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Research Article

On the Nonlinear Theory of Micropolar Bodies with Voids

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This paper is concerned with the nonlinear theory of micropolar, porous, and elastic solids. By using the theory of Langenbach, within this context, we obtain some existence and uniqueness results.

1. Introduction

Beju [1], and many other authors have established some existence and uniqueness theorems in the nonlinear classical theory of elastic bodies.

The theory of micropolar bodies has been introduced by Eringen and Suhubi in [2, 3]. Also, in [4] Eringen developed the theory of micromorphic continua. The theory of multipolar continuum was given by Green and Rivlin in [5]. A review of the field and further developments can be found in [6].

The concept of a porous material was introduced by Cowin and Nunziato, in the context of classical theory of Elasticity, in the paper [7] and also [8] by Goodman and Cowin. In these papers, the authors introduce an additional degree of freedom in order to develop the mechanical behavior of porous solids in which the matrix material is elastic and the interstices are voids of material.

The basic premise underling this theory is the concept of a material for which the bulk density is written as the product of two fields, the matrix material density field and the volume fraction field.

The intended applications of this theory are to geological materials like rocks and soil and to manufactured porous materials, like ceramics and pressed powders.

In the present paper we restrict our considerations to the case of isothermal processes. Our intention is to extend the result of Beju established in the classical theory of Elasticity. To obtain these results, we must make certain assumptions on the material response relating the convexity of internal energy to be compatible with the principle of objectivity.

2. Notations and basic equations

Consider that a bounded region B of three-dimensional Euclidean space R^3 is occupied by a porous micropolar body in mechanical equilibrium referred to the reference configuration and a fixed system of rectangular Cartesian axes. Let ∂B be the boundary of the domain B. With $\overline{B}$ we denote the closure of B. Suppose ∂B is a sufficiently smooth surface such that we can use the divergence theorem and Friedrich's inequality.

Letters in bold face stand for vector fields. The components of a vector $\mathbf{v}$ are denoted by v_i. The italic indices will always assume the values 1, 2, 3, whereas Greek indices will range over the values 1, 2. The Einstein convention regarding the summation on repeated indices is implied. A comma followed by a subscript denotes partial differentiation with respect to the spatial corresponding Cartesian coordinates.

Also, the spatial argument of a function will be omitted when there is no likelihood of confusion.

As usual, we will denote (X_K) the material coordinates of a typical particle and (x_i) the spatial coordinates of the same particle and we have

$$x_i = x_i(X_K), \quad X_K \in \overline{B}. \tag{2.1}$$

Suppose the continuous differentiability of the functions x_i with respect to each of the variables X_K, as many times, is required. Also we assume that

$$\det\left(\frac{\partial x_i}{\partial X_K}\right) > 0. \tag{2.2}$$

As in the paper [8] of Goodman and Cowin, we introduce the following additional kinematic variables:

$$m_i = m_i(X_K), \quad X_K \in \overline{B}, \tag{2.3}$$

with the property

$$\det\left(\frac{\partial m_i}{\partial X_K}\right) \neq 0. \tag{2.4}$$

This is to characterize the micropolar aspects of the bodies. To characterize the voids of material we consider that the bulk density ϱ of the material is the product of two fields, the density field of the matrix material γ and the volume fraction ν, that is,

$$\varrho = \gamma\nu, \tag{2.5}$$

and this relation also holds for the reference configuration:

$$\varrho_0 = \gamma_0\nu_0. \tag{2.6}$$

In this way, the deformation of a micropolar body with voids is characterized by the following independent kinematic variables:

$$x_i = x_i(X_K), \quad m_i = m_i(X_K), \quad \nu = \nu(X_K), \quad X_K \in \overline{B}. \tag{2.7}$$

