{\infty}$ and $K_n^{(i,j)}(x,y) = \partial_x{}^i \partial_y{}^j K_n(x,y)$. We need the following lemma which is easy to prove.

Lemma 2.1. *Let* $V = (x_1, x_2, \ldots, x_n)^t$ *and* $W = (y_1, y_2, \ldots, y_n)^t$ *be two vectors in* $\mathbb{C}^n$. *Then*

$$\det\big(I_n + VW^t\big) = 1 + \sum_{j=1}^{n} x_j y_j, \quad n \geq 1, \tag{2.11}$$

*where* $I_n$ *is the* $n \times n$ *identity matrix.*

Theorem 2.2. *The moment functional* $\tau$ *is quasi-definite if and only if* $d_n \neq 0$, $n \geq 0$, *where* $d_n$ *is the determinant of* $(\sum_{l=1}^{m}(m_l + 1)) \times (\sum_{l=1}^{m}(m_l + 1))$ *matrix* $D_n$:

$$D_n := \big[A_{tl}(n)\big]_{t,l=1}^{m}, \quad n \geq 0, \tag{2.12}$$

*where*

$$A_{tl}(n) = \left[\delta_{tl}\delta_{ki} + \lambda \sum_{j=0}^{m_l - i} \frac{u_{l,i+j}}{i!j!} K_n^{(k,j)}\big(c_t, c_l\big)\right]_{k=0,\ i=0}^{m_t \quad m_l}. \tag{2.13}$$

*If* $\tau$ *is quasi-definite, then the MOPS* $\{R_n(x)\}_{n=0}^{\infty}$ *relative to* $\tau$ *is given by*

$$R_n(x) = P_n(x) - \lambda \sum_{l=1}^{m} \sum_{i=0}^{m_l} \sum_{j=0}^{m_l - i} \frac{u_{l,i+j}}{i!j!} K_{n-1}^{(0,j)}\big(x, c_l\big) R_n^{(i)}\big(c_l\big), \tag{2.14}$$

*where* $\{R_n^{(i)}(c_l)\}_{l=1,i=0}^{m\ \ m_l}$ *are given by*

$$D_{n-1}\begin{bmatrix} R_n(c_1) \\ R_n'(c_1) \\ \vdots \\ R_n^{(m_1)}(c_1) \\ R_n(c_2) \\ \vdots \\ R_n^{(m_m)}(c_m) \end{bmatrix} = \begin{bmatrix} P_n(c_1) \\ P_n'(c_1) \\ \vdots \\ P_n^{(m_1)}(c_1) \\ P_n(c_2) \\ \vdots \\ P_n^{(m_m)}(c_m) \end{bmatrix}, \quad n \geq 0 \ (D_{-1} = I). \tag{2.15}$$

*Moreover*,

$$\langle \tau, R_n^2 \rangle = \frac{d_n}{d_{n-1}} \langle \sigma, P_n^2 \rangle, \quad n \geq 0 \ (d_{-1} = 1). \tag{2.16}$$

*Proof.* (⇒). Assume that $\tau$ is quasi-definite and expand $R_n(x)$ as

$$R_n(x) = P_n(x) + \sum_{j=0}^{n-1} C_{nj} P_j(x), \quad n \geq 1, \tag{2.17}$$

where $C_{nj} = \langle \sigma, R_n P_j \rangle / \langle \sigma, P_j^2 \rangle$, with $0 \leq j \leq n-1$. Here,

$$\begin{aligned} \langle \sigma, R_n P_j \rangle &= \left\langle \tau - \lambda \sum_{l=1}^{m} \sum_{k=0}^{m_l} \frac{(-1)^k u_{lk}}{k!} \delta^{(k)}(x - c_l), R_n P_j \right\rangle \\ &= -\lambda \sum_{l=1}^{m} \sum_{k=0}^{m_l} \frac{u_{lk}}{k!} \sum_{i=0}^{k} \binom{k}{i} R_n^{(i)}(c_l) P_j^{(k-i)}(c_l) \end{aligned} \tag{2.18}$$

so that

$$\begin{aligned} R_n(x) &= P_n(x) - \lambda \sum_{j=0}^{n-1} \frac{P_j(x)}{\langle \sigma, P_j^2 \rangle} \sum_{l=1}^{m} \sum_{k=0}^{m_l} \frac{u_{lk}}{k!} \sum_{i=0}^{k} \binom{k}{i} R_n^{(i)}(c_l) P_j^{(k-i)}(c_l) \\ &= P_n(x) - \lambda \sum_{l=1}^{m} \sum_{k=0}^{m_l} \frac{u_{lk}}{k!} \sum_{i=0}^{k} \binom{k}{i} R_n^{(i)}(c_l) K_{n-1}^{(0,k-i)}(x, c_l) \\ &= P_n(x) - \lambda \sum_{l=1}^{m} \sum_{i=0}^{m_l} \sum_{j=0}^{m_l - i} \frac{u_{l,i+j}}{i!j!} K_{n-1}^{(0,j)}(x, c_l) R_n^{(i)}(c_l). \end{aligned} \tag{2.19}$$

Hence, we have (2.14). Set the matrices $B_l$ and $E_l$ to be

$$B_l = \begin{bmatrix} R_n(c_l) \\ R_n'(c_l) \\ \vdots \\ R_n^{(m_l)}(c_l) \end{bmatrix}, \quad E_l = \begin{bmatrix} P_n(c_l) \\ P_n'(c_l) \\ \vdots \\ P_n^{(m_l)}(c_l) \end{bmatrix}, \quad 1 \leq l \leq m. \tag{2.20}$$

Then,

$$\begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_m \end{bmatrix} = \left[A_{tl}(n-1)\right]_{t,l=1}^{m} \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \end{bmatrix}, \tag{2.21}$$

which gives (2.15). Now,

$$\begin{aligned}
D_n &= \left[A_{tl}(n)\right]_{t,l=1}^{m} \\
&= D_{n-1} + \frac{\lambda}{\langle \sigma, P_n^2 \rangle} \left[ \left[ \sum_{j=0}^{m_l-i} \frac{u_{l,i+j}}{i!j!} P_n^{(j)}(c_l) P_n^{(k)}(c_t) \right]_{k=0,\ i=0}^{m_t \quad m_l} \right]_{t,l=1}^{m} \\
&= D_{n-1} + \frac{\lambda}{\langle \sigma, P_n^2 \rangle} \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_m \end{bmatrix} \left[ \sum_{j=0}^{m_1} \frac{u_{1j}}{j!} P_n^{(j)}(c_1),\ \sum_{j=0}^{m_1-1} \frac{u_{1,j+1}}{j!} P_n^{(j)}(c_1), \ldots, \right. \\
&\qquad \frac{u_{1,m_1}}{m_1!} P_n(c_1), \sum_{j=0}^{m_2} \frac{u_{2j}}{j!} P_n^{(j)}(c_2), \ldots, \\
&\qquad \left. \frac{u_{m,m_m}}{m_m!} P_n(c_m) \right] \\
&= D_{n-1} \left[ I + \frac{\lambda}{\langle \sigma, P_n^2 \rangle} \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \end{bmatrix} \left[ \sum_{j=0}^{m_1} \frac{u_{1j}}{j!} P_n^{(j)}(c_1),\ \sum_{j=0}^{m_1-1} \frac{u_{1,j+1}}{j!} P_n^{(j)}(c_1), \ldots, \right. \right. \\
&\qquad \frac{u_{1,m_1}}{m_1!} P_n(c_1), \sum_{j=0}^{m_2} \frac{u_{2j}}{j!} P_n^{(j)}(c_2), \ldots, \\
&\qquad \left. \left. \frac{u_{m,m_m}}{m_m!} P_n(c_m) \right] \right]
\end{aligned} \tag{2.22}$$

