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Research Article

Invariant Regions and Global Existence of Solutions for Reaction-Diffusion Systems with a Tridiagonal Matrix of Diffusion Coefficients and Nonhomogeneous Boundary Conditions

Abdelmalek Salem

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The purpose of this paper is the construction of invariant regions in which we establish the global existence of solutions for reaction-diffusion systems (three equations) with a tridiagonal matrix of diffusion coefficients and with nonhomogeneous boundary conditions after the work of Kouachi (2004) on the system of reaction diffusion with a full 2-square matrix. Our techniques are based on invariant regions and Lyapunov functional methods. The nonlinear reaction term has been supposed to be of polynomial growth.

1. Introduction

We consider the reaction-diffusion system

$$\frac{\partial u}{\partial t} - a\Delta u - b\Delta v = f(u,v,w) \quad \text{in } \mathbb{R}^+ \times \Omega, \tag{1.1}$$

$$\frac{\partial v}{\partial t} - c\Delta u - a\Delta v - b\Delta w = g(u,v,w) \quad \text{in } \mathbb{R}^+ \times \Omega, \tag{1.2}$$

$$\frac{\partial w}{\partial t} - c\Delta v - a\Delta w = h(u,v,w) \quad \text{in } \mathbb{R}^+ \times \Omega, \tag{1.3}$$

with the boundary conditions

$$\begin{aligned} \lambda u + (1-\lambda)\partial_\eta u &= \beta_1 \quad \text{in } \mathbb{R}^+ \times \partial\Omega, \\ \lambda v + (1-\lambda)\partial_\eta v &= \beta_2 \quad \text{in } \mathbb{R}^+ \times \partial\Omega, \\ \lambda w + (1-\lambda)\partial_\eta w &= \beta_3 \quad \text{in } \mathbb{R}^+ \times \partial\Omega, \end{aligned} \tag{1.4}$$

and the initial data

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad w(0,x) = w_0(x) \quad \text{in } \Omega. \tag{1.5}$$

(i) For nonhomogeneous Robin boundary conditions, we use

$$0 < \lambda < 1, \quad \beta_i \in \mathbb{R} \; i = 1, 2, 3. \tag{1.6}$$

(ii) For homogeneous Neumann boundary conditions, we use

$$\lambda = \beta_i = 0, \quad i = 1, 2, 3. \tag{1.7}$$

(iii) For homogeneous Dirichlet boundary conditions, we use

$$1 - \lambda = \beta_i = 0, \quad i = 1, 2, 3. \tag{1.8}$$

Here, Ω is an open bounded domain of class $\mathbb{C}^1$ in $\mathbb{R}^N$, with boundary $\partial\Omega$ and $\partial/\partial\eta$ denotes the outward normal derivative on $\partial\Omega$. The a, b, and c are positive constants satisfying the condition $\sqrt{2}a \geq (b+c)$ which reflects the parabolicity of the system and implies at the same time that the matrix of diffusion

$$A = \begin{pmatrix} a & b & 0 \\ c & a & b \\ 0 & c & a \end{pmatrix} \tag{1.9}$$

is positive definite; that is, the eigenvalues λ_1, λ_2, and λ_3 $(\lambda_1 < \lambda_2 < \lambda_3)$ of its transposed are positive.

The initial data are assumed to be in the following region:

$$\Sigma = \left\{ \begin{array}{lll} (u_0, v_0, w_0) \in \mathbb{R}^3 \text{ such that } \begin{cases} \sqrt{2\mu}|v_0| \leq u_0 + \mu w_0, \\ u_0 \leq \mu w_0 \end{cases} & \text{if } \begin{cases} \sqrt{2\mu}|\beta_2| \leq \beta_1 + \mu\beta_3, \\ \beta_1 \leq \mu\beta_3 \end{cases} \\ (u_0, v_0, w_0) \in \mathbb{R}^3 \text{ such that } \begin{cases} |u_0 + \mu w_0| \leq \sqrt{2\mu} v_0, \\ u_0 \leq \mu w_0 \end{cases} & \text{if } \begin{cases} |\beta_1 + \mu\beta_3| \leq \sqrt{2\mu}\beta_2, \\ \beta_1 \leq \mu\beta_3 \end{cases} \\ (u_0, v_0, w_0) \in \mathbb{R}^3 \text{ such that } \begin{cases} \sqrt{2\mu}|v_0| \leq u_0 + \mu w_0, \\ \mu w_0 \leq u_0 \end{cases} & \text{if } \begin{cases} \sqrt{2\mu}|\beta_2| \leq \beta_1 + \mu\beta_3, \\ \mu\beta_3 \leq \beta_1 \end{cases} \\ (u_0, v_0, w_0) \in \mathbb{R}^3 \text{ such that } \begin{cases} |u_0 + \mu w_0| \leq \sqrt{2\mu} v_0, \\ \mu w_0 \leq u_0 \end{cases} & \text{if } \begin{cases} |\beta_1 + \mu\beta_3| \leq \sqrt{2\mu}\beta_2, \\ \mu\beta_3 \leq \beta_1 \end{cases} \end{array} \right\}, \tag{1.10}$$

where

$$\mu = \frac{b}{c}. \tag{1.11}$$

One will treat the first case, the others will be discussed in the last section.