We will denote by u_i and φ_i the components of the displacement field and microrotation vector, respectively. As it is well known, we have

$$\begin{aligned} u_i &= x_i - \delta_{iK} X_K, \\ \varphi_i &= m_i - \delta_{iK} M_K, \end{aligned} \tag{2.8}$$

where δ_{iK} is the Kronecker symbol and M_K is the value of m_i in the reference state. Using the known procedure of Green and Rivlin, it is easy to prove that the equations of the equilibrium theory can be written in the following form:

$$\begin{gathered} T_{Ki,K} + \varrho_0 F_i = 0, \\ S_{iK,K} + \varepsilon_{ijK} T_{Kj} + \varrho_0 G_i = 0, \\ H_{K,K} + g + \varrho_0 L = 0. \end{gathered} \tag{2.9}$$

In these equations we have used the following notations:

(i) ϱ_0: the constant mass density (in the reference configuration),
(ii) F_i: the body force,
(iii) L_i: the components of the body couple,
(iv) L: the extrinsic equilibrated body force,
(v) g: the intrinsic equilibrated body force,
(vi) T_{Ki}: the first Piola-Kirchhoff stress tensor,
(vii) S_{Ki}: the couple stress tensor,
(viii) H_K: the components of the equilibrated stress vector associated with surface in the domain B, which were originally coordinate planes perpendicular to the X_K-axes through the point (X_K) measured per unit area of these planes.

In the context of the nonlinear theory of micropolar bodies with voids, the constitutive equations are:

$$\begin{gathered} \sigma = \sigma(u_{i,K}, \varphi_i, \varphi_{i,K}, \nu, \nu_{,K}), \\ T_{Ki} = \frac{\partial \sigma}{\partial u_{i,K}}, \qquad S_{iK} = \frac{\partial \sigma}{\partial \varphi_{i,K}}, \qquad H_K = \frac{\partial \sigma}{\partial \nu_{,K}}, \qquad g = \frac{\partial \sigma}{\partial \nu}, \end{gathered} \tag{2.10}$$

where σ is the internal energy density, considered as a smooth function.

In all that follows, we consider a materially homogeneous body.

To the equation of equilibrium (2.9) we add the following boundary conditions:

$$u_i = \widetilde{u}_i, \quad \varphi_i = \widetilde{\varphi}_i, \quad \nu = \widetilde{\nu} \quad \text{on } \partial B, \tag{2.11}$$

where $\widetilde{u}_i$, $\widetilde{\varphi}_i$, and $\widetilde{\nu}$ are prescribed functions.

By summarizing, the boundary-value problem in the context of micropolar bodies with voids consists in finding the functions u_i, φ_i, and ν which satisfy (2.9) and (2.10) in B and the boundary conditions (2.11) on ∂B.

3. Basic results

In the beginning of this section, we formulate some results due to Langenbach. These results will be used in order to prove the existence and uniqueness theorems in our context.

Consider a bounded domain Ω in n-dimensional Euclidean space R^n. We denote by $\partial\Omega$ the boundary of Ω and consider this surface to be sufficiently smooth such that we can apply the divergence theorem. By $\mathcal{H}(\Omega)$ we denote a Hilbert space on Ω.

Let T be an operator $T : D(T) \to \mathcal{H}(\Omega)$, where $D(T) \subset \mathcal{H}(\Omega)$ is the effective domain of the operator T and is a linear subset, dense in $\mathcal{H}(\Omega)$. Suppose that the operator T has a linear Gateaux differential on the set $\omega \subset D(T)$, which means there exists an operator:

$$(DT) : \omega \longrightarrow L(D(T), \mathcal{H}(\Omega)), \tag{3.1}$$

such that

$$\lim_{t\to 0} \frac{1}{t}\left[T(x+th) - T(x)\right] = (DT)(x)h, \quad x \in \omega,\ h \in D(T). \tag{3.2}$$

Here, as usual, we have denoted by $L(D(T), \mathcal{H}(\Omega))$ the set of all linear operators defined on $D(T)$, having the values in the Hilbert space $\mathcal{H}(\Omega)$.