so that

$$d_n = d_{n-1}\left(1+\frac{\lambda}{\langle\sigma,P_n^2\rangle}\sum_{l=1}^{m}\sum_{i=0}^{m_l}\sum_{j=0}^{m_l-i}\frac{u_{l,i+j}}{i!j!}P_n^{(j)}(c_l)R_n^{(i)}(c_l)\right) \tag{2.23}$$

by Lemma 2.1. On the other hand,

$$\begin{aligned}\langle\tau,R_n^2\rangle &= \langle\tau,R_nP_n\rangle\\ &= \langle\sigma,R_nP_n\rangle+\lambda\left\langle\sum_{l=1}^{m}\sum_{k=0}^{m_l}\frac{(-1)^k u_{lk}}{k!}\delta^{(k)}(x-c_l),R_nP_n\right\rangle\\ &= \langle\sigma,P_n^2\rangle+\lambda\sum_{l=1}^{m}\sum_{k=0}^{m_l}\frac{u_{lk}}{k!}\sum_{j=0}^{k}\binom{k}{j}R_n^{(j)}(c_l)P_n^{(k-j)}(c_l)\\ &= \langle\sigma,P_n^2\rangle+\lambda\sum_{l=1}^{m}\sum_{j=0}^{m_l}\sum_{k=j}^{m_l}\frac{u_{lk}}{k!}\binom{k}{j}R_n^{(j)}(c_l)P_n^{(k-j)}(c_l)\end{aligned} \tag{2.24}$$

so that

$$\langle\tau,R_n^2\rangle = \langle\sigma,P_n^2\rangle+\lambda\sum_{l=1}^{m}\sum_{j=0}^{m_l}\sum_{k=0}^{m_l-j}\frac{u_{l,j+k}}{j!k!}R_n^{(j)}(c_l)P_n^{(k)}(c_l). \tag{2.25}$$

Hence, from (2.23) and (2.25), we have

$$\langle\sigma,P_n^2\rangle d_n = d_{n-1}\langle\tau,R_n^2\rangle,\quad n\geq 0. \tag{2.26}$$

Note that (2.26) also holds for $n=0$ if we take $d_{-1}=1$. Hence, $d_n\neq 0$, $n\geq 0$ inductively and we have (2.16).

($\Leftarrow$). Assume that $d_n\neq 0$, with $n\geq 0$ and define $\{R_n(x)\}_{n=0}^{\infty}$ by (2.14). Then we have, by (2.14) and (2.23),

$$
\begin{aligned}
\langle \tau, R_n P_r \rangle &= \langle \sigma, R_n P_r \rangle + \lambda \left\langle \sum_{l=1}^{m} \sum_{k=0}^{m_l} \frac{(-1)^k u_{lk}}{k!} \delta^{(k)}(x-c_l), R_n P_r \right\rangle \\
&= \langle \sigma, R_n P_r \rangle + \lambda \sum_{l=1}^{m} \sum_{k=0}^{m_l} \frac{u_{lk}}{k!} \sum_{j=0}^{k} \binom{k}{j} R_n^{(j)}(c_l) P_r^{(k-j)}(c_l) \\
&= \langle \sigma, P_n P_r \rangle - \lambda \sum_{l=1}^{m} \sum_{j=0}^{m_l} \sum_{k=0}^{m_l-j} \frac{u_{l,j+k}}{j!k!} R_n^{(j)}(c_l) \langle \sigma, K_{n-1}^{(0,k)}(x,c_l) P_r(x) \rangle \\
&\quad + \lambda \sum_{l=1}^{m} \sum_{j=0}^{m_l} \sum_{k=0}^{m_l-j} \frac{u_{l,j+k}}{j!k!} R_n^{(j)}(c_l) P_r^{(k)}(c_l) \\
&= \langle \sigma, P_n P_r \rangle - \lambda \sum_{l=1}^{m} \sum_{j=0}^{m_l} \sum_{k=0}^{m_l-j} \frac{u_{l,j+k}}{j!k!} R_n^{(j)}(c_l) P_r^{(k)}(c_l) (1-\delta_{nr}) \\
&\quad + \lambda \sum_{l=1}^{m} \sum_{j=0}^{m_l} \sum_{k=0}^{m_l-j} \frac{u_{l,j+k}}{j!k!} R_n^{(j)}(c_l) P_r^{(k)}(c_l) \\
&= \begin{cases} 0, & 0 \le r \le n-1, \\ \langle \sigma, P_n^2 \rangle + \lambda \displaystyle\sum_{l=1}^{m} \sum_{j=0}^{m_l} \sum_{k=0}^{m_l-j} \frac{u_{l,j+k}}{j!k!} P_n^{(k)}(c_l) R_n^{(j)}(c_l), & r = n, \end{cases} \\
&= \begin{cases} 0, & 0 \le r \le n-1, \\ \dfrac{d_n}{d_{n-1}} \langle \sigma, P_n^2 \rangle \neq 0, & r = n, \end{cases}
\end{aligned} \tag{2.27}
$$

since $\langle \sigma, K_{n-1}^{(0,k)}(x,c_l) P_r(x) \rangle = P_r^{(k)}(c_l)(1-\delta_{nr})$. Hence,

$$
\langle \tau, R_n R_m \rangle = \begin{cases} 0, & \text{if } 0 \le m \le n-1, \\ \langle \tau, R_n P_n \rangle \neq 0, & \text{if } m = n, \end{cases} \tag{2.28}
$$

so that $\{R_n(x)\}_{n=0}^{\infty}$ is the MOPS relative to $\tau$ and so $\tau$ is also quasi-definite. □

General division problems of moment functionals

$$
D(x)\tau = A(x)\sigma \tag{2.29}
$$

is handled in [17], when $D(x)$ and $A(x)$ have no common zero. Theorem 2.2 includes the following as special cases.

- $m = 1$, $m_1 = 0$: Marcellán and Maroni [23],
- $m = 2$, $m_1 = m_2 = 0$: Draïdi and Maroni [9], Kwon and Park [20],

- $m=1$, $m_1=1$: Belmehdi and Marcellán [4],
- $m=1$: Kim, Kwon, and Park [12].