We suppose that the reaction terms f, g, and h are continuously differentiable, polynomially bounded on Σ satisfying

$$f\left(\sqrt{2\mu}v-\mu w,v,w\right)-\sqrt{2\mu}g\left(\sqrt{2\mu}v-\mu w,v,w\right)+\mu h\left(\sqrt{2\mu}v-\mu w,v,w\right)\geq 0, \tag{1.12}$$

$$-f(\mu w,v,w)+\mu h(\mu w,v,w)\geq 0, \tag{1.13}$$

$$f\left(-\sqrt{2\mu}v-\mu w,v,w\right)+\sqrt{2\mu}g\left(-\sqrt{2\mu}v-\mu w,v,w\right)+\mu h\left(-\sqrt{2\mu}v-\mu w,v,w\right)\geq 0 \tag{1.14}$$

for all $v,w\geq 0$. And for positive constants $C_1'\leq 1/\mu$, $C_1''\leq\sqrt{2/\mu}$, $\alpha_1\geq 1/\mu$ and $\beta_1\geq\sqrt{2/\mu}$,

$$C_1'f(u,v,w)+C_1''g(u,v,w)+h(u,v,w)\leq C_1(\alpha_1 u+\beta_1 v+w+1); \tag{1.15}$$

for all u, v, w in Σ, where C_1 is a positive constant.

In the trivial case where $b=c=0$, nonnegative solutions exist globally in time.

This class of systems motivated us to construct this type of functionals considered in this paper in the aim to prove global existence of solutions.

2. Existence

In this section, we prove that if (f,g,h) points into Σ on $\partial\Sigma$, then Σ is an invariant region for problem (1.1)–(1.5), that is, the solution remains in Σ for any intial data in Σ. Once the invariant regions are constructed, both problems of the local and global existensce become easier to be established.

2.1. Local existence. The usual norms in spaces $L^p(\Omega)$, $L^\infty(\Omega)$, and $C(\overline{\Omega})$ are denoted, respectively, by

$$\begin{aligned}\|u\|_p^p&=\frac{1}{|\Omega|}\int_\Omega|u(x)|^p\,dx,\\ \|u\|_\infty&=\max_{x\in\Omega}|u(x)|.\end{aligned} \tag{2.1}$$

For any initial data in $C(\overline{\Omega})$ or $L^p(\Omega)$, $p\in(1,+\infty)$ local existence and uniqueness of solutions to the initial value problem (1.1)–(1.5) follow from the basic existence theory for abstract semilinear differential equations (see Friedman [1], Henry [2], and Pazy [3]). The solutions are classical on $[0;T_{\max}[$; where $T_{\max}$ denotes the eventual blowing-up time in $L^\infty(\Omega)$.

2.2. Invariant regions. The main result of this subsection is the following.

Proposition 2.1. *Suppose that the functions f, g, and h point into the region Σ on $\partial\Sigma$, then for any $(u_0;v_0,w_0)$ in Σ, the solution $(u(t;\cdot);v(t;\cdot),w(t;\cdot))$ of the problem (1.1)–(1.5) remains in Σ for any time $[0;T^*[$.*

Proof. The proof follows the same way to that of Kouachi (see [4]). Then we multiply (1.1) by c, (1.2) by $\sqrt{2bc}$, and (1.3) by b. Adding the first result to the third one and subtracting the second result, we get (2.2). Subtracting the first result from the third one, we get (2.3). Adding the first, the second, and the third results to each other, we get (2.4):

$$\frac{\partial U}{\partial t} - \lambda_1 \Delta U = F(U,V,W) \quad \text{in }]0,T^*[\times\Omega, \tag{2.2}$$

$$\frac{\partial V}{\partial t} - \lambda_2 \Delta V = G(U,V,W) \quad \text{in }]0,T^*[\times\Omega, \tag{2.3}$$

$$\frac{\partial W}{\partial t} - \lambda_3 \Delta W = H(U,V,W) \quad \text{in }]0,T^*[\times\Omega \tag{2.4}$$

with the boundary conditions

$$\begin{aligned} \lambda U + (1-\lambda)\partial_\eta U &= \rho_1 \quad \text{in }]0,T^*[\times\partial\Omega, \\ \lambda V + (1-\lambda)\partial_\eta V &= \rho_2 \quad \text{in }]0,T^*[\times\partial\Omega, \\ \lambda W + (1-\lambda)\partial_\eta W &= \rho_3 \quad \text{in }]0,T^*[\times\partial\Omega, \end{aligned} \tag{2.5}$$

and the initial data

$$U(0,x) = U_0(x), \quad V(0,x) = V_0(x), \quad W(0,x) = W_0(x) \quad \text{in } \Omega, \tag{2.6}$$

where

$$\begin{aligned} U(t,x) &= cu(t,x) - \sqrt{2bc}v(t,x) + bw(t,x), \\ V(t,x) &= -cu(t,x) + bw(t,x), \\ W(t,x) &= cu(t,x) + \sqrt{2bc}v(t,x) + bw(t,x) \end{aligned} \tag{2.7}$$

for all (t,x) in $]0,T^*[\times\Omega$,

$$\begin{aligned} F(U,V,W) &= (cf - \sqrt{2bc}g + bh)(u,v,w) \\ G(U,V,W) &= (-cf + bh)(u,v,w) \\ H(U,V,W) &= (cf + \sqrt{2bc}g + bh)(u,v,w) \end{aligned} \tag{2.8}$$

for all (u,v,w) in Σ,

$$\begin{aligned} \lambda_1 &= a - \sqrt{2bc}, \\ \lambda_2 &= a, \\ \lambda_3 &= a + \sqrt{2bc}, \\ \rho_1 &= c\beta_1 - \sqrt{2bc}\beta_2 + b\beta_3, \\ \rho_2 &= -c\beta_1 + b\beta_3, \\ \rho_3 &= c\beta_1 + \sqrt{2bc}\beta_2 + b\beta_3. \end{aligned} \tag{2.9}$$

First, let us notice that the condition of the parabolicity of the system (1.1)–(1.3) implies the one of the (2.2)–(2.4) system; since $\sqrt{2}a \geq (b+c) \Rightarrow a > \sqrt{2bc}$.

