UPPER SEMICONTINUITY OF THE ATTRACTOR FOR LATTICE DYNAMICAL SYSTEMS OF PARTLY DISSIPATIVE REACTION-DIFFUSION SYSTEMS

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We investigate the existence of a global attractor and its upper semicontinuity for the infinite-dimensional lattice dynamical system of a partly dissipative reaction-diffusion system in the Hilbert space $l^2 \times l^2$. Such a system is similar to the discretized FitzHugh-Nagumo system in neurobiology, which is an adequate justification for its study.

1. Introduction

Lattice dynamical systems arise in various application fields, for instance, in chemical reaction theory, material science, biology, laser systems, image processing and pattern recognition, and electrical engineering (cf. [6, 7, 13]). In each field, they have their own forms, but in some other cases, they appear as spatial discretizations of partial differential equations (PDEs). Recently, many authors studied various properties of the solutions for several lattice dynamical systems. For instance, the chaotic properties have been investigated in [1, 7, 8, 10, 11, 17], and the travelling solutions have been carefully studied in [2, 3, 7, 8, 9, 22].

From [18], we know that it is difficult to estimate the attractor of the solution semiflow generated by the initial value problem of dissipative PDEs on unbounded domains because, in general, it is infinite dimensional. Therefore, it is significant to study the lattice dynamical systems corresponding to the initial value problem of PDEs on unbounded domains because of the importance of such systems and they can be regarded as an approximation to the corresponding continuous PDEs if they arise as spatial discretizations of PDEs.

The main idea of this work is originated from [4, 16]. In [4, 19, 20], the researchers proved the existence of global attractors for different lattice dynamical systems and they investigated the finite-dimensional approximations of these global attractors. In fact, Bates et al. [4] studied first-order lattice dynamical systems, and Zhou [20] gave a generalization of the result given by [4]. Zhou [19] studied a second-order lattice dynamical system and investigated the upper semicontinuity of the global attractor.

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For a positive integer r, consider the Hilbert space

$$l^2 = \left\{ u = (u_i)_{i\in\mathbb{Z}^r} : i = (i_1, i_2, \ldots, i_r) \in \mathbb{Z}^r,\ u_i \in \mathbb{R},\ \sum_{i\in\mathbb{Z}^r} u_i^2 < \infty \right\} \tag{1.1}$$

whose inner product and norm are given by, for all $u = (u_i)_{i\in\mathbb{Z}^r}$, $v = (v_i)_{i\in\mathbb{Z}^r} \in l^2$,

$$\langle u, v\rangle = \sum_{i\in\mathbb{Z}^r} u_i v_i, \qquad \|u\| = \langle u, u\rangle^{1/2} = \left(\sum_{i\in\mathbb{Z}^r} u_i^2 \right)^{1/2}. \tag{1.2}$$

We will study the following lattice dynamical system of a partly dissipative reaction-diffusion system:

$$\begin{gathered} \dot{u}_i + \nu(Au)_i + f_1(u_i) + g_1(u_i) + \alpha h_1(u_i, v_i) = q_{1,i}, \\ \dot{v}_i + f_2(v_i) + g_2(v_i) - \beta h_2(u_i, v_i) = q_{2,i}, \end{gathered} \tag{1.3}$$

$i = (i_1, i_2, \ldots, i_r) \in \mathbb{Z}^r$, $t > 0$, with the initial conditions

$$u_i(0) = u_{i,0}, \qquad v_i(0) = v_{i,0}, \tag{1.4}$$

where ν is a positive constant, α and β are real constants such that

$$\alpha\beta > 0, \tag{1.5}$$

the operator $A : l^2 \to l^2$ is defined by, for all $u = (u_i)_{i\in\mathbb{Z}^r} \in l^2$, $i = (i_1, i_2, \ldots, i_r)$,

$$\begin{aligned} (Au)_i = 2r u_{(i_1,i_2,\ldots,i_r)} &- u_{(i_1-1,i_2,\ldots,i_r)} - u_{(i_1,i_2-1,\ldots,i_r)} - \cdots - u_{(i_1,i_2,\ldots,i_r-1)} \\ &- u_{(i_1+1,i_2,\ldots,i_r)} - u_{(i_1,i_2+1,\ldots,i_r)} - \cdots - u_{(i_1,i_2,\ldots,i_r+1)}, \end{aligned} \tag{1.6}$$

and for $j = 1, 2$, $s, s_1, s_2 \in \mathbb{R}$,

$$q_j = (q_{j,i})_{i\in\mathbb{Z}^r} \in l^2, \tag{1.7}$$

$$f_j(s), g_j(s) \in C^1(\mathbb{R}, \mathbb{R}), \quad f_j(0) = g_j(0) = 0, \tag{1.8}$$

$$h_j(s_1, s_2) \in C^1(\mathbb{R}^2, \mathbb{R}), \quad h_j(0,0) = 0, \tag{1.9}$$

$$\varepsilon_j \le f_j'(s), \tag{1.10}$$

$$g_j(s)s \ge 0, \tag{1.11}$$

$$h_1(s_1, s_2)s_1 = h_2(s_1, s_2)s_2, \tag{1.12}$$

where ε_j is a positive constant.

For example, one can consider $f_j(s) = \delta_j s$, where δ_j is a positive constant, $g_j(s) = \sum_{i=0}^{n_j} \gamma_{ij} s^{2i+1}$, where n_j is a nonnegative integer and γ_{ij} is a nonnegative constant for each $i = 0,1,\dots,n_j$, $h_1(s_1,s_2) = s_1^{m_1-1} s_2^{m_2}$, and $h_2(s_1,s_2) = s_1^{m_1} s_2^{m_2-1}$, where $m_1,m_2 \geq 2$ are constants. In fact, for $f_1(s) = \lambda s$, $f_2(s) = \delta s$, $g_2(s) = 0$, $h_1(s_1,s_2) = s_2$, and $h_2(s_1,s_2) = s_1$, where λ and δ are positive constants, the system given by (1.3) and (1.4) can be regarded as a spatial discretization of the following partly dissipative reaction-diffusion system with continuous spatial variable $x \in \mathbb{R}^r$ and $t \in \mathbb{R}^+$:

$$u_t - \nu\Delta u + \lambda u + g_1(u) + \alpha v = q_1, \qquad v_t + \delta v - \beta u = q_2. \tag{1.13}$$

The system (1.13) describes the signal transmission across axons and is a model of FitzHugh-Nagumo equations in neurobiology, (cf. [5, 12, 15] and the references therein). The existence of global attractors of the system given by (1.13) has been proved in a bounded domain (cf. [14]) and in $\mathbb{R}^r$ (cf. [16]).