Some other special cases where $\sigma$ is a classical moment functional were handled in [2, 1, 3, 8, 11, 14].

From now on, we always assume that $d_n \neq 0$, with $n \geq 0$, so that $\tau$ is also quasi-definite.

Theorem 2.3. *For the MOPS* $\{R_n(x)\}_{n=0}^{\infty}$ *relative to* $\tau$, *we have*

(i) *(the three-term recurrence relation)*

$$R_{n+1}(x)=(x-\beta_n)R_n(x)-\gamma_n R_{n-1}(x), \quad n\geq 0, \tag{2.30}$$

*where*

$$\begin{aligned}\beta_n &= b_n+\frac{\lambda}{\langle\sigma,P_n^2\rangle}\sum_{l=1}^{m}\sum_{i=0}^{m_l}\sum_{j=0}^{m_l-i}\frac{u_{l,i+j}}{i!j!}\\ &\quad\times\left\{P_n^{(j)}(c_l)R_{n+1}^{(i)}(c_l)-P_{n-1}^{(j)}(c_l)R_n^{(i)}(c_l)\right\} \quad (n\geq 0),\end{aligned} \tag{2.31}$$

$$\gamma_n=\frac{d_n d_{n-2}}{d_{n-1}^2}c_n \quad (n\geq 1). \tag{2.32}$$

(ii) *(the quasi-orthogonality)*

$$\prod_{l=1}^{m}(x-c_l)^{m_l+1}R_n(x)=\sum_{j=n-r}^{n+r}C_{nj}P_j(x), \quad n\geq r, \tag{2.33}$$

*where* $r=\sum_{l=1}^{m}(m_l+1)$, $C_{n,n-r}\neq 0$, *and*

$$\begin{aligned}C_{nj} &= \frac{\langle\sigma,\prod_{l=1}^{m}(x-c_l)^{m_l+1}R_nP_j\rangle}{\langle\sigma,P_j^2\rangle}\\ &= \frac{\langle\tau,\prod_{l=1}^{m}(x-c_l)^{m_l+1}R_nP_j\rangle}{\langle\sigma,P_j^2\rangle}, \quad \text{where } n-r\leq j\leq n+r.\end{aligned} \tag{2.34}$$

*Proof.* For (i), by (2.14), we can rewrite (2.30) as

$$\begin{aligned}&P_{n+1}(x)-\lambda\sum_{l=1}^{m}\sum_{i=0}^{m_l}\sum_{j=0}^{m_l-i}\frac{u_{l,i+j}}{i!j!}K_n^{(0,j)}(x,c_l)R_{n+1}^{(i)}(c_l)\\ &\quad=(x-\beta_n)\left\{P_n(x)-\lambda\sum_{l=1}^{m}\sum_{i=0}^{m_l}\sum_{j=0}^{m_l-i}\frac{u_{l,i+j}}{i!j!}K_{n-1}^{(0,j)}(x,c_l)R_n^{(i)}(c_l)\right\}\\ &\qquad-\gamma_n\left\{P_{n-1}(x)-\lambda\sum_{l=1}^{m}\sum_{i=0}^{m_l}\sum_{j=0}^{m_l-i}\frac{u_{l,i+j}}{i!j!}K_{n-2}^{(0,j)}(x,c_l)R_{n-1}^{(i)}(c_l)\right\}.\end{aligned} \tag{2.35}$$

After multiplying (2.35) by $P_n(x)$ and applying $\sigma$, we have

$$\begin{aligned}&-\lambda\sum_{l=1}^{m}\sum_{i=0}^{m_l}\sum_{j=0}^{m_l-i}\frac{u_{l,i+j}}{i!j!}P_n^{(j)}(c_l)R_{n+1}^{(i)}(c_l)\\&\quad=(b_n-\beta_n)\langle\sigma,P_n^2\rangle-\lambda\sum_{l=1}^{m}\sum_{i=0}^{m_l}\sum_{j=0}^{m_l-i}\frac{u_{l,i+j}}{i!j!}P_{n-1}^{(j)}(c_l)R_n^{(i)}(c_l).\end{aligned}\tag{2.36}$$

Hence, we have (2.31) and by (2.16)

$$\gamma_n=\frac{\langle\tau,R_n^2\rangle}{\langle\tau,R_{n-1}^2\rangle}=\frac{d_n d_{n-2}}{d_{n-1}^2}c_n\quad(n\geq 1).\tag{2.37}$$

For (ii), $\prod_{l=1}^{m}(x-c_l)^{m_l+1}R_n(x)=\sum_{j=0}^{n+r}C_{nj}P_j(x)$, where $r=\sum_{l=1}^{m}(m_l+1)$ and

$$\begin{aligned}C_{nk}\langle\sigma,P_k^2\rangle&=\left\langle\sigma,\prod_{l=1}^{m}(x-c_l)^{m_l+1}R_n(x)P_k(x)\right\rangle\\&=\left\langle\prod_{l=1}^{m}(x-c_l)^{m_l+1}\tau,R_n(x)P_k(x)\right\rangle\\&=\left\langle\tau,\prod_{l=1}^{m}(x-c_l)^{m_l+1}R_n(x)P_k(x)\right\rangle=0,\quad\text{if } r+k<n.\end{aligned}\tag{2.38}$$

Hence, $C_{nk}=0$, $0\leq k\leq n-r-1$ and $C_{n,n-r}\neq 0$ so that we have (2.33) and (2.34). □

Corollary 2.4. *Assume that* $\sigma$ *is positive-definite and let* $[\xi,\eta]$ *be the true interval of the orthogonality of* $\sigma$. *Then*

(i) $\prod_{l=1}^{m}(x-c_l)^{m_l+1}R_n(x)$ *has at least* $n-r$ *distinct nodal zeros (i.e., zeros of odd multiplicities) in* $(\xi,\eta)$.

(ii) $R_n(x)$ *has at least* $n-r-m$ *distinct nodal zeros in* $(\xi,\eta)$.

*If furthermore* $m_l$ $(1\leq l\leq m)$ *are odd or* $\xi\geq c_l$ $(1\leq l\leq m)$, *then*

(iii) $R_n(x)$ *has at least* $n-r$ *distinct nodal zeros in* $(\xi,\eta)$.

*Proof.* (i) and (ii) are trivial by (2.33).