Now, it suffices to prove that the region

$$\{(U_0, V_0, W_0) \text{ such that } U_0 \geq 0, V_0 \geq 0, W_0 \geq 0\} = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \tag{2.10}$$

is invariant for system (2.2)–(2.4).

Since, from (1.12)–(1.14), we have that $F(0,V,W) \geq 0$ suffices to be $f(\sqrt{2\mu}v - \mu w, v, w) - \sqrt{2\mu}g(\sqrt{2\mu}v - \mu w, v, w) + \mu h(\sqrt{2\mu}v - \mu w, v, w) \geq 0$, for all $V, W \geq 0$ and all $v, w \geq 0$, then $U(t,x) \geq 0$, for all (t,x) in $]0, T^*[\times\Omega$, thanks to the invariant region method (see Smoller [5]), and Because $G(U,0,W) \geq 0$ suffices to be $-f(\mu w, v, w) + \mu h(\mu w, v, w) \geq 0$ for all $U, W \geq 0$ and all $v, w \geq 0$, then $V(t,x) \geq 0$ for all (t,x) in $]0, T^*[\times\Omega$, and $H(U,V,0) \geq 0$ suffices to be $f(-\sqrt{2\mu}v - \mu w, v, w) + \sqrt{2\mu}g(-\sqrt{2\mu}v - \mu w, v, w) + \mu h(-\sqrt{2\mu}v - \mu w, v, w) \geq 0$ for all $U, V \geq 0$ and all $v, w \geq 0$, then $W(t,x) \geq 0$ for all (t,x) in $]0, T^*[\times\Omega$, then Σ is an invariant region for the system (1.1)–(1.3). □

Then system (1.1)–(1.3), with boundary conditions (1.4) and initial data in Σ, is equivalent to system (2.2)–(2.4) with boundary conditions (2.5) and positive initial data (2.6). As it has been mentioned at the beginning of this section and since ρ_1, ρ_2, and ρ_3 are positive, for any initial data in $C(\overline{\Omega})$ and $L^p(\Omega)$, $p \in (1, +\infty)$. (The local existence and uniquness of the solutions to the initial value problem (2.2)–(2.6) gives us directly those of (1.1)–(1.5).)

Once invariant regions are constructed, one can apply Lyapunov technique and establish global existence of unique solutions for (1.1)–(1.5).

2.3. Global existence. As the determinant of the linear algebraic system (2.7) with regard to variables u, v, and w, is different from zero, then to prove global existence of solutions of problem (1.1)–(1.5), one needs to prove it for problem (2.2)–(2.6). To this subject; it is well-known that (see Henry [2]) it sufficies to derive anuniform estimate of $\|F(U,V,W)\|_p$, $\|G(U,V,W)\|_p$, and $\|H(U,V,W)\|_p$ on $[0, T^*[$ for some $p > N/2$.

Let us put $A_{12} = (a - \sqrt{bc/2})/\sqrt{a(a - \sqrt{2bc})}$, $A_{13} = a/2\sqrt{a^2 - 2bc}$, $A_{23} = (a + \sqrt{bc/2})/\sqrt{a(a + \sqrt{2bc})}$, θ, and σ are as two positive constants sufficiently large such that

$$\theta > A_{12}, \tag{2.11}$$

$$\left(\theta^2 - A_{12}^2\right)\left(\sigma^2 - A_{23}^2\right) > \left(A_{13} - A_{12}A_{23}\right)^2. \tag{2.12}$$

Let us define, for any positive integer n, the two finite sequences

$$\begin{aligned} \theta_q &= \theta^{(p-q)^2}, \quad q = 0, \ldots, p, \\ \sigma_p &= \sigma^{(n-p)^2}, \quad p = 0, \ldots, n. \end{aligned} \tag{2.13}$$

The main result of the paper is as follows.

THEOREM 2.2. *Let* $(U(t,\cdot),V(t,\cdot),W(t,\cdot))$ *be any positive solution of (2.2)–(2.6) and assume the functional*

$$L(t)=\int_{\Omega}H_n(U(t,x),V(t,x),W(t,x))\,dx, \tag{2.14}$$

where

$$H_n(U,V,W)=\sum_{p=0}^{n}\sum_{q=0}^{p}C_n^pC_p^q\theta_q\sigma_pU^qV^{p-q}W^{n-p}. \tag{2.15}$$

Then, the functional L is uniformly bounded on the interval $[0,T^*]$, $T^*<T_{\max}$.

COROLLARY 2.3. *Suppose that the functions* f, g, *and* h *are continuously differentiable on* Σ, *point into* $\partial\Sigma$ *and satisfy condition (1.15).Then all solutions of (1.1)–(1.5) with initial data in* Σ *and uniformly bounded on* Ω *are in* $L^\infty(0,T^*;L^p(\Omega))$ *for all* $p\geq 1$.

PROPOSITION 2.4. *Under the hypothesis of Corollary 2.3, if the reactions* f, g, *and* h *are polynomially bounded, then all solutions of (1.1)–(1.5) with the initial data in* Σ *and uniformly bounded on* Ω *are global time.*

Proofs. For the proof of Theorem 2.2, we need some preparatory lemmas.