2. Preliminaries

We can write the operator A in the following form:

$$A = A_1 + A_2 + \cdots + A_r \tag{2.1}$$

such that for $j = 1,2,\dots,r$, and $u = (u_i)_{i\in\mathbb{Z}^r} \in l^2$,

$$(A_j u)_i = 2u_{(i_1,i_2,\dots,i_r)} - u_{(i_1,i_2,\dots,i_{j-1},i_j-1,i_{j+1},\dots,i_r)} - u_{(i_1,i_2,\dots,i_{j-1},i_j+1,i_{j+1},\dots,i_r)}. \tag{2.2}$$

For $j = 1,2,\dots,r$, define the operators $B_j, B_j^* : l^2 \to l^2$ as follows: for $u = (u_i)_{i\in\mathbb{Z}^r} \in l^2$,

$$\begin{aligned}(B_j u)_i &= u_{(i_1,i_2,\dots,i_{j-1},i_j+1,i_{j+1},\dots,i_r)} - u_{(i_1,i_2,\dots,i_r)},\\ (B_j^* u)_i &= u_{(i_1,i_2,\dots,i_{j-1},i_j-1,i_{j+1},\dots,i_r)} - u_{(i_1,i_2,\dots,i_r)}.\end{aligned} \tag{2.3}$$

Then, it follows that for $j = 1,2,\dots,r$,

$$A_j = B_j^* B_j = B_j B_j^*, \tag{2.4}$$

there exists a constant $C_0 = C_0(r)$ such that

$$\|Au\|^2 \leq C_0\|u\|^2, \quad \|B_j u\|^2 = \|B_j^* u\|^2 \leq 4\|u\|^2, \quad \forall u \in l^2, \tag{2.5}$$

$$\langle B_j u, v\rangle = \langle u, B_j^* v\rangle, \quad \forall u,v \in l^2. \tag{2.6}$$

It is clear that A, A_j, B_j, and B_j^*, $j = 1,2,\dots,r$, are bounded linear operators from l^2 into l^2.

We can represent the initial value problem, (1.3) and (1.4), in the following form:

$$\begin{gathered}\dot{u} + \nu Au + f_1(u) + g_1(u) + \alpha h_1(u,v) = q_1,\\ \dot{v} + f_2(v) + g_2(v) - \beta h_2(u,v) = q_2,\\ u(0) = (u_{i,0})_{i\in\mathbb{Z}^r}, \qquad v(0) = (v_{i,0})_{i\in\mathbb{Z}^r},\end{gathered} \tag{2.7}$$

where $u = (u_i)_{i\in\mathbb{Z}^r}$, $v = (v_i)_{i\in\mathbb{Z}^r}$, $Au = (Au_i)_{i\in\mathbb{Z}^r}$, and for $j = 1,2$, $f_j(u) = (f_j(u_i))_{i\in\mathbb{Z}^r}$, $g_j(u) = (g_j(u_i))_{i\in\mathbb{Z}^r}$, $h_j(u,v) = (h_j(u_i,v_i))_{i\in\mathbb{Z}^r}$, $q_j = (q_{j,i})_{i\in\mathbb{Z}^r}$.

Consider the Hilbert space $E := l^2 \times l^2$, endowed with the inner product and norm as follows: for $\varphi_j = (u^{(j)}, v^{(j)})^T = ((u_i^{(j)}),(v_i^{(j)}))_{i\in\mathbb{Z}^r}^T \in E$, $j = 1,2$,

$$\begin{aligned} \langle \varphi_1, \varphi_2 \rangle_E &= \langle u^{(1)}, u^{(2)} \rangle + \langle v^{(1)}, v^{(2)} \rangle, \\ \|\varphi\|_E &= \langle \varphi, \varphi \rangle_E^{1/2}, \quad \forall \varphi \in E. \end{aligned} \tag{2.8}$$

Now, the system given by (2.7) is equivalent to the following initial value problem in the Hilbert space $E := l^2 \times l^2$:

$$\dot{\varphi} + C(\varphi) = F(\varphi), \qquad \varphi(0) = (u_0, v_0)^T, \tag{2.9}$$

where $\varphi = (u,v)^T$, $C(\varphi) = (\nu A\mathrm{u}, 0)^T$, $F(\varphi) = (G_1(\varphi), G_2(\varphi))^T$,

$$\begin{aligned} G_1(\varphi) &= q_1 - f_1(u) - g_1(u) - \alpha h_1(u,v), \\ G_2(\varphi) &= q_2 - f_2(v) - g_2(v) + \beta h_2(u,v). \end{aligned} \tag{2.10}$$

For a given function $f(s_1,s_2) \in C^1(\mathbb{R}^2,\mathbb{R})$, let $D_1 f(a,b)$ represent the partial derivative of f with respect to the first independent variable, s_1, at $(s_1,s_2) = (a,b)$, and let $D_2 f(a,b)$ represent the partial derivative of f with respect to the second independent variable, s_2, at $(s_1,s_2) = (a,b)$. From (1.9) and the mean value theorem, it follows that given $u = (u_i)_{i\in\mathbb{Z}^r}$, $v = (v_i)_{i\in\mathbb{Z}^r} \in l^2$, there exist $\xi_{1i}, \xi_{2i} \in (0,1)$, for each $i \in \mathbb{Z}^r$, such that

$$\begin{aligned} ||h_1(u,v)||^2 &= \sum_{i\in\mathbb{Z}^r} \left(h_1(u_i,v_i)\right)^2 = \sum_{i\in\mathbb{Z}^r} \left(D_1 h_1(\xi_{1i}u_i, v_i)u_i + D_2 h_1(0,\xi_{2i}v_i)v_i\right)^2 \\ &\le 2\left(\max_{|a|\le\|u\|}\max_{|b|\le\|v\|} \left(D_1 h_1(a,b)\right)^2\right)\|u\|^2 \\ &\quad + 2\left(\max_{|b|\le\|v\|} \left(D_2 h_1(0,b)\right)^2\right)\|v\|^2. \end{aligned} \tag{2.11}$$

Thus, for $u,v \in l^2$, we have $h_1(u,v) \in l^2$. Similarly one can show that for $u,v \in l^2$, $h_2(u,v) \in l^2$. By using (1.8) and the mean value theorem, it is easy to show that f_1, f_2, g_1, and g_2 map l^2 into l^2. From the above discussion, it is obvious that F maps E into E.

3. The existence of an absorbing set

First we will prove that there exists a unique local solution of the system given by (2.9) in E. Suppose that (1.7), (1.8), and (1.9) are satisfied. Let G be a bounded set in E, and

$\varphi_j = (u^{(j)}, v^{(j)}) = ((u_i^{(j)}), (v_i^{(j)}))_{i\in\mathbb{Z}^r} \in G$, $j = 1,2$, then

$$\begin{aligned}
&||F(\varphi_1) - F(\varphi_2)||_E^2 \\
&\quad = ||G_1(\varphi_1) - G_1(\varphi_2)||^2 + ||G_2(\varphi_1) - G_2(\varphi_2)||^2 \\
&\quad \le 2||f_1(u^{(1)}) - f_1(u^{(2)})||^2 + 4||g_1(u^{(1)}) - g_1(u^{(2)})||^2 \\
&\qquad + 4\alpha^2||h_1(u^{(1)}, v^{(1)}) - h_1(u^{(2)}, v^{(2)})||^2 + 2||f_2(v^{(1)}) - f_2(v^{(2)})||^2 \\
&\qquad + 4||g_2(v^{(1)}) - g_2(v^{(2)})||^2 + 4\beta^2||h_2(u^{(1)}, v^{(1)}) - h_2(u^{(2)}, v^{(2)})||^2.
\end{aligned} \tag{3.1}$$