For (iii), assume that $m_l$ $(1\leq l\leq m)$ are odd. Then $\tilde{\sigma}:=\prod_{l=1}^{m}(x-c_l)^{m_l+1}\sigma$ is also positive-definite on $[\xi,\eta]$. Let $\{\tilde{P}_n(x)\}_{n=0}^{\infty}$ be the MOPS relative to $\tilde{\sigma}$. Then we may write $R_n(x)=\sum_{j=0}^{n}\tilde{C}_{nj}\tilde{P}_j(x)$, where

$$\begin{aligned}\tilde{C}_{nk}\langle\tilde{\sigma},\tilde{P}_k^2\rangle=\langle\tilde{\sigma},R_n\tilde{P}_k\rangle&=\left\langle\prod_{l=1}^{m}(x-c_l)^{m_l+1}\tau,R_n\tilde{P}_k\right\rangle\\&=\left\langle\tau,\prod_{l=1}^{m}(x-c_l)^{m_l+1}R_n\tilde{P}_k\right\rangle.\end{aligned}\tag{2.39}$$

Hence, $\tilde{C}_{nk}=0$, $0\le k\le n-r-1$ so that $R_n(x)=\sum_{j=n-r}^{n}\tilde{C}_{nj}\tilde{P}_j(x)$. Hence, $R_n(x)$ has at least $n-r$ distinct nodal zeros in $(\xi,\eta)$. In case $\xi\ge c_l$ $(1\le l\le m)$, $\tilde{\sigma}=\prod_{l=1}^{m}(x-c_l)^{m_l+1}\sigma$ is also positive-definite on $[\xi,\eta]$ so that by the same reasoning as above $R_n(x)$ has at least $n-r$ distinct nodal zeros in $(\xi,\eta)$. □

Theorem 2.5. *For any polynomial* $p(x)$ *of degree at most* $n$, *we have*

$$\left\langle\tau,L_n^{(0,k)}(x,y)p(x)\right\rangle=p^{(k)}(y),\tag{2.40}$$

*where* $L_n(x,y)=\sum_{i=0}^{n}R_i(x)R_i(y)/\langle\tau,R_i^2\rangle$, $n\ge 0$, *is the* $n$*th kernel polynomial for* $\{R_n(x)\}_{n=0}^{\infty}$ *and*

$$L_n(x,y)=K_n(x,y)-\frac{\lambda}{d_n}\sum_{l=1}^{m}\sum_{i=0}^{m_l}\left|D_n^u\right|\sum_{j=0}^{m_l-i}\frac{u_{l,i+j}}{i!j!}K_n^{(0,j)}\left(x,c_l\right),\tag{2.41}$$

*where* $u=\sum_{k=1}^{l-1}(m_k+1)+(i+1)$ *and* $D_n^i$ *is the matrix obtained from* $D_n$ *by replacing the* $i$*th column of* $D_n$ *by*

$$\begin{aligned}&\left[K_n\left(c_1,y\right),K_n^{(1,0)}\left(c_1,y\right),\dots,K_n^{(m_1,0)}\left(c_1,y\right),\right.\\&\left.\;K_n\left(c_2,y\right),K_n^{(1,0)}\left(c_2,y\right),\dots,K_n^{(m_m,0)}\left(c_m,y\right)\right]^T.\end{aligned}\tag{2.42}$$

*Proof.* If deg$(p)\le n$, then $p(x)=\sum_{i=0}^{n}(\langle\tau,pR_i\rangle/\langle\tau,R_i^2\rangle)R_i(x)$ so that

$$\begin{aligned}\left\langle\tau,L_n^{(0,k)}(x,y)p(x)\right\rangle&=\sum_{i=0}^{n}\frac{\left\langle\tau,pR_i\right\rangle}{\left\langle\tau,R_i^2\right\rangle}\left\langle\tau,L_n^{(0,k)}(x,y)R_i(x)\right\rangle\\&=\sum_{i=0}^{n}\frac{\left\langle\tau,pR_i\right\rangle}{\left\langle\tau,R_i^2\right\rangle}\sum_{j=0}^{n}\frac{R_j^{(k)}(y)}{\left\langle\tau,R_j^2\right\rangle}\left\langle\tau,R_j(x)R_i(x)\right\rangle\\&=\sum_{i=0}^{n}\frac{\left\langle\tau,pR_i\right\rangle}{\left\langle\tau,R_i^2\right\rangle}R_i^{(k)}(y)=p^{(k)}(y).\end{aligned}\tag{2.43}$$

Expand $L_n(x,y)$ as $L_n(x,y)=\sum_{j=0}^{n}a_{nj}(y)P_j(x)$, where

$$\begin{aligned}a_{nj}(y)&=\frac{\left\langle\sigma,L_n(x,y)P_j(x)\right\rangle}{\left\langle\sigma,P_j^2\right\rangle}\\&=\frac{\left\langle\tau,L_n(x,y)P_j(x)\right\rangle}{\left\langle\sigma,P_j^2\right\rangle}-\frac{\lambda}{\left\langle\sigma,P_j^2\right\rangle}\\&\quad\times\sum_{l=1}^{m}\sum_{k=0}^{m_l}\frac{(-1)^k u_{lk}}{k!}\left\langle\delta^{(k)}\left(x-c_l\right),L_n(x,y)P_j(x)\right\rangle\\&=\frac{P_j(y)}{\left\langle\sigma,P_j^2\right\rangle}-\frac{\lambda}{\left\langle\sigma,P_j^2\right\rangle}\sum_{l=1}^{m}\sum_{k=0}^{m_l}\frac{u_{lk}}{k!}\sum_{i=0}^{k}\binom{k}{i}L_n^{(i,0)}\left(c_l,y\right)P_j^{(k-i)}\left(c_l\right)\end{aligned}\tag{2.44}$$

by (2.40). Hence

$$
\begin{aligned}
L_n(x,y) &= K_n(x,y) - \lambda \sum_{l=1}^{m} \sum_{k=0}^{m_l} \frac{u_{lk}}{k!} \sum_{i=0}^{k} \binom{k}{i} L_n^{(i,0)}(c_l,y) K_n^{(0,k-i)}(x,c_l) \\
&= K_n(x,y) - \lambda \sum_{l=1}^{m} \sum_{i=0}^{m_l} \sum_{j=0}^{m_l-i} \frac{u_{l,i+j}}{i!j!} K_n^{(0,j)}(x,c_l) L_n^{(i,0)}(c_l,y),
\end{aligned}
\tag{2.45}
$$

and so

$$
\begin{aligned}
&D_n\big[L_n(c_1,y), L_n^{(1,0)}(c_1,y), \ldots, L_n^{(m_1,0)}(c_1,y), \\
&\quad L_n(c_2,y), \ldots, L_n^{(m_m,0)}(c_m,y)\big]^T \\
&\quad = \big[K_n(c_1,y), K_n^{(1,0)}(c_1,y), \ldots, K_n^{(m_1,0)}(c_1,y), \\
&\qquad K_n(c_2,y), \ldots, K_n^{(m_m,0)}(c_m,y)\big]^T.
\end{aligned}
\tag{2.46}
$$

Hence, we have (2.41) from (2.45) and (2.46). □

## 3. Semi-classical case

Since $\tau$ is a linear spectral transform (see [29]) of $\sigma$, if $\sigma$ is a Laguerre-Hahn form (see [22]) or a semi-classical form (see [24]) or a second degree form (see [26]), then so is $\tau$. Here, we consider the semi-classical case more closely.