LEMMA 2.5. *Let* H_n *be the homogeneous polynomial defined by (2.15). Then*

$$\begin{aligned}
\partial_UH_n&=n\sum_{p=0}^{n-1}\sum_{q=0}^{p}C_{n-1}^pC_p^q\theta_{(q+1)}\sigma_{(p+1)}U^qV^{p-q}W^{(n-1)-p},\\
\partial_VH_n&=n\sum_{p=0}^{n-1}\sum_{q=0}^{p}C_{n-1}^pC_p^q\theta_q\sigma_{(p+1)}U^qV^{p-q}W^{(n-1)-p},\\
\partial_WH_n&=n\sum_{p=0}^{n-1}\sum_{q=0}^{p}C_{n-1}^pC_p^q\theta_q\sigma_pU^qV^{p-q}W^{(n-1)-p}.
\end{aligned} \tag{2.16}$$

Proof of Lemma 2.5. See Kouachi [6]. □

LEMMA 2.6. *The second partial derivatives of* H_n *are given by*

$$\begin{aligned}
\partial_{U^2}H_n&=n(n-1)\sum_{p=0}^{n-2}\sum_{q=0}^{p}C_{n-2}^pC_p^q\theta_{(q+2)}\sigma_{(p+2)}U^qV^{p-q}W^{(n-2)-p},\\
\partial_{UV}H_n&=n(n-1)\sum_{p=0}^{n-2}\sum_{q=0}^{p}C_{n-2}^pC_p^q\theta_{(q+1)}\sigma_{(p+2)}U^qV^{p-q}W^{(n-2)-p},\\
\partial_{UW}H_n&=n(n-1)\sum_{p=0}^{n-2}\sum_{q=0}^{p}C_{n-2}^pC_p^q\theta_{(q+1)}\sigma_{(p+1)}U^qV^{p-q}W^{(n-2)-p},\\
\partial_{V^2}H_n&=n(n-1)\sum_{p=0}^{n-2}\sum_{q=0}^{p}C_{n-2}^pC_p^q\theta_q\sigma_{(p+2)}U^qV^{p-q}W^{(n-2)-p},
\end{aligned}$$

$$\partial_{VW}H_n = n(n-1)\sum_{p=0}^{n-2}\sum_{q=0}^{p} C_{n-2}^{p}C_{p}^{q}\theta_q\sigma_{(p+1)}U^{q}V^{p-q}W^{(n-2)-p},$$
$$\partial_{W^2}H_n = n(n-1)\sum_{p=0}^{n-2}\sum_{q=0}^{p} C_{n-2}^{p}C_{p}^{q}\theta_q\sigma_{q}U^{q}V^{p-q}W^{(n-2)-p}. \tag{2.17}$$

Proof of Lemma 2.6. See Kouachi [6]. □

Proof of Theorem 2.2. Differentiating L with respect to t yields

$$\begin{aligned} L'(t) &= \int_\Omega \left(\partial_U H_n \frac{\partial U}{\partial t} + \partial_V H_n \frac{\partial V}{\partial t} + \partial_W H_n \frac{\partial W}{\partial t}\right) dx \\ &= \int_\Omega (\lambda_1 \partial_U H_n \Delta U + \lambda_2 \partial_V H_n \Delta V + \lambda_3 \partial_W H_n \Delta W)\, dx \\ &\quad + \int_\Omega (F\partial_U H_n + G\partial_V H_n + H\partial_W H_n)\, dx = I + J. \end{aligned} \tag{2.18}$$

Using Green's formula and applying Lemma 2.5, we get $I = I_1 + I_2$, where

$$\begin{aligned} I_1 &= \int_{\partial\Omega} (\lambda_1 \partial_U H_n \partial_\eta u + \lambda_2 \partial_V H_n \partial_\eta v + \lambda_3 \partial_W H_n \partial_\eta w)\, dx, \\ I_2 &= -n(n-1)\int_\Omega \sum_{p=0}^{n-2}\sum_{q=0}^{p} C_{n-2}^{p}C_{p}^{q}(B_{qp}T)\cdot T\, dx, \end{aligned} \tag{2.19}$$

where

$$B_{qp} = \begin{bmatrix} \lambda_1\sigma_{(p+2)}\theta_{(q+2)} & \dfrac{\lambda_1+\lambda_2}{2}\sigma_{(p+2)}\theta_{(q+1)} & \dfrac{\lambda_1+\lambda_3}{2}\sigma_{(p+1)}\theta_{(q+1)} \\ \dfrac{\lambda_1+\lambda_2}{2}\sigma_{(p+2)}\theta_{(q+1)} & \lambda_2\sigma_{(p+2)}\theta_{q} & \dfrac{\lambda_2+\lambda_3}{2}\sigma_{(p+1)}\theta_{q} \\ \dfrac{\lambda_1+\lambda_3}{2}\sigma_{(p+1)}\theta_{(q+1)} & \dfrac{\lambda_2+\lambda_3}{2}\sigma_{(p+1)}\theta_{q} & \lambda_3\sigma_{p}\theta_{q} \end{bmatrix} \tag{2.20}$$

for $q = \overline{0,p}$, $p = \overline{0,n-2}$ and $T = (\nabla U, \nabla V, \nabla W)^t$.

We prove that there exists a positive constant C_2 independent of $t \in [0, T_{\max}[$ such that

$$I_1 \le C_2 \quad \forall\, t \in [0, T_{\max}[\tag{2.21}$$

and that

$$I_2 \le 0 \tag{2.22}$$

for several boundary conditions.

(i) If $0<\lambda<1$, using the boundary conditions (1.4), we get

$$I_1=\int_{\partial\Omega}\left(\lambda_1\partial_u H_n(\gamma_1-\alpha U)+\lambda_2\partial_v H_n(\gamma_2-\alpha V)+\lambda_3\partial_w H_n(\gamma_3-\alpha Z)\right)dx, \tag{2.23}$$

where $\alpha=\lambda/(1-\lambda)$ and $\gamma_i=\rho_i/(1-\lambda)$, $i=1,2,3$.