Using (1.9) and the mean value theorem, it follows that there exist $\xi_{3i}, \xi_{4i} \in (0,1)$, for each $i \in \mathbb{Z}^r$, and $L_1 = L_1(G)$ such that

$$\begin{aligned}
&||h_1(u^{(1)}, v^{(1)}) - h_1(u^{(2)}, v^{(2)})||^2 \\
&\quad = \sum_{i\in\mathbb{Z}^r} \left(h_1\left(u_i^{(1)}, v_i^{(1)}\right) - h_1\left(u_i^{(2)}, v_i^{(2)}\right)\right)^2 \\
&\quad = \sum_{i\in\mathbb{Z}^r} \begin{pmatrix} D_1h_1\left(u_i^{(1)} + \xi_{3i}\left(u_i^{(2)} - u_i^{(1)}\right), v_i^{(1)}\right)\left(u_i^{(1)} - u_i^{(2)}\right) \\ +D_2h_1\left(u_i^{(2)}, v_i^{(1)} + \xi_{4i}\left(v_i^{(2)} - v_i^{(1)}\right)\right)\left(v_i^{(1)} - v_i^{(2)}\right) \end{pmatrix}^2 \\
&\quad \le 2\Big(\max_{|a|\le\|u^{(1)}\|+\|u^{(2)}\|, |b|\le\|v^{(1)}\|} (D_1h_1(a,b))^2\Big)||u^{(1)} - u^{(2)}||^2 \\
&\qquad + 2\Big(\max_{|a|\le\|u^{(2)}\|, |b|\le\|v^{(1)}\|+\|v^{(2)}\|} (D_2h_1(a,b))^2\Big)||v^{(1)} - v^{(2)}||^2 \\
&\quad \le L_1\left(||u^{(1)} - u^{(2)}||^2 + ||v^{(1)} - v^{(2)}||^2\right).
\end{aligned} \tag{3.2}$$

Thus

$$||h_1(u^{(1)}, v^{(1)}) - h_1(u^{(2)}, v^{(2)})||^2 \le L_1||\varphi_1 - \varphi_2||^2. \tag{3.3}$$

We can obtain similar results, as (3.3), for f_1, f_2, g_1, g_2, and h_2 by using (1.8), (1.9), and the mean value theorem. In such a case, using (3.1), there exists $L_2 = L_2(G)$ such that

$$||F(\varphi_1) - F(\varphi_2)||_E^2 \le L_2||\varphi_1 - \varphi_2||^2. \tag{3.4}$$

Thus F is locally Lipschitz from E into E. In such a case from the standard theory of ordinary differential equations, we get the following lemma.

Lemma 3.1. *If (1.7), (1.8), and (1.9) are satisfied, then for any initial data $\varphi(0) = (u_0, v_0)^T \in E$, there exists a unique local solution $\varphi(t) = (u(t), v(t))^T$ of (2.9) such that $\varphi \in C^1([0,T), E)$ for some $T > 0$. If $T < \infty$, then $\lim_{t\to T^-} \|\varphi\|_E^2 = \infty$.*

Assume that (1.5), (1.7), (1.8), (1.9), (1.10), (1.11), and (1.12) are satisfied. Now we are ready to prove that the solution of the system given by (2.9) exists globally and there exists an absorbing set.

Let $\varphi = (u,v)^T \in E$ be a solution of (2.9). If we consider the inner product of (2.9) with φ in E, taking into account (2.4) and (2.6), we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \nu\sum_{j=1}^{r}\|B_j u\|^2 + \langle f_1(u),u\rangle + \langle g_1(u),u\rangle + \alpha\langle h_1(u,v),u\rangle = \langle q_1,u\rangle, \tag{3.5}$$

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 + \langle f_2(v),v\rangle + \langle g_2(v),v\rangle - \beta\langle h_2(u,v),v\rangle = \langle q_2,v\rangle, \tag{3.6}$$

If we multiply (3.5) by $|\beta|$, (3.6) by $|\alpha|$, and we sum up the results, taking into account (1.5) and (1.12), we find that

$$\begin{aligned}\frac{|\beta|}{2}\frac{d}{dt}\|u\|^2 + \frac{|\alpha|}{2}\frac{d}{dt}\|v\|^2 + |\beta|\nu\sum_{j=1}^{r}\|B_j u\|^2 + |\beta|\langle f_1(u),u\rangle + |\alpha|\langle f_2(v),v\rangle \\ + |\beta|\langle g_1(u),u\rangle + |\alpha|\langle g_2(v),v\rangle = |\beta|\langle q_1,u\rangle + |\alpha|\langle q_2,v\rangle.\end{aligned} \tag{3.7}$$

By using (1.8), (1.10), and the mean value theorem, for each $i \in \mathbb{Z}^r$, there exists a constant $\xi_{5i} \in (0,1)$ such that

$$\langle f_1(u),u\rangle = \sum_{i\in\mathbb{Z}^r}\left(f_1(u_i)u_i\right) = \sum_{i\in\mathbb{Z}^r}\left(f_1'(\xi_{5i}u_i)u_i^2\right) \geq \varepsilon_1\|u\|^2. \tag{3.8}$$

Thus we have

$$\langle f_1(u),u\rangle \geq \varepsilon_1\|u\|^2, \qquad \langle f_2(v),v\rangle \geq \varepsilon_2\|v\|^2. \tag{3.9}$$

From (1.11), we obtain

$$\langle g_1(u),u\rangle = \sum_{i\in\mathbb{Z}^r}\left(g_1(u_i)u_i\right) \geq 0, \qquad \langle g_2(v),v\rangle \geq 0. \tag{3.10}$$

Now if we substitute (3.9) and (3.10) into (3.7), we find that

$$\begin{aligned}&\frac{|\beta|}{2}\frac{d}{dt}\|u\|^2 + \frac{|\alpha|}{2}\frac{d}{dt}\|v\|^2 + \varepsilon_1|\beta|\|u\|^2 + \varepsilon_2|\alpha|\|v\|^2 \\ &\quad\leq |\beta|\langle q_1,u\rangle + |\alpha|\langle q_2,v\rangle \leq \frac{|\beta|}{2\varepsilon_1}\|q_1\|^2 + \frac{\varepsilon_1|\beta|}{2}\|u\|^2 + \frac{|\alpha|}{2\varepsilon_2}\|q_2\|^2 + \frac{\varepsilon_2|\alpha|}{2}\|v\|^2.\end{aligned} \tag{3.11}$$

Thus if we choose

$$\sigma = \min\{|\beta|,|\alpha|,\varepsilon_1|\beta|,\varepsilon_2|\alpha|\}, \tag{3.12}$$

then it follows that

$$\frac{d}{dt}\|\varphi\|_E^2 + \|\varphi\|_E^2 \le \left(\frac{\beta}{\sigma}\|q_1\|\right)^2 + \left(\frac{\alpha}{\sigma}\|q_2\|\right)^2. \tag{3.13}$$

From the Gronwall lemma, we obtain

$$\|\varphi\|_E^2 \le e^{-t}\|\varphi_0\|_E^2 + \frac{1-e^{-t}}{\sigma^2}\left((\beta\|q_1\|)^2 + (\alpha\|q_2\|)^2\right), \tag{3.14}$$

$$\lim_{t\longrightarrow\infty}\|\varphi\|_E^2 \le \left(\frac{\beta}{\sigma}\|q_1\|\right)^2 + \left(\frac{\alpha}{\sigma}\|q_2\|\right)^2. \tag{3.15}$$

Inequality (3.15) implies that the solution semigroup $\{S(t)\}_{t\ge 0}$ of (2.9) exists globally and possesses a bounded absorbing set in E. In such a case, the maps

$$S(t) : \varphi(0) \in E \longrightarrow S(t)\varphi(0) = \varphi(t) \in E, \quad t \ge 0, \tag{3.16}$$

generate a continuous semigroup $\{S(t)\}_{t\ge 0}$ on E. Now from Lemma 3.1 and (3.15), we are ready to present the following lemma.