The connection between T and (ST) is given by

$$Tx - Tx_0 = \int_0^t (DT)\left(x_0 + \tau(x - x_0)\right)(x - x_0)\,d\tau. \tag{3.3}$$

We remember that the operator T is called *monotone* if it satisfies the relation

$$\langle Tu - Tv, u - v\rangle \geq 0, \quad \forall u, v \in D(T), \tag{3.4}$$

and T is a *strictly monotone* operator if it is monotone and, in addition, satisfies the relation

$$\langle Tu - Tv, u - v\rangle = 0, \Longleftrightarrow u = v. \tag{3.5}$$

Now, we consider the operatorial equation:

$$Tu = f, \tag{3.6}$$

with the linear and homogeneous boundary value conditions:

$$L_i u = 0, \quad i = \overline{1, m}. \tag{3.7}$$

Consider the set D_0 defined by

$$D_0 = \{u \in D(T) : L_i u = 0\}. \tag{3.8}$$

The following theorem allows us to associate a variational problem with our boundary-value problem formulated in Section 2.

Theorem 3.1. *Consider the following five conditions to be satisfied:*

(1) *$D_0(T)$ and $D(T)$ are linear sets and $D_0(T)$ is dense in the Hilbert space $\mathcal{H}(\Omega)$;*

(2) *the operator T has linear Gateaux differential for all $u,h \in D(T)$ and the mapping $(DT)(u)h$ is continuous with respect to u, the value $(DT)(u)h$ belonging to a two-dimensional hyperplane which contains the point u;*

(3) *the operator T satisfies the condition*

$$T(0) = 0; \tag{3.9}$$

(4) *for all $u \in D(T)$, $h,g \in D_0(T)$, one has*

$$\langle (DT)(u)h,g\rangle = \langle (DT)(u)g,h\rangle; \tag{3.10}$$

(5) *for all $u \in D(T)$, $h \in D_0(T)$, $h \neq 0$, one has*

$$\langle (DT)(u)h,h\rangle > 0. \tag{3.11}$$

Then one has the following:

(i) *if there exists a solution $u_0 \in D_0(T)$ of (3.6), it is unique and attains on $D_0(T)$ the minimum of the functional*

$$\Phi(u) = \int_0^t (T(\tau u),u)\,d\tau - (f,u), \tag{3.12}$$

where $f \in \mathcal{H}(\Omega)$;

(ii) *conversely, if an element of $D_0(T)$ attains the minimum of the functional defined in (3.12), then this element is a solution of (3.6).*

The following theorem has been proved also by Langenbach and assures the conditions for the existence and uniqueness of a generalized solution for the boundary value problems (3.6), (3.7).

Theorem 3.2. *If one says that*

$$\langle (DT)(u)h,h\rangle \geq c|h|^2, \quad u \in D(T),\ h \in D_0(T),\ c = \text{constant},\ c > 0, \tag{3.13}$$

then,

(i) *the functional (3.12) is bounded below on $D_0(T)$,*

(ii) *the functional (3.12) is strictly convex on $D_0(T)$,*

(iii) *any minimizing sequence of the functional (3.12) is convergent in $\mathcal{H}(\Omega)$.*

The limit of a minimizing sequence of the functional (3.12) is called *generalized solution* of the boundary value problems (3.6), (3.7).

Langenbach has proved that the generalized solution of (3.6), (3.7) is unique.

Theorem 3.3. *Assume that there exists an element $u_0 \in D_0(T)$ such that*

$$\langle (DT)(u)h,h\rangle \geq c_1\langle (DT)(u_0)h,h\rangle \geq c_2|h|^2, \tag{3.14}$$

where c_1 and c_2 are positive constants.

Then the generalized solution of the boundary value problems (3.6), (3.7) is an element of the energetic space of the linear operator $(DT)(u_0)$.