Since $K(U,V,W)=\lambda_1\partial_u H_n(\gamma_1-\alpha U)+\lambda_2\partial_v H_n(\gamma_2-\alpha V)+\lambda_3\partial_w H_n(\gamma_3-\alpha W)$, where P_{n-1} and Q_n are polynomials with positive coefficients and respective degrees $(n-1)$ and n. Since the solution is positive, then

$$\limsup_{(|U|+|V|+|W|)\to+\infty} K(U,V,W)=-\infty \tag{2.24}$$

which proves that K is uniformly bounded on $\mathbb{R}_+^3$ and consequently (2.21).

(ii) If $\lambda=0$, then $I_1=0$ on $[0,T_{\max}[$.

(iii) The case of homogeneous Dirichlet conditions is trivial, since in this case the positivity of the solution on $[0,T_{\max}[\times\Omega$ implies that $\partial_\eta U\le 0,\partial_\eta V\le 0$ and $\partial_\eta W\le 0$ on $[0,T_{\max}[\times\partial\Omega$. Consequently, one gets again (2.21) with $C_3=0$.

Now, we prove (2.22). The quadratic forms (with respect to ∇U, ∇V, and ∇W) associated with the matrixes B_{qp}, $q=0,\dots,p$, and $p=0,\dots,n-2$ are positive since their main determinants Δ_1, Δ_2, and Δ_3 are positive, too. To see this, we have

(1) $\Delta_1=\lambda_1\sigma_{(p+2)}\theta_{(q+2)}>0$ for $q=0,\dots,p$ and $p=0,\dots,n-2$;

(2)

$$\begin{aligned}\Delta_2&=\begin{vmatrix}\lambda_1\sigma_{(p+2)}\theta_{(q+2)} & \dfrac{\lambda_1+\lambda_2}{2}\sigma_{(p+2)}\theta_{(q+1)}\\ \dfrac{\lambda_1+\lambda_2}{2}\sigma_{(p+2)}\theta_{(q+1)} & \lambda_2\sigma_{(p+2)}\theta_q\end{vmatrix}\\ &=\lambda_1\lambda_2\sigma_{(p+2)}^2\theta_{(q+1)}^2\left(\theta^2-A_{12}^2\right),\end{aligned} \tag{2.25}$$

for $q=0,\dots,p$, and $p=0,\dots,n-2$. Using (2.11), we get $\Delta_2>0$.

(3) $\Delta_3=|B_{qp}|=\lambda_1\lambda_2\lambda_3\sigma_{(p+2)}^2\sigma_{(p+1)}^2\theta_{(q+1)}^2\theta_q[(\theta^2-A_{12}^2)(\sigma^2-A_{23}^2)-(A_{13}-A_{12}A_{23})^2]$ for $q=0,\dots,p$ and $p=0,\dots,n-2$. Using (2.12), we get $\Delta_3>0$ (see Kouachi [6]).

Substituting the expressions of the partial derivatives given by Lemma 2.5 in the second integralyields

$$\begin{aligned}J&=\int_\Omega n\sum_{p=0}^{n-1}\sum_{q=0}^{p}C_{n-1}^pC_p^q\left[\sigma_{(p+1)}\theta_{(q+1)}F+\sigma_{(p+1)}\theta_qG+\sigma_p\theta_qH\right]U^qV^{p-q}W^{(n-1)-p}dx\\ &=\int_\Omega n\sum_{p=0}^{n-1}\sum_{q=0}^{p}C_{n-1}^pC_p^qU^qV^{p-q}W^{(n-1)-p}\left[F+\frac{\theta_q}{\theta_{(q+1)}}G+\frac{\sigma_p}{\sigma_{(p+1)}}\frac{\theta_q}{\theta_{(q+1)}}H\right]\sigma_{(p+1)}\theta_{(q+1)}dx.\end{aligned} \tag{2.26}$$

Using the expressions (2.8), we obtain

$$J = \int_\Omega n\sum_{p=0}^{n-1}\sum_{q=0}^{p} C_{n-1}^{p} C_p^q U^q V^{p-q} W^{(n-1)-p} \times\Bigg(\bigg(1-\frac{\theta_q}{\theta_{(q+1)}}+\frac{\sigma_p}{\sigma_{(p+1)}}\frac{\theta_q}{\theta_{(q+1)}}\bigg)cf+\bigg(-1+\frac{\sigma_p}{\sigma_{(p+1)}}\frac{\theta_q}{\theta_{(q+1)}}\bigg)\sqrt{2bc}g + \bigg(1+\frac{\theta_q}{\theta_{(q+1)}}+\frac{\sigma_p}{\sigma_{(p+1)}}\frac{\theta_q}{\theta_{(q+1)}}\bigg)bh\Bigg)\sigma_{(p+1)}\theta_{(q+1)}dx. \tag{2.27}$$

And so, we get the following inequality:

$$J \le \int_\Omega n\sum_{p=0}^{n-1}\sum_{q=0}^{p} C_{n-1}^{p} C_p^q U^q V^{p-q} W^{(n-1)-p} \times\Bigg(\frac{1-\theta_q/\theta_{(q+1)}+(\sigma_p/\sigma_{(p+1)})(\theta_q/\theta_{(q+1)})}{1+\theta_q/\theta_{(q+1)}+(\sigma_p/\sigma_{(p+1)})(\theta_q/\theta_{(q+1)})}cf + \frac{-1+(\sigma_p/\sigma_{(p+1)})(\theta_q/\theta_{(q+1)})}{1+\theta_q/\theta_{(q+1)}+(\sigma_p/\sigma_{(p+1)})(\theta_q/\theta_{(q+1)})}\sqrt{2bc}g+bh\Bigg)\sigma_{(p+1)}\theta_{(q+1)}dx, \tag{2.28}$$

using condition (1.15) and relation (2.7) successively, we get

$$J \le C_4\int_\Omega \sum_{p=0}^{n-1}\sum_{q=0}^{p} C_{n-1}^{p} C_p^q U^q V^{p-q} W^{(n-1)-p}(U+V+W+1)dx. \tag{2.29}$$