Lemma 3.2. *If (1.5), (1.7), (1.8), (1.9), (1.10), (1.11), and (1.12) are satisfied, then for any initial data in E, the solution $\varphi(t)$ of (2.9) exists globally for all $t \ge 0$. That is, $\varphi \in C^1([0,\infty),E)$. Moreover, there exists a bounded ball $O = O_E(0,r_0)$ in E, centered at 0 with radius r_0, such that for every bounded set G of E, there exists $T(G) \ge 0$ such that*

$$S(t)G \subset O, \quad \forall t \ge T(G), \tag{3.17}$$

where $r_0^2 > ((\beta/\sigma)\|q_1\|)^2 + ((\alpha/\sigma)\|q_2\|)^2$. Therefore, there exists a constant $T_0 \ge 0$ depending on O such that

$$S(t)O \subset O, \quad \forall t \ge T_0. \tag{3.18}$$

4. The existence of the global attractor

To prove the existence of the global attractor for the solution semigroup $\{S(t)\}_{t\ge 0}$ of (2.9), we need to prove the asymptotic compactness of $\{S(t)\}_{t\ge 0}$. Along the lines of [4], the key idea of showing the asymptotic compactness for such a lattice system is to establish uniform estimates on "Tail Ends" of solutions.

Lemma 4.1. *If (1.5), (1.7), (1.8), (1.9), (1.10), (1.11), and (1.12) are satisfied and $\varphi(0) = (u_0,v_0) \in O$, where O is the bounded absorbing ball given by Lemma 3.2, then for any $\eta > 0$, there exist positive constants $T(\eta)$ and $K(\eta)$ such that the solution $\varphi(t) = (u(t),v(t))^T = (\varphi_i(t))_{i\in\mathbb{Z}^r} = ((u_i(t)),(v_i(t)))_{i\in\mathbb{Z}^r}^T \in E$ of (2.9) satisfies*

$$\sum_{\|i\|_m \ge K(\eta)} \|\varphi_i(t)\|_E^2 = \sum_{\|i\|_m \ge K(\eta)} \left(u_i^2(t) + v_i^2(t)\right) \le \eta \tag{4.1}$$

for all $t \ge T(\eta)$, where $\|i\|_m = \max_{1\le j\le r}|i_j|$ for $i = (i_1,i_2,\dots,i_r) \in \mathbb{Z}^r$.

Proof. Consider a smooth increasing function $\theta \in C^1(\mathbb{R}^+, \mathbb{R})$ such that

$$\begin{aligned} \theta(s) &= 0, \quad 0 \le s < 1, \\ 0 \le \theta(s) &\le 1, \quad 1 \le s < 2, \\ \theta(s) &= 1, \quad s \ge 2, \end{aligned} \tag{4.2}$$

and there exists a constant M_0 such that $\theta'(s) \le M_0$ for all $s \in \mathbb{R}^+$.

Let L be an arbitrary positive integer. Set $w_i = \theta(\|i\|_m/L)u_i$, $z_i = \theta(\|i\|_m/L)v_i$, $w = (w_i)_{i\in\mathbb{Z}^r}$, $z = (z_i)_{i\in\mathbb{Z}^r}$, and $\psi = (w,z)^T$. Following [4], we take the inner product of (2.9) with ψ in E, then it follows that

$$\begin{aligned} \sum_{i\in\mathbb{Z}^r} \theta\Big(\frac{\|i\|_m}{L}\Big)\Big(\frac{1}{2}\frac{d}{dt}u_i^2 + f_1(u_i)u_i + g_1(u_i)u_i + \alpha h_1(u_i,v_i)u_i\Big)& \\ + \nu \sum_{i\in\mathbb{Z}^r}\sum_{j=1}^{r}(B_j u)_i (B_j w)_i = \sum_{i\in\mathbb{Z}^r}\theta\Big(\frac{\|i\|_m}{L}\Big)q_{1,i}u_i,& \end{aligned} \tag{4.3}$$

$$\sum_{i\in\mathbb{Z}^r} \theta\Big(\frac{\|i\|_m}{L}\Big)\Big(\frac{1}{2}\frac{d}{dt}v_i^2 + f_2(v_i)v_i + g_2(v_i)v_i - \beta h_2(u_i,v_i)v_i\Big) = \sum_{i\in\mathbb{Z}^r}\theta\Big(\frac{\|i\|_m}{L}\Big)q_{2,i}v_i. \tag{4.4}$$

If we multiply (4.3) by $|\beta|$, (4.4) by $|\alpha|$, and we sum up the results, taking into account (1.5) and (1.12), we get

$$\begin{aligned} \sum_{i\in\mathbb{Z}^r} \theta\Big(\frac{\|i\|_m}{L}\Big)\begin{pmatrix} \dfrac{|\beta|}{2}\dfrac{d}{dt}u_i^2 + \dfrac{|\alpha|}{2}\dfrac{d}{dt}v_i^2 + |\beta| f_1(u_i)u_i \\ +|\alpha| f_2(v_i)v_i + |\beta| g_1(u_i)u_i + |\alpha| g_2(v_i)v_i \end{pmatrix}& \\ + |\beta|\nu \sum_{i\in\mathbb{Z}^r}\sum_{j=1}^{r}(B_j u)_i (B_j w)_i = \sum_{i\in\mathbb{Z}^r}\theta\Big(\frac{\|i\|_m}{L}\Big)\big(|\beta| q_{1,i}u_i + |\alpha| q_{2,i}v_i\big).& \end{aligned} \tag{4.5}$$

Using (1.8), (1.10), and the mean value theorem, it follows that for each $i \in \mathbb{Z}^r$,

$$f_1(u_i)u_i \ge \varepsilon_1 u_i^2, \qquad f_2(v_i)v_i \ge \varepsilon_2 v_i^2. \tag{4.6}$$

From (1.11), we obtain

$$g_1(u_i)u_i \ge 0, \qquad g_2(v_i)v_i \ge 0. \tag{4.7}$$

Recalling (69) of [21], taking into account that $\|B_j\| \le 2$, $j = 1,2,\dots,r$, one can see that

$$|\beta|\nu \sum_{i\in\mathbb{Z}^r}\sum_{j=1}^{r}(B_j u)_i (B_j w)_i \ge -\frac{16|\beta|\nu n M_0}{L}r_0^2, \quad \forall t \ge T_0. \tag{4.8}$$