In the following, we will use these theorems to characterize our boundary value problems (2.9), (2.10), (2.11). Equation (2.9) can be rewritten in the following form:

$$\begin{gathered}\left(\frac{\partial\sigma}{\partial u_{i,K}}\right)_{,K} = -\varrho_0 F_i,\\ \left(\frac{\partial\sigma}{\partial\varphi_{i,K}}\right)_{,K} + \varepsilon_{ijK}\frac{\partial\sigma}{\partial u_{j,K}} = -\varrho_0 G_i,\\ \left(\frac{\partial\sigma}{\partial\nu_{,K}}\right)_{,K} + \frac{\partial\sigma}{\partial\nu} = -\varrho_0 L.\end{gathered} \tag{3.15}$$

The ordered triplets $U = (u_i, \varphi_i, \nu)$ are elements of the real vector space

$$\mathcal{V}_{(7)} = \mathcal{V}_{(3)} \bigoplus \mathcal{V}_{(3)} \bigoplus \mathcal{V}_{(1)}. \tag{3.16}$$

Of course, the space $\mathcal{V}_{(7)}$ is seven-dimensional and is defined on $\overline{B}$. We now introduce the notations

$$\begin{aligned} M_i\mathbf{U} &= -\left(\frac{\partial\sigma}{\partial u_{i,K}}\right)_{,K},\\ N_i\mathbf{U} &= -\left(\frac{\partial\sigma}{\partial\varphi_{i,K}}\right)_{,K} - \varepsilon_{ijK}\frac{\partial\sigma}{\partial u_{j,K}},\\ P_0\mathbf{U} &= -\left(\frac{\partial\sigma}{\partial\nu_{,K}}\right)_{,K} - \frac{\partial\sigma}{\partial\nu},\\ M\mathbf{U} &= (M_i\mathbf{U},\, N_i\mathbf{U},\, P_0\mathbf{U}),\\ \mathbf{F} &= (\varrho_0 F_i,\, \varrho_0 G_i,\, \varrho_0 L). \end{aligned} \tag{3.17}$$

Taking into account the notations (3.17), the system of (3.15) can be written in the form

$$M\mathbf{U} = \mathbf{F}, \quad \text{on } B. \tag{3.18}$$

For the sake of simplicity, we denote by $\mathcal{V}$ the space $\mathcal{V}_{(7)}$. Let $\mathbf{V} \in \mathcal{V}$, $\mathbf{V} = (v_i, \psi_i, v)$ such that

$$v_i = \overline{u}_i, \qquad \psi_i = \overline{\varphi}_i, \qquad v = \overline{\nu}, \quad \text{on } \partial B, \tag{3.19}$$

where $\overline{u}_i$, $\overline{\varphi}_i$, and $\overline{\nu}$ are prescribed functions defined in (2.11).

Let us define $\mathbf{W}$, $A\mathbf{W}$, and $\mathcal{F}$ by

$$\begin{gathered} \mathbf{W} = \mathbf{U} - \mathbf{V}, \qquad \mathcal{F} = \mathbf{F} - M\mathbf{V}, \\ A\mathbf{W} = (A_i\mathbf{W}, B_i\mathbf{W}, C_0\mathbf{W}) = M(\mathbf{W}+\mathbf{V}) - M\mathbf{V}. \end{gathered} \tag{3.20}$$

Then the boundary value problems (2.9), (2.10), (2.11) receive the form

$$A\mathbf{W} = \mathcal{F}, \quad \text{on } B, \tag{3.21}$$

$$\mathbf{W} = 0, \quad \text{on } \partial B. \tag{3.22}$$

Let $L_2(B)$ be the Hilbert space of all vector fields $\mathbf{U} = (u_i, \varphi_i, \nu)$ whose components are square integrable on B, with the norm generated by the scalar product:

$$\langle \mathbf{U}, \mathbf{V} \rangle = \int_B (u_i v_i + \varphi_i \psi_i + \nu v)\, dV, \tag{3.23}$$

where $\mathbf{U} = (u_i, \varphi_i, \nu)$ and $\mathbf{V} = (v_i, \varphi_i, \nu)$.