Following the same reasoning as in Kouachi [6], a straightforward calculation shows that

$$L'(t) \le C_5 L(t) + C_6 L^{(n-1)/n}(t) \quad \text{on } [0,T^*], \tag{2.30}$$

which for $Z = L^{1/n}$ can be written as

$$nZ' \le C_5 Z + C_6. \tag{2.31}$$

A simple integration gives the uniform bound of the functional L on the interval $[0,T^*]$; this ends the proof of the Theorem 2.2. □

Proof of corollary. The proof of this corollary is an immediate consequence of Theorem 2.2 and the inequality

$$\int_\Omega (U+V+W)^p dx \le C_8 L(t) \quad \text{on } [0,T^*[\tag{2.32}$$

for some $p \ge 1$, taking into consideration expressions (2.7). □

Proof of proposition. As it has been mentioned above; it suffices to derive a uniform estimate of $\|F(U,V,W)\|_p$, $\|G(U,V,W)\|_p$, and $\|H(U,V,W)\|_p$ on $[0,T^*[$ for some $p > N/2$. Since the functions f, g, and h are polynomially bounded on Σ, then using relations (2.5), (2.7), and (2.8), we get that F, G, and H are polynomially bounded, too, and the proof becomes an immediate consequence of Corollary 2.3. □

3. Final remarks

The second, the third, and the fourth cases are to be studied in the same way as we have done with the first case.

The second case:.If $\{|\beta_1 + \mu\beta_3| \leq \sqrt{2\mu}\beta_2 \text{ and } \beta_1 \leq \mu\beta_3\}$, then system (1.1)–(1.3) can be rewritten as follows:

$$\begin{aligned}\frac{\partial u}{\partial t} - a\Delta u - b\Delta v &= f(u,v,w) \quad \text{in } \mathbb{R}^+ \times \Omega,\\ \frac{\partial w}{\partial t} - a\Delta w - c\Delta v &= h(u,v,w) \quad \text{in } \mathbb{R}^+ \times \Omega,\\ \frac{\partial v}{\partial t} - c\Delta u - b\Delta w - a\Delta v &= g(u,v,w) \quad \text{in } \mathbb{R}^+ \times \Omega\end{aligned} \tag{3.1}$$

with the same boundary conditions (1.4) and initial data (1.5).

In this case, the diffusion matrix of the system becomes

$$A = \begin{pmatrix} a & 0 & b \\ 0 & a & c \\ c & b & a \end{pmatrix}. \tag{3.2}$$

Then all the previous results remain valid in the region

$$\Sigma = \left\{(u_0,v_0,w_0) \in \mathbb{R}^3 \text{ such that: } |u_0 + \mu w_0| \leq \sqrt{2\mu}v_0 \text{ and } u_0 \leq \mu w_0\right\}, \tag{3.3}$$

where

$$\mu = \frac{b}{c}. \tag{3.4}$$

And system (2.2)–(2.4) becomes

$$\begin{aligned}\frac{\partial U}{\partial t} - a\Delta U &= F_1(U,V;W),\\ \frac{\partial V}{\partial t} - (a + \sqrt{2}\sqrt{bc})\Delta V &= G_1(U,V;W),\\ \frac{\partial W}{\partial t} - (a - \sqrt{2}\sqrt{bc})\Delta W &= H_1(U,V;W),\end{aligned} \tag{3.5}$$

where

$$\begin{aligned} U(t,x) &= -cu(t,x) + bw(t,x), \\ V(t,x) &= cu(t,x) + \sqrt{2bc}v(t,x) + bw(t,x), \\ W(t,x) &= -cu(t,x) + \sqrt{2bc}v(t,x) - bw(t,x) \end{aligned} \tag{3.6}$$

for all (t,x) in $]0,T^*[\times\Omega$,

$$\begin{aligned} F_1(U,V;W) &= (-cf + bh)(u,v,w), \\ G_1(U,V;W) &= \left(cf + \sqrt{2}\sqrt{bc}g + bh\right)(u,v,w), \\ H_1(U,V;W) &= \left(-cf + \sqrt{2}\sqrt{bc}g - bh\right)(u,v,w) \end{aligned} \tag{3.7}$$

for all (u,v,w) in Σ; with the boundary conditions

$$\begin{aligned} \lambda U + (1-\lambda)\partial_\eta U &= \rho_1 \quad \text{in }]0,T^*[\times\partial\Omega, \\ \lambda V + (1-\lambda)\partial_\eta V &= \rho_2 \quad \text{in }]0,T^*[\times\partial\Omega, \\ \lambda W + (1-\lambda)\partial_\eta W &= \rho_3 \quad \text{in }]0,T^*[\times\partial\Omega, \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} \rho_1 &= -c\beta_1 + b\beta_3, \\ \rho_2 &= c\beta_1 + \sqrt{2bc}\beta_2 + b\beta_3, \\ \rho_3 &= -c\beta_1 + \sqrt{2bc}\beta_2 - b\beta_3, \end{aligned} \tag{3.9}$$

and initial data (1.5).