*Definition 3.1* (see Maroni [24]). A moment functional $\sigma$ is said to be semi-classical if $\sigma$ is quasi-definite and satisfies a Pearson type functional equation

$$
\big(\phi(x)\sigma\big)' - \psi(x)\sigma = 0 \tag{3.1}
$$

for some polynomials $\phi(x)$ and $\psi(x)$ with $\deg(\phi) \geq 0$ and $\deg(\psi) \geq 1$.

For a semi-classical moment functional $\sigma$, we call

$$
s := \min \max\big(\deg(\phi) - 2, \deg(\psi) - 1\big) \tag{3.2}
$$

the class number of $\sigma$, where the minimum is taken over all pairs $(\phi, \psi)$ of polynomials satisfying (3.1). We then call the MOPS $\{P_n(x)\}_{n=0}^{\infty}$ relative to $\sigma$ a semi-classical OPS (SCOPS) of class $s$.

From now on, we assume that $\sigma$ is a semi-classical moment functional satisfying (3.1) and set $s := \max(\deg(\phi) - 2, \deg(\psi) - 1)$. Then $\tau$ satisfies the functional equation

$$
\big(T(x)\phi(x)\tau\big)' = \big(T'(x)\phi(x) + T(x)\psi(x)\big)\tau, \tag{3.3}
$$

where

$$T(x)=\prod_{l=1}^{m}\left(x-c_l\right)^{m_l+2}. \tag{3.4}$$

We now determine the class number of $\tau$. By (3.3), if $\sigma$ is of class $s$, then $\tau$ is of class $\leq s+\sum_{l=1}^{m}(m_l+2)$.

Lemma 3.2 (see [25]). *The semi-classical moment functional $\sigma$ satisfying (3.1) is of class $s$ if and only if for any zero $c$ of $\phi(x)$,*

$$\mathbb{N}(\sigma;c):=\left|r_c\right|+\left|\langle\sigma,q_c(x)\rangle\right|\neq 0, \tag{3.5}$$

*where $\phi(x)=(x-c)\phi_c(x)$ and $\phi_c(x)-\psi(x)=(x-c)q_c(x)+r_c$.*

Theorem 3.3. *Assume that $\sigma$ is of class $s=\max(\deg(\phi)-2,\deg(\psi)-1)$. Then $\tau$ is of class $s+\sum_{l=1}^{m}(m_l+2)$ if $\phi(c_l)\neq 0$, $1\leq l\leq m$.*

*Proof.* Assume $\phi(c_l)\neq 0$, $1\leq l\leq m$. Let $\tilde{\phi}(x)=T(x)\phi(x)$ and $\tilde{\psi}(x)=T'(x)\phi(x)+T(x)\psi(x)$. For any zero $c$ of $\tilde{\phi}(x)$, let $\tilde{\phi}(x)=(x-c)\tilde{\phi}_c(x)$ and $\tilde{\phi}_c(x)-\tilde{\psi}(x)=(x-c)\tilde{q}_c(x)+\tilde{r}_c$. Then either $c=c_t$ $(1\leq t\leq m)$ or $\phi(c)=0$.

If $c=c_t$ $(1\leq t\leq m)$, then

$$\tilde{\phi}_c(x)-\tilde{\psi}(x)=\frac{T(x)\phi(x)}{x-c_t}-T'(x)\phi(x)-T(x)\psi(x)=\left(x-c_t\right)\tilde{q}_c(x). \tag{3.6}$$

Hence, $\tilde{r}_c=0$ but

$$\begin{aligned}
\langle\tau,\tilde{q}_c(x)\rangle &= \left\langle \sigma+\lambda\sum_{l=1}^{m}\sum_{k=0}^{m_l}\frac{(-1)^k u_{lk}}{k!}\delta^{(k)}\left(x-c_l\right),\tilde{q}_c(x)\right\rangle \\
&= \left\langle \sigma+\lambda\sum_{l=1}^{m}\sum_{k=0}^{m_l}\frac{(-1)^k u_{lk}}{k!}\delta^{(k)}\left(x-c_l\right),\right. \\
&\qquad \left.\frac{T(x)\phi(x)}{\left(x-c_t\right)^2}-\frac{T'(x)\phi(x)+T(x)\psi(x)}{x-c_t}\right\rangle \\
&= \left\langle (\phi\sigma)'-\psi\sigma,\frac{T(x)}{x-c_t}\right\rangle \\
&\quad +\left\langle \lambda\sum_{l=1}^{m}\sum_{k=0}^{m_l}\frac{(-1)^k u_{lk}}{k!}\delta^{(k)}\left(x-c_l\right),\right. \\
&\qquad \left.\frac{T(x)\phi(x)}{\left(x-c_t\right)^2}-\frac{T'(x)\phi(x)+T(x)\psi(x)}{x-c_t}\right\rangle \\
&= -\lambda u_{t,m_t}\left(m_t+1\right)\prod_{\substack{l=1\\ l\neq t}}^{m}\left(c_t-c_l\right)\phi\left(c_t\right)\neq 0,
\end{aligned} \tag{3.7}$$

so that $\mathbb{N}(\tau,c)\neq 0$.

If $c \neq c_t$ $(1 \leq t \leq m)$, then $\phi(c) = 0$, $\tilde{\phi}_c(x) = T(x)\phi_c(x)$, and

$$\tilde{\phi}_c(x) - \tilde{\psi}(x) = T(x)\phi_c(x) - T^{'}(x)\phi(x) - T(x)\psi(x). \tag{3.8}$$

Hence, $\tilde{r}_c = T(c)\phi_c(c) - T(c)\psi(c) = T(c)r_c$. If $r_c \neq 0$, then $\tilde{r}_c \neq 0$ so that $\mathbb{N}(\tau; c) \neq 0$.

If $r_c = 0$, then $\langle \sigma, q_c(x)\rangle \neq 0$ and $\tilde{r}_c = 0$ so that

$$\tilde{q}_c(x) = T(x)q_c(x) - T^{'}(x)\phi_c(x). \tag{3.9}$$

Then

$$\big\langle \tau, \tilde{q}_c(x)\big\rangle = \big\langle \sigma, \tilde{q}_c(x)\big\rangle = \big\langle \sigma, T(x)q_c(x) - T^{'}(x)\phi_c(x)\big\rangle. \tag{3.10}$$

Set $Q_1(x)$, $Q_2(x)$, $Q_3(x)$, and $r_1$, $r_2$, $r_3$ to be

$$\begin{aligned} T(x) &= (x-c)Q_1(x) + r_1; \\ T^{'}(x) &= (x-c)Q_2(x) + r_2; \\ Q_1(x) &= (x-c)Q_3(x) + r_3. \end{aligned} \tag{3.11}$$

Then $Q_2(x) = Q_1^{'}(x) + Q_3(x)$ and $r_2 = r_3 = Q_1(c)$.