We denote by $W_0^2(B)$ the Sobolev space of all elements from $L_2(B)$ belonging to $C^2(B)$ which satisfy the boundary condition (3.22). This space will be the domain of definition for the operator A defined in (3.20), that is,

$$A : W_0^2(B) \longrightarrow L_2(B). \tag{3.24}$$

Also, we suppose that $\mathcal{F} \in L_2(B)$.

Theorem 3.4. *One assumes that the function σ is of class C^2 with respect to each variables $u_{i,K}$, φ_j, $\varphi_{s,M}$, ν, $\nu_{,N}$ and satisfies the inequality*

$$\begin{aligned} \Gamma(\mathbf{W}) = \int_B \bigg(& \frac{\partial^2\sigma}{\partial u_{i,K}\partial u_{j,M}} f_{i,K} f_{j,M} + 2\frac{\partial^2\sigma}{\partial u_{i,K}\partial \varphi_j} f_{i,K} g_j \\ & + 2\frac{\partial^2\sigma}{\partial u_{i,K}\partial \varphi_{j,M}} f_{i,K} g_{j,M} + 2\frac{\partial^2\sigma}{\partial u_{i,K}\partial \nu} f_{i,K} h + 2\frac{\partial^2\sigma}{\partial u_{i,K}\partial \nu_{,M}} f_{i,K} h_{,M} \\ & + \frac{\partial^2\sigma}{\partial \varphi_{i,K}\partial \varphi_{j,M}} g_{i,K} g_{j,M} + 2\frac{\partial^2\sigma}{\partial \varphi_{i,K}\partial \varphi_j} g_{i,K} g_j + 2\frac{\partial^2\sigma}{\partial \varphi_{i,K}\partial \nu} g_{i,K} h \\ & + \frac{\partial^2\sigma}{\partial \varphi_{i,K}\partial \nu_{,M}} g_{i,K} h_{,M} + \frac{\partial^2\sigma}{\partial \varphi_i\partial \varphi_j} g_i g_j + 2\frac{\partial^2\sigma}{\partial \varphi_i\partial \nu} g_i h + \frac{\partial^2\sigma}{\partial \varphi_i\partial \nu_{,K}} g_i h_{,K} \\ & + \frac{\partial^2\sigma}{\partial \nu\partial \nu} h^2 + \frac{\partial^2\sigma}{\partial \nu_{,K}\partial \nu_{,M}} h_{,K} h_{,M} + \frac{\partial^2\sigma}{\partial \nu\partial \nu_{,K}} h h_{,K} \bigg) dv > 0, \end{aligned} \tag{3.25}$$

for all $\mathbf{W} = (u_i, \varphi_i, \nu)$, $\mathbf{G} = (f_i, g_i, h)$, $\mathbf{G} \neq 0$, which possess the partial derivatives of first order with respect to the variable X_K.

Then one has the following:

(i) *if there exists a solution* $\mathbf{W} \in W_0^2(B)$ *of (3.21), it is unique and attains on* $W_0^2(B)$ *the minimum of the functional*

$$\Phi(\mathbf{W}) = \int_0^1 \langle A(t\mathbf{W},\mathbf{W}\rangle dt - \langle \mathcal{F},\mathbf{W}\rangle; \tag{3.26}$$

(ii) *conversely, if the minimum of the functional (3.26), on the space* $W_0^2(B)$*, is attained in an element* $\mathbf{W}_0 \in W_0^2(B)$*, then this element is a solution of (3.21).*

Proof. The two assertions of the theorem will be proved if we show that the hypotheses of Theorem 3.1 are satisfied.