The conditions (1.12)–(1.15) become, respectively,

$$\begin{gathered} -f(\mu w,v,w) + \mu h(\mu w,v,w) \geq 0, \\ f\left(-\sqrt{2\mu}v - \mu w,v,w\right) + \sqrt{2\mu}g\left(-\sqrt{2\mu}v - \mu w,v,w\right) + \mu h\left(-\sqrt{2\mu}v - \mu w,v,w\right) \geq 0, \\ -f\left(\sqrt{2\mu}v - \mu w,v,w\right) + \sqrt{2\mu}g\left(\sqrt{2\mu}v - \mu w,v,w\right) - \mu h\left(\sqrt{2\mu}v - \mu w,v,w\right) \geq 0 \end{gathered} \tag{3.10}$$

for all $v,w \geq 0$, and for positive constants $C_2' \leq \sqrt{1/2\mu}$, $C_1'' \leq \sqrt{\mu/2}$, $\alpha_2 \geq \sqrt{1/2\mu}$, and $\beta_2 \geq \sqrt{\mu/2}$

$$C_2' f(u,v,w) + g(u,v,w) + C_2'' h(u,v,w) \leq C_1\left(\alpha_2 u + v + \beta_2 w + 1\right); \tag{3.11}$$

for all (u,v,w) in Σ, where C_1 is a positive constant.

The third case: If $\{\sqrt{2\mu}|\beta_2| \le \beta_1 + \mu\beta_3$ and $\mu\beta_3 \le \beta_1\}$, then system (1.1)–(1.3) can be rewritten as follows:

$$\begin{aligned}\frac{\partial v}{\partial t} - a\Delta v - b\Delta w - c\Delta u &= g(u,v,w) \quad \text{in } \mathbb{R}^+ \times \Omega,\\ \frac{\partial w}{\partial t} - c\Delta v - a\Delta w &= h(u,v,w) \quad \text{in } \mathbb{R}^+ \times \Omega,\\ \frac{\partial u}{\partial t} - b\Delta v - a\Delta u &= f(u,v,w) \quad \text{in } \mathbb{R}^+ \times \Omega,\end{aligned} \tag{3.12}$$

with the same boundary conditions (1.4) and initial data (1.5).

In this case, the diffusion matrix of the system becomes

$$A = \begin{pmatrix} a & b & c \\ c & a & 0 \\ b & 0 & a \end{pmatrix}. \tag{3.13}$$

Then all the previous results remain valid in the region

$$\Sigma = \left\{ (u_0, v_0, w_0) \in \mathbb{R}^3 \text{ such that: } \sqrt{2\mu}\,|\,v_0\,| \le u_0 + \mu w_0 \text{ and } \mu w_0 \le u_0 \right\}, \tag{3.14}$$

where

$$\mu = \frac{b}{c}. \tag{3.15}$$

And systems (2.2)–(2.4) become

$$\begin{aligned}\frac{\partial U}{\partial t} - a\Delta U &= F_2(U,V;W),\\ \frac{\partial V}{\partial t} - (a + \sqrt{2}\sqrt{bc})\Delta V &= G_2(U,V;W),\\ \frac{\partial W}{\partial t} - (a - \sqrt{2}\sqrt{bc})\Delta W &= H_2(U,V;W),\end{aligned} \tag{3.16}$$

where

$$\begin{aligned}U(t,x) &= cu(t,x) - bw(t,x),\\ V(t,x) &= cu(t,x) + \sqrt{2bc}v(t,x) + bw(t,x),\\ W(t,x) &= cu(t,x) - \sqrt{2bc}v(t,x) + bw(t,x)\end{aligned} \tag{3.17}$$

for all (t,x) in $]0,T^*[\times\Omega$,

$$\begin{aligned}F_2(U,V;W) &= (cf - bh)(u,v,w),\\ G_2(U,V;W) &= \left(cf + \sqrt{2}\sqrt{bc}g + bh\right)(u,v,w),\\ H_2(U,V;W) &= \left(cf - \sqrt{2}\sqrt{bc}g + bh\right)(u,v,w)\end{aligned} \tag{3.18}$$

for all (u,v,w) in Σ; with the boundary conditions

$$\begin{aligned} \lambda U+(1-\lambda)\partial_\eta U &= \rho_1 \quad \text{in }]0,T^*[\times\partial\Omega, \\ \lambda V+(1-\lambda)\partial_\eta V &= \rho_2 \quad \text{in }]0,T^*[\times\partial\Omega, \\ \lambda W+(1-\lambda)\partial_\eta W &= \rho_3 \quad \text{in }]0,T^*[\times\partial\Omega, \end{aligned} \tag{3.19}$$

where

$$\begin{aligned} \rho_1 &= c\beta_1 - b\beta_3, \\ \rho_2 &= c\beta_1 + \sqrt{2bc}\beta_2 + b\beta_3, \\ \rho_3 &= c\beta_1 - \sqrt{2bc}\beta_2 + b\beta_3 \end{aligned} \tag{3.20}$$

and the initial data (1.5).

The conditions (1.12)–(1.15) become, respectively,

$$\begin{gathered} f(\mu w,v,w) - \mu h(\mu w,v,w) \geq 0, \\ f\left(-\sqrt{2\mu}v-\mu w,v,w\right) + \sqrt{2\mu}g\left(-\sqrt{2\mu}v-\mu w,v,w\right) + \mu h\left(-\sqrt{2\mu}v-\mu w,v,w\right) \geq 0, \\ f\left(\sqrt{2\mu}v-\mu w,v,w\right) - \sqrt{2\mu}g\left(\sqrt{2\mu}v-\mu w,v,w\right) + \mu h\left(\sqrt{2\mu}v-\mu w,v,w\right) \geq 0 \end{gathered} \tag{3.21}$$

for all $v,w \geq 0$, and for positive constants $C_3' \leq \sqrt{2\mu}$, $C_1'' \leq \mu$, $\alpha_3 \geq \sqrt{2\mu}$, and $\beta_3 \geq \mu$,

$$f(u,v,w) + C_3' g(u,v,w) + C_3'' h(u,v,w) \leq C_1\left(u+\alpha_3 v+\beta_3 w+1\right); \tag{3.22}$$

for all (u,v,w) in Σ, where C_1 is a positive constant.