Now if we substitute (4.6), (4.7), and (4.8), into (4.5), we find that for $t \geq T_0$,

$$\begin{aligned}
&\sum_{i\in\mathbb{Z}^r}\theta\left(\frac{\|i\|_m}{L}\right)\left(\frac{|\beta|}{2}\frac{d}{dt}u_i^2+\frac{|\alpha|}{2}\frac{d}{dt}v_i^2+\varepsilon_1|\beta|u_i^2+\varepsilon_2|\alpha|v_i^2\right)\\
&\quad\leq \sum_{i\in\mathbb{Z}^r}\theta\left(\frac{\|i\|_m}{L}\right)\left(|\beta|q_{1,i}u_i+|\alpha|q_{2,i}v_i\right)+\frac{16|\beta|\nu nM_0}{L}r_0^2\\
&\quad\leq \sum_{i\in\mathbb{Z}^r}\theta\left(\frac{\|i\|_m}{L}\right)\left(\frac{|\beta|}{\varepsilon_1}q_{1,i}^2+\frac{\varepsilon_1|\beta|}{4}u_i^2+\frac{|\alpha|}{\varepsilon_2}q_{2,i}^2+\frac{\varepsilon_2|\alpha|}{4}v_i^2\right)+\frac{16|\beta|\nu nM_0}{L}r_0^2.
\end{aligned} \tag{4.9}$$

Thus if we choose

$$\sigma=\min\{|\beta|,|\alpha|,\varepsilon_1|\beta|,\varepsilon_2|\alpha|\}, \tag{4.10}$$

then it follows that for $t \geq T_0$,

$$\begin{aligned}
&\sum_{i\in\mathbb{Z}^r}\theta\left(\frac{\|i\|_m}{L}\right)\left(\frac{d}{dt}\|\varphi_i(t)\|_E^2+\|\varphi_i(t)\|_E^2\right)\\
&\quad\leq 2\sum_{i\in\mathbb{Z}^r}\theta\left(\frac{\|i\|_m}{L}\right)\left(\left(\frac{\beta}{\sigma}q_{1,i}\right)^2+\left(\frac{\alpha}{\sigma}q_{2,i}\right)^2\right)+\frac{32|\beta|\nu nM_0}{L}r_0^2\\
&\quad\leq 2\sum_{\|i\|_m\geq L}\left(\left(\frac{\beta}{\sigma}q_{1,i}\right)^2+\left(\frac{\alpha}{\sigma}q_{2,i}\right)^2\right)+\frac{32|\beta|\nu nM_0}{L}r_0^2.
\end{aligned} \tag{4.11}$$

Since $q_1, q_2 \in l^2$, then for a given $\eta > 0$, we can fix L such that

$$2\sum_{\|i\|_m\geq L}\left(\left(\frac{\beta}{\sigma}q_{1,i}\right)^2+\left(\frac{\alpha}{\sigma}q_{2,i}\right)^2\right)+\frac{32|\beta|\nu nM_0}{L}r_0^2\leq\frac{\eta}{2}, \tag{4.12}$$

and in such a case we get that

$$\sum_{i\in\mathbb{Z}^r}\theta\left(\frac{\|i\|_m}{L}\right)\left(\frac{d}{dt}\|\varphi_i(t)\|_E^2+\|\varphi_i(t)\|_E^2\right)\leq\frac{\eta}{2},\quad\forall t\geq T_0. \tag{4.13}$$

From the Gronwall lemma, we obtain

$$\sum_{i\in\mathbb{Z}^r}\left(\theta\left(\frac{\|i\|_m}{L}\right)\|\varphi_i(t)\|_E^2\right)\leq e^{-t}\sum_{i\in\mathbb{Z}^r}\left(\theta\left(\frac{\|i\|_m}{L}\right)\|\varphi_i(0)\|_E^2\right)+\frac{\eta}{2},\quad\forall t\geq T_0. \tag{4.14}$$

Since $\varphi(0)=(u_0,v_0)^T\in O$, we have

$$\|\varphi(0)\|_E\leq r_0. \tag{4.15}$$

Therefore,

$$\sum_{i\in\mathbb{Z}^r}\left(\theta\left(\frac{\|i\|_m}{L}\right)\|\varphi_i(t)\|_E^2\right)\leq r_0^2e^{-t}+\frac{\eta}{2},\quad\forall t\geq T_0. \tag{4.16}$$

But again for $\eta > 0$, there exists a constant $T_1 = T_1(\eta)$ such that

$$r_0^2 e^{-t} \le \frac{\eta}{2}, \quad \forall t \ge T_1. \tag{4.17}$$

Using (4.16) and (4.17) with $K(\eta) = 2L$, $T(\eta) = \max\{T_0, T_1\}$, we obtain

$$\begin{aligned} \sum_{\|i\|_m \ge K(\eta)} \|\varphi_i\|_E^2 &= \sum_{\|i\|_m \ge K(\eta)} \left(\theta\left(\frac{\|i\|_m}{L}\right)\|\varphi_i\|_E^2\right) \\ &\le \sum_{i\in\mathbb{Z}^r} \left(\theta\left(\frac{\|i\|_m}{L}\right)\|\varphi_i\|_E^2\right) \le \eta, \quad \forall t \ge T(\eta). \end{aligned} \tag{4.18}$$

The proof is completed. □

Lemma 4.2. *If (1.5), (1.7), (1.8), (1.9), (1.10), (1.11), and (1.12) are satisfied, then the solution semigroup $\{S(t)\}_{t\ge 0}$ of (2.9) is asymptotically compact in E, that is, if $\{\varphi_n\}$ is bounded in E and $t_n \to \infty$, then $\{S(t_n)\varphi_n\}$ is precompact in E.*

Proof. By using Lemmas 3.2 and 4.1, above, the proof of this lemma will be similar to that of [19, Lemma 3.2]. □

Theorem 4.3. *If (1.5), (1.7), (1.8), (1.9), (1.10), (1.11), and (1.12) are satisfied, then the solution semigroup $\{S(t)\}_{t\ge 0}$ of (2.9) possesses a global attractor $\mathcal{A}$ in E.*

Proof. From the existence theorem of global attractors, (cf. [18, Lemmas 2 and 4]), we get the result. □

5. Upper semicontinuity of the global attractor

Here we will study the upper semicontinuity of the global attractor $\mathcal{A}$ of the solution semigroup $\{S(t)\}_{t\ge 0}$ of (2.9), in the sense that $\mathcal{A}$ is approximated by the global attractors of finite-dimensional versions of (2.9), as was done in [4] for a simpler system.

Let n be a positive integer, and

$$\mathbb{Z}_n^r = \{i \in \mathbb{Z}^r : \|i\|_m \le n\}. \tag{5.1}$$

For $i = (i_1, i_2, \dots, i_r) \in \mathbb{Z}_n^r$, consider $w = (w_i)_{\|i\|_m \le n} \in \mathbb{R}^{(2n+1)^r}$. For convenience, we reorder the subscripts of components of w as follows:

$$\begin{aligned} w = \big(& w_{(-n,-n,\dots,-n,-n)}, w_{(-n,-n,\dots,-n,-n+1)}, \dots, w_{(-n,-n,\dots,-n,n)}, \\ & w_{(-n,-n,\dots,-n+1,-n)}, w_{(-n,-n,\dots,-n+1,-n+1)}, \dots, \\ & w_{(-n,-n,\dots,-n+1,n)}, \dots, w_{(n,n,\dots,n,-n)}, w_{(n,n,\dots,n,-n+1)}, \dots, w_{(n,n,\dots,n,n)}\big)^T. \end{aligned} \tag{5.2}$$

Let

$$X = w = (w_i)_{\|i\|_m \le n} : w \in \mathbb{R}^{(2n+1)^r}, \tag{5.3}$$

where subscripts of components of w are ordered as in (5.2).