Therefore, we have the following:

(1) as it is known, $W_0^2(B)$ is a linear set, dense in the space $L_2(B)$ (see, e.g., Minty [9]);

(2) for all $\mathbf{W},\mathbf{G} \in W_0^2(B)$, the operator A defined in (3.20) has a linear Gateaux differential given by

$$\begin{aligned}
(DA_i)(\mathbf{W})\mathbf{G} = &-\Big(\frac{\partial^2\sigma}{\partial u_{i,K}\partial u_{j,M}} f_{j,M} + \frac{\partial^2\sigma}{\partial u_{i,K}\partial \varphi_j} g_j \\
&+ \frac{\partial^2\sigma}{\partial u_{i,K}\partial \varphi_{j,M}} g_{j,M} + \frac{\partial^2\sigma}{\partial u_{i,K}\partial \nu} h + \frac{\partial^2\sigma}{\partial u_{i,K}\partial \nu_{,M}} h_{,M}\Big)_{,K},\\
(DB_i)(\mathbf{W})\mathbf{G} = &-\Big(\frac{\partial^2\sigma}{\partial \varphi_{i,K}\partial u_{j,M}} f_{j,M} + \frac{\partial^2\sigma}{\partial \varphi_{i,K}\partial \varphi_j} g_j \\
&+ \frac{\partial^2\sigma}{\partial \varphi_{i,K}\partial \varphi_{j,M}} g_{j,M} + \frac{\partial^2\sigma}{\partial \varphi_{i,K}\partial \nu} h + \frac{\partial^2\sigma}{\partial \varphi_{i,K}\partial \nu_{,M}} h_{,M}\Big)_{,K}\\
&+ \frac{\partial^2\sigma}{\partial \varphi_i\partial u_{j,K}} f_{j,K} + \frac{\partial^2\sigma}{\partial \varphi_i\partial \varphi_j} g_j + \frac{\partial^2\sigma}{\partial \varphi_i\partial \varphi_{j,K}} g_{j,K} + \frac{\partial^2\sigma}{\partial \varphi_i\partial \nu} h + \frac{\partial^2\sigma}{\partial \varphi_i\partial \nu_{,K}} h_{,K},\\
(DC_0)(\mathbf{W})\mathbf{G} = &-\Big(\frac{\partial^2\sigma}{\partial \nu_{,K}\partial u_{i,M}} f_{i,M} + \frac{\partial^2\sigma}{\partial \nu_{,K}\partial \varphi_j} g_j \\
&+ \frac{\partial^2\sigma}{\partial \nu_{,K}\partial \varphi_{j,M}} g_{j,M} + \frac{\partial^2\sigma}{\partial \nu_{,K}\partial \nu} h + \frac{\partial^2\sigma}{\partial \nu_{,K}\partial \nu_{,M}} h_{,M}\Big)_{,K}\\
&+ \frac{\partial^2\sigma}{\partial \nu\partial u_{i,K}} f_{i,K} + \frac{\partial^2\sigma}{\partial \nu\partial \varphi_i} g_i + \frac{\partial^2\sigma}{\partial \nu\partial \varphi_{i,K}} g_{i,K} + \frac{\partial^2\sigma}{\partial \nu\partial \nu} h + \frac{\partial^2\sigma}{\partial \nu\partial \nu_{,K}} h_{,K},
\end{aligned} \tag{3.27}$$

it is easy to verify that for a given $\mathbf{G}$, the mapping $(DA)(\mathbf{W})\mathbf{G}$ is continuous with respect to $\mathbf{W}$ in every two-dimensional hyperplane which contains the point $\mathbf{W}$;

(3) this hypothesis is satisfied because from (3.20) we deduce that $A(0) = 0$;

(4) for $\mathbf{W}, \mathbf{G}, \mathbf{H} \in W_0^2(B)$, provided that $\mathbf{H} = (h_i, \chi_i, \mu)$ possess the partial derivatives of first order with respect to the variable X_K, we get

$$\begin{aligned}\langle (DA)(\mathbf{W})\mathbf{G}, \mathbf{H}\rangle &= \int_B [(DA_i)(\mathbf{W})\mathbf{G}h_i + (DB_i)(\mathbf{W})\mathbf{G}\chi_i + (DC_0)(\mathbf{W})\mathbf{G}\mu]dV \\ &= \int_B [(DA_i)(\mathbf{W})\mathbf{H}f_i + (DB_i)(\mathbf{W})\mathbf{H}g_i + (DC_0)(\mathbf{W})\mathbf{H}h]dV \\ &= \langle (DA)(\mathbf{W})\mathbf{H}, \mathbf{G}\rangle;\end{aligned} \tag{3.28}$$