Hence,

$$\begin{aligned} \big\langle \tau, \tilde{q}_c(x)\big\rangle &= \big\langle \sigma, Q_1(x)\big(\phi_c(x) - \psi(x)\big)\big\rangle + \big\langle \sigma, r_1 q_c(x)\big\rangle \\ &\quad - \big\langle \sigma, Q_2(x)\phi(x)\big\rangle - \big\langle \sigma, r_2\phi_c(x)\big\rangle \\ &= \big\langle \sigma, Q_3(x)\phi(x)\big\rangle + \big\langle \sigma, r_3\phi_c(x)\big\rangle \\ &\quad - \big\langle \sigma, Q_1(x)\psi(x)\big\rangle + \big\langle \sigma, r_1 q_c(x)\big\rangle \\ &\quad - \big\langle \sigma, Q_2(x)\phi(x)\big\rangle - \big\langle \sigma, r_2\phi_c(x)\big\rangle \\ &= \big\langle \phi(x)\sigma, Q_3(x)\big\rangle + \big\langle \phi(x)\sigma, Q_1^{'}(x)\big\rangle \\ &\quad - \big\langle \phi(x)\sigma, Q_2(x)\big\rangle + r_1\big\langle \sigma, q_c(x)\big\rangle \\ &= r_1\big\langle \sigma, q_c(x)\big\rangle = \prod_{l=1}^{m}\big(c - c_l\big)^{m_l+2}\big\langle \sigma, q_c(x)\big\rangle \neq 0. \end{aligned} \tag{3.12}$$

Hence $\mathbb{N}(\tau; c) \neq 0$. □

## 4. Examples

As illustrating examples, we consider the following example.

*Example 4.1.* Let

$$\tau := \sigma + \lambda\big(u_{10}\delta(x-1) + u_{20}\delta(x+1) + u_{11}\delta'(x-1) + u_{21}\delta^{'}(x+1)\big), \tag{4.1}$$

where $\lambda \neq 0$, $|u_{10}|+|u_{20}|+|u_{11}|+|u_{21}| \neq 0$, and $\sigma$ is the Jacobi moment functional defined by

$$\langle \sigma, \pi \rangle = \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta}\pi(x)\,dx \quad (\alpha > -1, \beta > -1), \quad \pi \in \mathbb{P}. \tag{4.2}$$

Then

$$\begin{aligned} P_n(x) &= P_n^{(\alpha,\beta)}(x) \\ &= \binom{2n+\alpha+\beta}{n}^{-1} \sum_{k=0}^{n} \binom{n+\alpha}{n-k}\binom{n+\beta}{k}(x-1)^k(x+1)^{n-k}, \quad n \geq 0 \end{aligned} \tag{4.3}$$

are the Jacobi polynomials satisfying

$$\begin{aligned} &\left(1-x^2\right)y''(x) + \left[\beta-\alpha-(\alpha+\beta+2)x\right]y'(x) + n(n+\alpha+\beta+1)y(x) = 0, \\ &\left\langle \sigma, P_n^{(\alpha,\beta)}(x)^2 \right\rangle := k_n \\ &\qquad = \frac{2^{\alpha+\beta+2n+1} n!\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)(2n+\alpha+\beta+1)(n+\alpha+\beta+1)_n^2}, \quad n \geq 0, \end{aligned} \tag{4.4}$$

where

$$(a)_k = \begin{cases} 1, & \text{if } k = 0 \\ a(a+1)\cdots(a+k-1), & \text{if } k \geq 1. \end{cases} \tag{4.5}$$

In this case, using the differential-difference relation

$$\left(P_n^{(\alpha,\beta)}(x)\right)^{(\nu)} = \frac{n!}{(n-\nu)!} P_{n-\nu}^{(\alpha+\nu,\beta+\nu)}(x), \quad \nu = 0,1,2,\ldots,\ n \geq \nu, \tag{4.6}$$

the structure relation

$$\left(1-x^2\right)P_n^{(\alpha,\beta)}(x)' = \tilde{\alpha}_n P_{n+1}^{(\alpha,\beta)}(x) + \tilde{\beta}_n P_n^{(\alpha,\beta)}(x) + \tilde{\gamma}_n P_{n-1}^{(\alpha,\beta)}(x), \quad n \geq 0, \tag{4.7}$$

where

$$\begin{aligned} \tilde{\alpha}_n &= -n, \\ \tilde{\beta}_n &= \frac{2(\alpha-\beta)n(n+\alpha+\beta+1)}{(2n+\alpha+\beta)(2n+2+\alpha+\beta)}, \\ \tilde{\gamma}_n &= \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)(n+\alpha+\beta+1)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}, \end{aligned} \tag{4.8}$$

and the three term recurrence relation

$$P_{n+1}^{(\alpha,\beta)}(x) = \left(x-\beta_n\right)P_n^{(\alpha,\beta)}(x) - \gamma_n P_{n-1}^{(\alpha,\beta)}(x), \tag{4.9}$$

where

$$\beta_n = \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+2+\alpha+\beta)},$$
$$\gamma_n = \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}, \tag{4.10}$$

we can easily obtain (see [1, equations (30)–(32)]):

$$K_{n-1}^{(0,0)}(1,1) = \frac{\left(P_n^{(\alpha,\beta)}(1)\right)^2 n(n+\beta)}{k_{n-1}(2n+\alpha+\beta+1)\gamma_n(\alpha+1)},$$
$$K_{n-1}^{(0,0)}(1,-1) = -\frac{nP_n^{(\alpha,\beta)}(-1)P_n^{(\alpha,\beta)}(1)}{k_{n-1}(2n+\alpha+\beta+1)\gamma_n},$$
$$K_{n-1}^{(0,1)}(1,1) = \frac{\left(P_n^{(\alpha,\beta)}\right)'(1)P_n^{(\alpha,\beta)}(1)(n+\beta)(n-1)}{k_{n-1}(2n+\alpha+\beta+1)\gamma_n(\alpha+2)},$$
$$K_{n-1}^{(0,1)}(1,-1) = -\frac{\left(P_n^{(\alpha,\beta)}\right)'(-1)P_n^{(\alpha,\beta)}(1)(n-1)}{k_{n-1}(2n+\alpha+\beta+1)\gamma_n},$$
$$K_{n-1}^{(1,1)}(1,1) = P_n^{(\alpha,\beta)}(1)\left(P_n^{(\alpha,\beta)}\right)'(1)(n-1)(n+\beta) \times \frac{\left[(\alpha+2)\left(n^2+n\alpha+n\beta\right)-(\alpha+1)(\alpha+\beta+2)\right]}{2k_{n-1}(2n+\alpha+\beta+1)\gamma_n(\alpha+1)(\alpha+2)(\alpha+3)},$$
$$K_{n-1}^{(0,0)}(1,1) = -\frac{P_n^{(\alpha,\beta)}(1)\left(P_n^{(\alpha,\beta)}\right)'(-1)(n-1)\left[n^2+n\alpha+n\beta-\alpha-\beta-2\right]}{2k_{n-1}(2n+\alpha+\beta+1)\gamma_n(\alpha+1)}, \tag{4.11}$$