Consider the $(2n+1)^r$-dimensional ordinary differential equations with the initial data in X:

$$\begin{gathered}\dot{w}_i + \nu(Aw)_i + f_1(w_i) + g_1(w_i) + \alpha h_1(w_i, z_i) = q_{1,i},\\ \dot{z}_i + f_2(z_i) + g_2(z_i) - \beta h_2(w_i, z_i) = q_{2,i},\\ w_i(0) = w_{i,0}, \qquad z_i(0) = z_{i,0}\\ i \in \mathbb{Z}_n^r,\ t > 0.\end{gathered} \tag{5.4}$$

The system (5.4) can be written as

$$\begin{gathered}\dot{w} + \nu\widetilde{A}w + \widetilde{f}_1(w) + \widetilde{g}_1(w) + \alpha\widetilde{h}_1(w,z) = \widetilde{q}_1,\\ \dot{z} + \widetilde{f}_2(z) + \widetilde{g}_2(z) - \beta\widetilde{h}_2(w,z) = \widetilde{q}_2,\\ w(0) = (w_{i,0})_{\|i\|_m \le n}, \qquad z(0) = (z_{i,0})_{\|i\|_m \le n},\end{gathered} \tag{5.5}$$

where $w = (w_i)_{\|i\|_m \le n}$, $z = (z_i)_{\|i\|_m \le n} \in X$, for $k = 1, 2, \dots, r$,

$$\begin{aligned}w_{(i_1,\dots,-n,i_k+1,\dots,i_r)} &= w_{(i_1,\dots,n+1,i_k+1,\dots,i_r)},\\ w_{(i_1,\dots,n,i_k+1,\dots,i_r)} &= w_{(i_1,\dots,-n-1,i_k+1,\dots,i_r)},\\ (\widetilde{A}w)_{(i_1,i_2,\dots,i_r)} &= 2rw_{(i_1,i_2,\dots,i_r)} - w_{(i_1-1,i_2,\dots,i_r)} - w_{(i_1,i_2-1,\dots,i_r)} - \cdots - w_{(i_1,i_2,\dots,i_r-1)}\\ &\quad - w_{(i_1+1,i_2,\dots,i_r)} - w_{(i_1,i_2+1,\dots,i_r)} - \cdots - w_{(i_1,i_2,\dots,i_r+1)},\end{aligned} \tag{5.6}$$

and for $j = 1, 2$,

$$\begin{gathered}\widetilde{f}_j(w) = (f_j(w_i))_{\|i\|_m \le n}, \qquad \widetilde{g}_j(w) = (g_j(w_i))_{\|i\|_m \le n},\\ \widetilde{h_j}(w,z) = (h_j(w_i, z_i))_{\|i\|_m \le n}, \qquad \widetilde{q}_j = (q_{j,i})_{\|i\|_m \le n}.\end{gathered} \tag{5.7}$$

For $w = (w_i)_{\|i\|_m \le n} \in X$, define the linear operators $\widetilde{B}_j, \widetilde{B_j^*}, \widetilde{A}_j : X \to X$, $j = 1, 2, \dots, r$, by

$$\begin{aligned}(\widetilde{B}_j w)_i &= w_{(i_1,\dots,i_j+1,\dots,i_r)} - w_{(i_1,\dots,i_j,\dots,i_r)},\\ (\widetilde{B_j^*} w)_i &= w_{(i_1,\dots,i_j-1,\dots,i_r)} - w_{(i_1,\dots,i_j,\dots,i_r)},\\ (\widetilde{A}_j w)_i &= 2w_{(i_1,\dots,i_j,\dots,i_r)} - w_{(i_1,\dots,i_j-1,\dots,i_r)} - w_{(i_1,\dots,i_j+1,\dots,i_r)},\end{aligned} \tag{5.8}$$

then

$$\widetilde{A} = \widetilde{A}_1 + \widetilde{A}_2 + \cdots + \widetilde{A}_r, \quad \widetilde{A}_j = \widetilde{B}_j \widetilde{B_j^*} = \widetilde{B_j^*}\widetilde{B}_j, \quad j = 1, 2, \dots, r. \tag{5.9}$$

For $w^{(j)} = (w_i^{(j)})_{\|i\|_m \le n} \in X$, $j = 1, 2$, $i = (i_1, i_2, \dots, i_r) \in \mathbb{Z}_n^r$, define

$$\langle w^{(1)}, w^{(2)} \rangle_X = \sum_{\|i\|_m \le n} w_i^{(1)} w_i^{(2)}, \qquad \|w^{(1)}\|_X = \langle w^{(1)}, w^{(1)} \rangle_X^{1/2}. \tag{5.10}$$

In such a case, it is clear that $X = (X, \|\cdot\|_X)$ is a Hilbert space, and $\widetilde{E} = X \times X$ is a Hilbert space with the following inner product and norm: for $Y_j = (w^{(j)}, z^{(j)})^T = ((w_i^{(j)}), (z_i^{(j)}))^T_{\|i\|_m \le n} \in \widetilde{E}$, $j = 1, 2$,

$$\langle Y_1, Y_2\rangle_{\widetilde{E}} = \langle w^{(1)}, w^{(2)}\rangle_X + \langle z^{(1)}, z^{(2)}\rangle_X, \qquad \|Y_1\|_{\widetilde{E}} = \left(\|w^{(1)}\|_X^2 + \|z^{(1)}\|_X^2\right)^{1/2}. \quad (5.11)$$

It is easy to check that problem (5.5) can be formulated to the following first-order system in the Hilbert space $\widetilde{E}$:

$$\dot{Y} + \widetilde{C}(Y) = \widetilde{F}(Y), \qquad Y(0) = (w(0), z(0))^T \in \widetilde{E}, \quad (5.12)$$

where

$$\begin{gathered} Y = (w, z)^T, \qquad \widetilde{C}(Y) = (\nu \widetilde{A} w, 0)^T, \qquad \widetilde{F}(Y) = (\widetilde{G}_1(Y), \widetilde{G}_2(Y))^T, \\ \widetilde{G}_1(Y) = \widetilde{q}_1 - \lambda \widetilde{f}_1(w) - \widetilde{g}_1(w) - \alpha \widetilde{h}_1(w, z), \\ \widetilde{G}_2(Y) = \widetilde{q}_2 - \delta \widetilde{f}_2(z) - \widetilde{g}_2(z) + \beta \widetilde{h}_2(w, z). \end{gathered} \quad (5.13)$$

Similar to Section 2, one can see that, if (1.5), (1.7), (1.8), (1.9), (1.10), (1.11), and (1.12) are satisfied, then (5.12) is a well-posed problem in $\widetilde{E}$. Thus for any $Y(0) \in \widetilde{E}$, there exists a unique solution $Y \in C([0, +\infty), \widetilde{E}) \cap C^1((0, +\infty), \widetilde{E})$, see Lemmas 3.1 and 3.2, also there exist maps of solutions $S_n(t) : Y(0) \in \widetilde{E} \to Y(t) = S_n(t)Y(0) \in \widetilde{E}$, $t \ge 0$, generating a continuous semigroup $\{S_n(t)\}_{t \ge 0}$ on $\widetilde{E}$.