(5) taking into account the inequality (3.25) and the equality (3.28), we deduce that

$$\langle (DA)(\mathbf{W})\mathbf{H}, \mathbf{H}\rangle > 0, \quad \forall \mathbf{W}, \mathbf{H} \in W_0^2(B), \ \mathbf{H} \neq 0, \tag{3.29}$$

that is, the last hypothesis of Theorem 3.1 is satisfied and the demonstration of the theorem is complete. □

Theorem 3.5. *One supposes that (3.25) holds. Then the boundary value problems (2.9), (2.10), (2.11) have at most one solution* $\mathbf{U} \in C^0(B)$.

Proof. The demonstration of this theorem will be based on the following result (see, e.g., the paper [9] of Minty).

If the domain $D(T)$ of the operator T is convex, then a sufficient condition for T to be strictly monotone on $D(T)$ is that the derivative

$$\frac{d}{dt}[\langle T(\mathbf{U} + t\mathbf{G}), \mathbf{G}\rangle]_{t=0} \tag{3.30}$$

exists and is positive for all $\mathbf{U}, \mathbf{V} \in D(T)$, $\mathbf{G} = \mathbf{V} - \mathbf{V}$, $\mathbf{G} \neq 0$.

In view of this result, consider Z to be the set of all vector fields $\mathbf{U} = (u_i, \varphi_i, \nu)$ that satisfy the boundary conditions (2.11).

We will prove that the operator M defined by (3.17) is strictly monotone on Z.

Let $\mathbf{U}, \mathbf{V} \in Z$, $0 \leq t \leq 1$. It is easy to verify that

$$t\mathbf{U} + (1 - t)\mathbf{V} \in Z. \tag{3.31}$$

Then, by using (3.25) and (3.28), we can prove that

$$\begin{aligned}&\frac{d}{dt}[\langle M(\mathbf{U} + t\mathbf{G}), \mathbf{G}\rangle]_{t=0} \\ &\quad = \left[\frac{d}{dt}\int_B [M_i(\mathbf{U} + t\mathbf{G})f_i + N_i(\mathbf{U} + t\mathbf{G})g_i + P_0(\mathbf{U} + t\mathbf{G})h]dV\right]_{t=0} \\ &\quad = \langle (DA)(\mathbf{U})\mathbf{G}, \mathbf{G}\rangle,\end{aligned} \tag{3.32}$$

for all $\mathbf{U}, \mathbf{V} \in Z$, $\mathbf{G} = \mathbf{V} - \mathbf{U}$, $\mathbf{G} \neq 0$ on ∂B.

Therefore, we deduce that the operator M is strictly monotone on the set Z. As a consequence, if $\mathbf{U}_1$ and $\mathbf{U}_2$ are two solutions of our problem, then by direct calculations we get

$$\langle M\mathbf{U}_1 - M\mathbf{U}_2, \mathbf{U}_1 - \mathbf{U}_2\rangle = \langle 0, \mathbf{U}_1 - \mathbf{U}_2\rangle = 0, \tag{3.33}$$

such that we deduce that $\mathbf{U}_1 = \mathbf{U}_2$, according to the definition of a strictly monotone operator. □

Following the proof of Theorem 3.2, we immediately obtain the next results.

Theorem 3.6. *One supposes that the hypotheses of Theorem 3.4 are satisfied. Moreover, assume that*

$$\Gamma(\mathbf{W}) > c\int_B (f_i f_i + g_i g_i + h^2)\,dV, \tag{3.34}$$

for $\mathbf{W} = (u_i, \varphi_i, v)$, $\mathbf{G} = (f_i, g_i, h)$ *having the partial derivatives of first order with respect to the variable* X_K, *and* $c =$ *constant*, $c > 0$.