where

$$K_n(x,y) = \sum_{k=0}^{n} \frac{P_k^{(\alpha,\beta)}(x)P_k^{(\alpha,\beta)}(y)}{k_n} \tag{4.12}$$

is the $n$th kernel polynomial of $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ and $K_n^{(i,j)}(x,y) = \partial_x^i \partial_y^j K_n(x,y)$. Using the symmetry of the Jacobi kernels, we obtain that the moment functional $\tau$ in (4.1) is quasi-definite if and only if

$$d_n = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} \neq 0, \quad n \geq 0, \tag{4.13}$$

where

$$A_{11} = \begin{pmatrix} 1+\lambda u_{10}K_n^{(0,0)}(1,1)+\lambda u_{11}K_n^{(0,1)}(1,1) & \lambda u_{11}K_n^{(0,0)}(1,1) \\ \lambda u_{10}K_n^{(1,0)}(1,1)+\lambda u_{11}K_n^{(1,1)}(1,1) & 1+\lambda u_{11}K_n^{(1,0)}(1,1) \end{pmatrix}$$

$$A_{12} = \begin{pmatrix} \lambda u_{20}K_n^{(0,0)}(1,-1)+\lambda u_{21}K_n^{(0,1)}(1,-1) & \lambda u_{21}K_n^{(0,0)}(1,-1) \\ \lambda u_{20}K_n^{(1,0)}(1,-1)+\lambda u_{21}K_n^{(1,1)}(1,-1) & \lambda u_{21}K_n^{(1,0)}(1,-1) \end{pmatrix}$$

$$A_{21} = \begin{pmatrix} \lambda u_{10}K_n^{(0,0)}(-1,1)+\lambda u_{11}K_n^{(0,1)}(-1,1) & \lambda u_{11}K_n^{(0,0)}(-1,1) \\ \lambda u_{10}K_n^{(1,0)}(-1,1)+\lambda u_{11}K_n^{(1,1)}(-1,1) & \lambda u_{11}K_n^{(1,0)}(-1,1) \end{pmatrix}$$

$$A_{22} = \begin{pmatrix} 1+\lambda u_{20}K_n^{(0,0)}(-1,-1)+\lambda u_{21}K_n^{(0,1)}(-1,-1) & \lambda u_{21}K_n^{(0,0)}(-1,-1) \\ \lambda u_{20}K_n^{(1,0)}(-1,-1)+\lambda u_{21}K_n^{(1,1)}(-1,-1) & 1+\lambda u_{21}K_n^{(1,0)}(-1,-1) \end{pmatrix}. \tag{4.14}$$

Álvarez-Nodarse, J. Arvesú, and F. Marcellán [1] showed that for any values of $\lambda$ and $u_{10}$, $u_{20}$, $u_{11}$, $u_{21}$, $d_n \neq 0$ for large $n$ so that $R_n(x)$ exists for large $n$. Moreover, they express $R_n(x)$ as

$$\begin{aligned} R_n(x) = {}& \left(1+n\zeta_n+n\eta_n\right)P_n^{(\alpha,\beta)}(x)+\left[\zeta_n(1-x)-\eta_n(1+x)+\theta_n\right]\left(P_n^{(\alpha,\beta)}(x)\right)' \\ & +\left[\chi_n(1+x)-\omega_n(1-x)\right]\left(P_n^{(\alpha,\beta)}(x)\right)'', \end{aligned} \tag{4.15}$$

where $\zeta_n$, $\eta_n$, $\theta_n$, $\chi_n$, and $\omega_n$ are constants depending on $n$, $\lambda$, $u_{10}$, $u_{20}$, and $u_{11}$, (see [1, equations (47)–(50)]). They also express $R_n(x)$ as a generalized hypergeometric series ${}_6F_5$ (see [1, Proposition 2]).

*Example 4.2*. Consider a moment functional $\tau$ given by

$$\tau := \sigma+\lambda\sum_{k=0}^{N}\frac{(-1)^k u_k}{k!}\delta^{(k)}(x), \tag{4.16}$$

where $\lambda \neq 0$, $u_k \in \mathbb{C}$, $N \in \{0,1,2,\dots\}$ and $\sigma$ is the Laguerre moment functional defined by

$$\langle\sigma,p\rangle = \int_0^\infty x^\alpha e^{-x}\pi(x)dx \quad (\alpha>-1),\ \pi\in\mathbb{P}. \tag{4.17}$$

Then

$$\begin{aligned} P_n(x) = L_n^{(\alpha)}(x) &= (-1)^n n!\sum_{k=0}^{n}\binom{n+\alpha}{n-k}\frac{(-x)^k}{k!} \\ &= (-1)^n(\alpha+1)_n\,{}_1F_1(-n;\alpha+1;x), \quad n\geq 0 \end{aligned} \tag{4.18}$$

are the monic Laguerre polynomials satisfying

$$\begin{aligned} &xy''(x)+(1+\alpha-x)y'(x)+ny(x)=0, \\ &\left\langle \sigma, L_n^{(\alpha)}(x)^2 \right\rangle = n!\Gamma(n+\alpha+1), \quad n \geq 0. \end{aligned} \tag{4.19}$$

Hence

$$L_n^{(\alpha)}(0) = \frac{(-1)^n\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)} = (-1)^n \binom{n+\alpha}{n} n!. \tag{4.20}$$

Hence, by Theorem 2.2, the moment functional $\tau$ in (4.16) is quasi-definite if and only if $d_n \neq 0$, where $d_n$ is the determinant of the $(N+1)\times(N+1)$ matrix $D_n$,

$$D_n := \left[ \delta_{ij} + \lambda \sum_{k=0}^{N-j} \frac{u_{j+k}}{j!k!} K_n^{(i,k)}(0,0) \right]_{i,j=0}^{N}, \quad n \geq 0, \tag{4.21}$$

where $K_n(x,y) = \sum_{k=0}^{n} L_k^{(\alpha)}(x) L_k^{(\alpha)}(y) / \langle \sigma, L_k^{(\alpha)}(x)^2 \rangle$.