The fourth case. If $\{|\beta_1+\mu\beta_3| \leq \sqrt{2\mu}\beta_2$ and $\mu\beta_3 \leq \beta_1\}$, then system (1.1)–(1.3) can be rewritten as follows:

$$\begin{aligned} \frac{\partial w}{\partial t} - a\Delta w - c\Delta v &= h(u,v,w) \quad \text{in } \mathbb{R}^+\times\Omega, \\ \frac{\partial u}{\partial t} - a\Delta u - b\Delta v &= f(u,v,w) \quad \text{in } \mathbb{R}^+\times\Omega, \\ \frac{\partial v}{\partial t} - b\Delta w - c\Delta u - a\Delta v &= g(u,v,w) \quad \text{in } \mathbb{R}^+\times\Omega \end{aligned} \tag{3.23}$$

with the same boundary conditions (1.4) and initial data (1.5).

In this case, the diffusion matrix of the system becomes

$$A = \begin{pmatrix} a & 0 & c \\ 0 & a & b \\ b & c & a \end{pmatrix}. \tag{3.24}$$

Then all the previous results remain valid in the region

$$\Sigma = \left\{(u_0, v_0, w_0) \in \mathbb{R}^3 \text{ such that: } |u_0 + \mu w_0| \leq \sqrt{2\mu} v_0 \text{ and } \mu w_0 \leq u_0\right\}, \tag{3.25}$$

where

$$\mu = \frac{b}{c}. \tag{3.26}$$

And system (2.2)–(2.4) becomes

$$\begin{aligned}
\frac{\partial U}{\partial t} - a\Delta U &= F_3(U, V; W), \\
\frac{\partial V}{\partial t} - (a + \sqrt{2}\sqrt{bc})\Delta V &= G_3(U, V; W), \\
\frac{\partial W}{\partial t} - (a - \sqrt{2}\sqrt{bc})\Delta W &= H_3(U, V; W),
\end{aligned} \tag{3.27}$$

where

$$\begin{aligned}
U(t,x) &= cu(t,x) - bw(t,x), \\
V(t,x) &= cu(t,x) + \sqrt{2bc}v(t,x) + bw(t,x), \\
W(t,x) &= -cu(t,x) + \sqrt{2bc}v(t,x) - bw(t,x)
\end{aligned} \tag{3.28}$$

for all (t,x) in $]0, T^*[\times\Omega$,

$$\begin{aligned}
F_3(U, V; W) &= (cf - bh)(u, v, w), \\
G_3(U, V; W) &= (cf + \sqrt{2}\sqrt{bc}g + bh)(u, v, w), \\
H_3(U, V; W) &= (-cf + \sqrt{2}\sqrt{bc}g - bh)(u, v, w)
\end{aligned} \tag{3.29}$$

for all (u, v, w) in Σ; with the boundary conditions

$$\begin{aligned}
\lambda U + (1-\lambda)\partial_\eta U &= \rho_1 \quad \text{in }]0, T^*[\times\partial\Omega, \\
\lambda V + (1-\lambda)\partial_\eta V &= \rho_2 \quad \text{in }]0, T^*[\times\partial\Omega, \\
\lambda W + (1-\lambda)\partial_\eta W &= \rho_3 \quad \text{in }]0, T^*[\times\partial\Omega,
\end{aligned} \tag{3.30}$$

where

$$\begin{aligned}
\rho_1 &= c\beta_1 - b\beta_3, \\
\rho_2 &= c\beta_1 + \sqrt{2bc}\beta_2 + b\beta_3, \\
\rho_3 &= -c\beta_1 + \sqrt{2bc}\beta_2 - b\beta_3.
\end{aligned} \tag{3.31}$$

The conditions (1.12)–(1.15) become, respectively:

$$\begin{gathered}
f(\mu w, v, w) - \mu h(\mu w, v, w) \geq 0, \\
f(-\sqrt{2\mu}v - \mu w, v, w) + \sqrt{2\mu}g(-\sqrt{2\mu}v - \mu w, v, w) + \mu h(-\sqrt{2\mu}v - \mu w, v, w) \geq 0, \\
-f(\sqrt{2\mu}v - \mu w, v, w) + \sqrt{2\mu}g(\sqrt{2\mu}v - \mu w, v, w) - \mu h(\sqrt{2\mu}v - \mu w, v, w) \geq 0
\end{gathered} \tag{3.32}$$

for all $v, w \geq 0$, and for positive constants $C_4' \leq \sqrt{1/2\mu}$, $C_4'' \leq \sqrt{\mu/2}$, $\alpha_4 \geq \sqrt{1/2\mu}$, and $\beta_4 \geq \sqrt{\mu/2}$,

$$C_4' f(u, v, w) + g(u, v, w) + C_4'' h(u, v, w) \leq C_1(\alpha_4 u + v + \beta_4 w + 1); \tag{3.33}$$

for all (u, v, w) in Σ, where C_1 is a positive constant.

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Abdelmalek Salem: Department of Mathematics and Informatiques, University Center of Tebessa, Tebessa 12002, Algeria

Email address: a.salem@gawab.com