Similar to Lemma 3.2 and Theorem 4.3, we can prove the following Lemma.

Lemma 5.1. *If (1.5), (1.7), (1.8), (1.9), (1.10), (1.11), and (1.12) are satisfied, then there exists a bounded ball $\widetilde{O} = \widetilde{O}_{\widetilde{E}}(0, r_0)$ in $\widetilde{E}$, centered at 0 with radius r_0 such that for every bounded set $\widetilde{G}$ of $\widetilde{E}$, there exists $T(\widetilde{G}) \ge 0$ such that*

$$S_n(t)\widetilde{G} \subset \widetilde{O}, \quad \forall t \ge T(\widetilde{G}),\ n = 1, 2, \ldots, \quad (5.14)$$

where r_0 is the same constant given by Lemma 3.2, and it is independent of n. Moreover, the semigroup $\{S_n(t)\}_{t\ge 0}$ possesses a global attractor $\mathcal{A}_n$, $\mathcal{A}_n \subset \widetilde{O} \subset \widetilde{E}$.

Here we prove the upper semicontinuity of the global attractor $\mathcal{A}$ of the solution semigroup $\{S(t)\}_{t\ge 0}$ of (2.9). In such a case, we should extend the element $u = (u_i)_{\|i\|_m \le n} \in X$ to an element of l^2 such that $u_i = 0$ for $\|i\|_m > n$, still denoted by u.

Lemma 5.2. *If (1.5), (1.7), (1.8), (1.9), (1.10), (1.11), and (1.12) are satisfied, and $\varphi_n(0) \in \mathcal{A}_n$, then there exists a subsequence $\{\varphi_{n_k}(0)\}$ of $\{\varphi_n(0)\}$ and $\varphi_0 \in \mathcal{A}$ such that $\varphi_{n_k}(0)$ converges to φ_0 in E.*

Proof. Consider $\varphi_n(t) = S_n(t)\varphi_n(0) = (u_n(t), v_n(t))^T \in \widetilde{E}$ to be a solution of (5.12). Since $\varphi_n(0) \in \mathcal{A}_n$, $\varphi_n(t) \in \mathcal{A}_n \subset \widetilde{O}$ for all $t \in \mathbb{R}^+$, and again the element $\varphi_n = (\varphi_{n,i})_{\|i\|_m \le n} \in \widetilde{E}$ can be extended to the element $\varphi_n = (\varphi_{n,i})_{i \in \mathbb{Z}^r} \in E$, where $\varphi_{n,i} = (0,0)^T$ for $\|i\|_m > n$, it follows that

$$\|\varphi_n(t)\|_{\widetilde{E}} = \|\varphi_n(t)\|_E = \left(\|u_n\|^2 + \|v_n\|^2\right)^{1/2} \\ \leq r_0, \quad \forall t \in \mathbb{R}^+,\ n = 1,2,\dots. \tag{5.15}$$

From (2.5) and (5.15), we get that

$$\|\widetilde{C}(\varphi_n(t))\|_{\widetilde{E}}^2 \leq \|C(\varphi_n(t))\|_E^2 = \|\nu A u_n\|^2 \leq \nu C_0 \|u_n\|^2 \leq \nu C_0 r_0^2, \tag{5.16}$$

for all $t \in \mathbb{R}^+$, $n = 1,2,\dots$.

Similarly, for $j = 1,2$, the element $\widetilde{q}_j = (q_{j,i})_{\|i\|_m \leq n} \in \mathbb{R}^{(2n+1)^r}$ can be extended to the element $q_{n,j} = (q_{n,j,i})_{i\in\mathbb{Z}^r} \in l^2$, where $q_{n,j,i} = q_{j,i}$ for $\|i\|_m \leq n$ and $q_{n,j,i} = 0$ for $\|i\|_m > n$. From (1.8) and (1.9), we know that for $j = 1,2$,

$$f_j(0) = g_j(0) = h_j(0,0) = 0. \tag{5.17}$$

Therefore,

$$\begin{aligned}\|\widetilde{F}(\varphi_n(t))\|_{\widetilde{E}}^2 = \|F(\varphi_n(t))\|_E^2 &= \|q_{n,1} - f_1(u_n) - g_1(u_n) - \alpha h_1(u_n,v_n)\|^2 \\ &\quad + \|q_{n,2} - f_2(v_n) - g_2(v_n) + \beta h_2(u_n,v_n)\|^2 \\ &\leq 2\|q_1\|^2 + 4\|f_1(u_n)\|^2 + 8\|g_1(u_n)\|^2 + 8\alpha^2\|h_1(u_n,v_n)\|^2 \\ &\quad + 2\|q_2\|^2 + 4\|f_2(v_n)\|^2 + 8\|g_2(v_n)\|^2 + 8\beta^2\|h_2(u_n,v_n)\|^2.\end{aligned} \tag{5.18}$$

Using (1.9), (5.15), and the mean value theorem, there exist $\xi_{6i}, \xi_{7i} \in (0,1)$ for each $i \in \mathbb{Z}^r$ such that

$$\begin{aligned}\|h_1(u_n,v_n)\|^2 &= \sum_{i\in\mathbb{Z}^r} \left(h_1(u_{n,i},v_{n,i})\right)^2 \\ &= \sum_{i\in\mathbb{Z}^r} \left(D_1h_1(\xi_{6i}u_{n,i},v_{n,i})u_{n,i} + D_2h_1(0,\xi_{7i}v_{n,i})v_{n,i}\right)^2 \\ &\leq 2\left(\max_{a,b\in[-r_0,r_0]} \left(D_1h_1(a,b)\right)^2\right) r_0^2 + 2\left(\max_{b\in[-r_0,r_0]} \left(D_2h_1(0,b)\right)^2\right) r_0^2\end{aligned} \tag{5.19}$$

for all $t \in \mathbb{R}^+$, $n = 1,2,\dots$. Again using (1.8), (1.9), (5.15), and the mean value theorem, we can get similar results, as (5.19), for f_1, f_2, g_1, g_2, and h_2. Thus from (5.18), it is obvious that there exists a constant $C_1 = C_1(r_0,q_1,q_2,f_1,f_2,g_1,g_2,h_1,h_2)$ such that

$$\|\widetilde{F}(\varphi_n(t))\|_{\widetilde{E}}^2 = \|F(\varphi_n(t))\|_E^2 \leq C_1, \quad \forall t \in \mathbb{R}^+,\ n = 1,2,\dots. \tag{5.20}$$

Now, by using (5.12), we obtain

$$\|\dot{\varphi}_n(t)\|_E^2 \leq 2\|C(\varphi_n(t))\|_E^2 + 2\|F(\varphi_n(t))\|_E^2. \tag{5.21}$$