When $d_n \neq 0$ for $n \geq 0$, we now claim that the MOPS $\{R_n(x)\}_{n=0}^{\infty}$ relative to $\tau$ can be given as

$$R_n(x) = \sum_{k=0}^{N+1} A_k^{(n)} \partial_x^k L_n^{(\alpha)}(x), \quad n \geq 0 \tag{4.22}$$

for suitable constants $A_k^{(n)}$ $(0 \leq k \leq N+1)$ with $A_0^{(n)} = 1$. For any fixed $n \geq 1$, set

$$\tilde{R}_n(x) := \sum_{k=0}^{N+1} A_k \partial_x^k L_n^{(\alpha)}(x), \tag{4.23}$$

where $\{A_k\}_{k=0}^{N+1}$ are constants to be determined. Note here that if $0 \leq n \leq N$, then $\partial_x^k L_n^{(\alpha)}(x) = 0$ for $n+1 \leq k \leq N+1$ so that we may take $A_k$ for $n+1 \leq k \leq N+1$ to be 0. Since $(L_n^{(\alpha)}(x))' = nL_{n-1}^{(\alpha+1)}(x)$, $n \geq 1$, we have

$$\tilde{R}_n(x) = \sum_{k=0}^{N+1} (n-k+1)_k A_k L_{n-k}^{(\alpha+k)}(x), \tag{4.24}$$

where $L_n^{(\alpha)}(x) = 0$ for $n < 0$. We now show that the coefficients $\{A_k\}_{k=0}^{N+1}$ can be chosen so that

$$\langle \tau, x^m \tilde{R}_n(x) \rangle = 0, \quad 0 \le m \le n-1. \tag{4.25}$$

If $N+1 \le m < n$, then by (4.24)

$$\begin{aligned} \langle \tau, x^m \tilde{R}_n(x) \rangle &= \int_0^\infty x^m \tilde{R}_n(x) x^\alpha e^{-x}\, dx + \lambda \sum_{k=0}^{N} \frac{u_k}{k!} (x^m \tilde{R}_n(x))^{(k)}(0) \\ &= \sum_{k=0}^{N+1} \frac{n!}{(n-k)!} A_k \int_0^\infty x^m L_{n-k}^{(\alpha+k)}(x) x^\alpha e^{-x}\, dx \\ &= \sum_{k=0}^{N+1} \frac{n!}{(n-k)!} A_k \int_0^\infty x^{m-k} L_{n-k}^{(\alpha+k)}(x) x^{\alpha+k} e^{-x}\, dx \\ &= 0. \end{aligned} \tag{4.26}$$

We now assume that $0 \le m \le \min(N, n-1)$. Then

$$\begin{aligned} \langle \sigma, x^m L_{n-k}^{(\alpha+k)}(x) \rangle &= \int_0^\infty x^m L_{n-k}^{(\alpha+k)}(x) x^\alpha e^{-x}\, dx \\ &= \int_0^\infty x^{m-k} L_{n-k}^{(\alpha+k)}(x) x^{\alpha+k} e^{-x}\, dx = 0, \end{aligned} \tag{4.27}$$

for $0 \le k \le m$. For $m+1 \le k \le n$,

$$\begin{aligned} \langle \sigma, x^m L_{n-k}^{(\alpha+k)}(x) \rangle &= (-1)^{n-k}(n-k)! \sum_{j=0}^{n-k} \frac{(-1)^j}{j!} \binom{n+\alpha}{n-k-j} \int_0^\infty x^{m+\alpha+j} e^{-x}\, dx \\ &= (-1)^{n-k}(n-k)! \sum_{j=0}^{n-k} \frac{(-1)^j}{j!} \binom{n+\alpha}{n-k-j} \Gamma(m+\alpha+j+1) \\ &= (-1)^{n-k}(n-k)! \binom{n+\alpha}{n-k} \Gamma(m+\alpha+1) \\ &\quad \times {}_2F_1(-n+k, m+\alpha+1; k+\alpha+1; 1) \\ &= (-1)^{n-k}(n-k)! \binom{n-m-1}{n-k} \Gamma(m+\alpha+1) \end{aligned} \tag{4.28}$$

by (4.18) and ${}_2F_1(-n,b;c;1) = (c-b)_n/(c)_n$. Hence by (4.20)

$$\begin{aligned}\left\langle \tau, x^m \tilde{R}_n(x) \right\rangle &= \sum_{k=0}^{N+1} (n-k+1)_k A_k \left\langle \sigma, x^m L_{n-k}^{(\alpha+k)}(x) \right\rangle \\ &\quad + \lambda \sum_{l=0}^{N} \frac{u_l}{l!} \sum_{k=0}^{N+1} (n-k+1)_k A_k \left( x^m L_{n-k}^{(\alpha+k)}(x) \right)^{(l)}(0) \\ &= n!\Gamma(m+\alpha+1) \sum_{k=m+1}^{N+1} (-1)^{n-k} \binom{n-m-1}{n-k} A_k \\ &\quad + \lambda n! \sum_{l=m}^{N} \frac{u_l}{(l-m)!} \sum_{k=0}^{N+1} (-1)^{n-k-l+m} \binom{n+\alpha}{n-k-l+m} A_k, \\ &\qquad 0 \le m \le \min(N, n-1),\end{aligned} \tag{4.29}$$

where $\binom{n}{k} = 0$, for $k < 0$. Hence $\langle \tau, x^m \tilde{R}_n(x) \rangle = 0$, with $0 \le m \le n-1$ if and only if

$$\begin{aligned}&\Gamma(m+\alpha+1) \sum_{k=m+1}^{N+1} (-1)^{n-k} \binom{n-m-1}{n-k} A_k \\ &\quad + \lambda \sum_{l=m}^{N} \frac{u_l}{(l-m)!} \sum_{k=0}^{N+1} (-1)^{n-k-l+m} \binom{n+\alpha}{n-k-l+m} A_k = 0, \\ &\qquad 0 \le m \le \min(N, n-1).\end{aligned} \tag{4.30}$$

Since (4.30) is a homogeneous system of $N+1$ (resp., $n$) equations for $N+2$ (resp., $n+1$) unknowns $\{A_k\}_{k=0}^{N+1}$ (resp., $\{A_k\}_{k=0}^{n}$) when $n \ge N+1$ (resp., $n \le N$ ), there always exists a nontrivial solution $\{A_k\}_{k=0}^{N+1}$. With this choice of $\{A_k\}_{k=0}^{N+1}$, $\tilde{R}_n(x)$ is a nonzero polynomial of degree $\le n$ and $\langle \tau, x^m \tilde{R}_n(x) \rangle = 0$ for $0 \le m \le n-1$ so that $\deg(\tilde{R}_n) = n$, that is, $A_0 \ne 0$. Then $A_0^{-1} \tilde{R}_n(x) = R_n(x)$ so that we have (4.22).

Now we can express $R_n(x)$ as a hypergeometric series (see [13]);

$$\begin{aligned}R_n(x) &= \frac{\beta_0 \beta_1 \cdots \beta_N}{(\alpha+1)_{N+1}} (-1)^n (\alpha+1)_n \left( A_0 + A_1 + \cdots + A_{N+1} \right) \\ &\quad \times {}_{N+2}F_{N+2} \left( \begin{matrix} -n, \beta_0+1, \beta_1+1, \ldots, \beta_N+1 \\ \alpha+N+2, \beta_0, \beta_1, \ldots, \beta_N \end{matrix} \middle| x \right)\end{aligned} \tag{4.31}$$

for suitable constants $\{\beta_j\}_{j=0}^{N}$.

## Acknowledgements

D. H. Kim was partially supported by BK-21 project and K. H. Kwon was partially supported by KOSEF (99-2-101-001-5).

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