Hence, from (5.16) and (5.20), it follows that there exists a constant $C_2 = C_2(r_0,C_0,C_1)$ such that

$$\|\dot{\varphi}_n(t)\|_E \leq C_2, \quad \forall t \in \mathbb{R}^+,\ n = 1,2,\dots. \tag{5.22}$$

Let J_k $(k = 1,2,\ldots)$ be a sequence of compact intervals of $\mathbb{R}^+$ such that $J_k \subset J_{k+1}$ and $\cup_k J_k = \mathbb{R}^+$. Then there exists a subsequence of $\{\varphi_n(t)\}$, still denoted by $\{\varphi_n(t)\}$, and $\varphi(t) \in C(\mathbb{R}^+, E)$ such that

$$\varphi_n(t) \longrightarrow \varphi(t) \text{ in } C(J,E) \text{ as } n \longrightarrow \infty \text{ for any compact set } J \subset \mathbb{R}^+, \tag{5.23}$$

$$\dot{\varphi}_n(t) \longrightarrow \dot{\varphi}(t) \text{ weak star in } L^\infty(\mathbb{R}^+,E) \text{ as } n \longrightarrow +\infty. \tag{5.24}$$

From (5.15) and (5.23), we obtain that there exists a constant $C_3 = C_3(r_0)$ such that for $\varphi(t) = (u(t),v(t))^T = ((u_i(t)),(v_i(t)))^T_{i\in\mathbb{Z}^r} \in E$,

$$\|\varphi(t)\|_E = \left(\|u\|^2 + \|v\|^2\right)^{1/2} \leq C_3, \quad \forall t \in \mathbb{R}^+. \tag{5.25}$$

Let $i \in \mathbb{Z}^r$ and $n \geq \|i\|_m$. Since $\varphi_n(t) = (u_n(t),v_n(t))^T = ((u_{n,i}(t)),(v_{n,i}(t)))^T_{\|i\|_m\leq n} \in \widetilde{E}$ is the solution of (5.12), it follows that for all $t \in \mathbb{R}^+$ and $i \in \mathbb{Z}^r_{n-1}$,

$$\begin{gathered} \dot{u}_{n,i} + \nu(Au_n)_i + f_1(u_{n,i}) + g_1(u_{n,i}) + \alpha h_1(u_{n,i},v_{n,i}) = q_{1,i}, \\ \dot{v}_{n,i} + f_2(v_{n,i}) + g_2(v_{n,i}) - \beta h_2(u_{n,i},v_{n,i}) = q_{2,i}. \end{gathered} \tag{5.26}$$

Therefore, for all $\psi \in C_0^\infty(J)$, we obtain

$$\begin{gathered} \int_J \left(\dot{u}_{n,i} + \nu(Au_n)_i + f_1(u_{n,i}) + g_1(u_{n,i}) + \alpha h_1(u_{n,i},v_{n,i})\right)\psi(t)dt = \int_J q_{1,i}\psi(t)dt, \\ \int_J \left(\dot{v}_{n,i} + f_2(v_{n,i}) + g_2(v_{n,i}) - \beta h_2(u_{n,i},v_{n,i})\right)\psi(t)dt = \int_J q_{2,i}\psi(t)dt. \end{gathered} \tag{5.27}$$

From (1.8), and by using the mean value theorem, there exists $\xi_{8i} \in (0,1)$ for each i such that

$$\begin{aligned} &\left|\int_J f_1(u_{n,i})\psi(t)dt - \int_J f_1(u_i)\psi(t)dt\right| \\ &\quad\leq \sup_{t\in J} \left| f_1(u_{n,i}) - f_1(u_i)\right| \int_J |\psi(t)|dt \\ &\quad\leq \sup_{t\in J} \left(\left| f_1'\left(u_{n,i} + \xi_{8i}(u_i - u_{n,i})\right)\right| |u_{n,i} - u_i|\right) \int_J |\psi(t)|dt. \end{aligned} \tag{5.28}$$

By using (5.15) and (5.25), it is clear that there exists a constant $C_4 = C_4(r_0)$ such that

$$\left|u_{n,i} + \xi_i(u_i - u_{n,i})\right| \leq |u_{n,i}| + |u_i| \leq \|u_n\| + \|u\| \leq C_4 \tag{5.29}$$

for all $t \in \mathbb{R}^+$ and $n = 1,2,\ldots$. In such a case, we obtain

$$\sup_{t\in J}\left| f_1'\left(u_{n,i} + \xi_{8i}(u_i - u_{n,i})\right)\right| \leq \sup_{a\in[-C_5,C_5]} \left|f_1'(a)\right| < +\infty. \tag{5.30}$$

From (5.23), we know that as $n \to \infty$, we have

$$\sup_{t\in J} |u_{n,i} - u_i| \longrightarrow 0. \tag{5.31}$$

Therefore, from (5.28), (5.30), and (5.31), it is clear that

$$\left| \int_J f_1(u_{n,i})\psi(t)dt - \int_J f_1(u_i)\psi(t)dt \right| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{5.32}$$

Similarly, one can get the same result, as (5.32), for f_2, g_1, g_2, h_1, and h_2. In such a case from (5.24), (5.26), and (5.27), it follows that

$$\begin{gathered} \dot{u}_i + \nu(Au)_i + \lambda f_1(u_i) + g_1(u_i) + \alpha h_1(u_i, v_i) = q_{1,i}, \\ \dot{v}_i + \delta f_2(v_i) + g_2(v_i) - \beta h_2(u_i, v_i) = q_{2,i}. \end{gathered} \tag{5.33}$$

But J is arbitrary, thus (5.33) holds for all $t \in \mathbb{R}^+$, which means that $\varphi(t)$ is a solution of (2.9). From (5.25), it follows that $\varphi(t)$ is bounded for $t \in \mathbb{R}^+$, that is, $\varphi(t) \in \mathcal{A}$, therefore $\varphi_n(0) \to \varphi(0) \in \mathcal{A}$, and the proof is completed. □

Now we are ready to represent the main result of this section. In fact, the following theorem shows that the global attractor $\mathcal{A}$ of the lattice dynamical system (2.9) is upper semicontinuous with respect to the (cutoff) approximate finite-dimensional dynamical system (5.12).

Theorem 5.3. *If (1.5), (1.7), (1.8), (1.9), (1.10), (1.11), and (1.12) are satisfied, then*

$$\lim_{n \to \infty} d_E(\mathcal{A}_n, \mathcal{A}) = 0, \tag{5.34}$$

where $d_E(\mathcal{A}_n, \mathcal{A}) = \sup_{a \in \mathcal{A}_n} \inf_{b \in \mathcal{A}} \|a - b\|_E$.

Proof. We argue by contradiction. If the conclusion is not true, then there exists a sequence $\varphi_{n_k} \in \mathcal{A}_{n_k}$ and a constant $K > 0$ such that

$$d_E(\varphi_{n_k}, \mathcal{A}) \geq K > 0. \tag{5.35}$$

However, by Lemma 5.2, we know that there exists a subsequence $\varphi_{n_{k_m}}$ of φ_{n_k} such that

$$d_E(\varphi_{n_{k_m}}, \mathcal{A}) \longrightarrow 0, \tag{5.36}$$

which contradicts (5.35). The proof is completed. □

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