Then one has the following:

(a) *the functional (3.26) is bounded below on* $W_0^2(B)$;

(b) *the functional (3.26) is strictly convex on* $W_0^2(B)$;

(c) *any minimizing sequence of the functional (3.26) is convergent in* $L_2(B)$ *and its limit is a generalized solution of the problems (3.21), (3.22);*

(d) *this generalized solution is unique.*

Regarding (3.34), we make the following observations.

Suppose there exists a positive constant c_1 such that for all

$$\mathbf{W} = (u_i, \varphi_i, v), \qquad \mathbf{G} = (f_i, g_i, h) \in W_0^2(B), \tag{3.35}$$

we have

$$\begin{aligned}
&\frac{\partial^2\sigma}{\partial u_{i,K}\partial u_{j,M}} f_{i,K} f_{j,M} + 2\frac{\partial^2\sigma}{\partial u_{i,K}\partial \varphi_j} f_{i,K} g_j + 2\frac{\partial^2\sigma}{\partial u_{i,K}\partial \varphi_{j,M}} f_{i,K} g_{j,M}\\
&\quad + 2\frac{\partial^2\sigma}{\partial u_{i,K}\partial v} f_{i,K} h + 2\frac{\partial^2\sigma}{\partial u_{i,K}\partial v_{,M}} f_{i,K} h_{,M} + \frac{\partial^2\sigma}{\partial \varphi_{i,K}\partial \varphi_{j,M}} g_{i,K} g_{j,M}\\
&\quad + 2\frac{\partial^2\sigma}{\partial \varphi_{i,K}\partial \varphi_j} g_{i,K} g_j + 2\frac{\partial^2\sigma}{\partial \varphi_{i,K}\partial v} g_{i,K} h + \frac{\partial^2\sigma}{\partial \varphi_{i,K}\partial v_{,M}} g_{i,K} h_{,M}\\
&\quad + \frac{\partial^2\sigma}{\partial \varphi_i\partial \varphi_j} g_i g_j + 2\frac{\partial^2\sigma}{\partial \varphi_i\partial v} g_i h + 2\frac{\partial^2\sigma}{\partial \varphi_i\partial v_{,K}} g_i h_{,K}\\
&\quad + \frac{\partial^2\sigma}{\partial v\partial v} h^2 + \frac{\partial^2\sigma}{\partial v_{,K}\partial v_{,M}} h_{,K} h_{,M} + 2\frac{\partial^2\sigma}{\partial v\partial v_{,K}} h h_{,K}\\
&> c_1 (f_{i,K} f_{i,K} + g_i g_i + g_{i,K} g_{i,K} + h_K h_K + h^2).
\end{aligned} \tag{3.36}$$

On the other hand, by using Friedrich's inequality, we deduce that there exists a real constant c_2 such that

$$\int_B (f_{i,K} f_{i,K} + g_i g_i + g_{i,K} g_{i,K} + h_K h_K + h^2)\,dV \ge c_2\int_B (f_i f_i + g_i g_i + h^2)\,dV. \tag{3.37}$$

Finally, taking into account (3.36) and (3.37), we deduce that the condition (3.34) is satisfied.

As a consequence of Theorem 3.3, it is easy to obtain the result from the following theorem.

Theorem 3.7. *One assumes that there exists* $\mathbf{W}_0 \in W_0^2(B)$ *and two positive constants* c_1 *and* c_2 *such that*

$$T(\mathbf{W}) \geq c_1 T(\mathbf{W}_0) \geq c_2 \int_B (f_i f_i + g_i g_i + h^2)\, dV, \tag{3.38}$$

for all $\mathbf{W}, \mathbf{G} \in W_0^2(B)$, $\mathbf{G} = (f_i, g_i, h)$.

Then the generalized solution of the boundary value problems (3.21), (3.22) belongs to the energetic space of the linear operator $(DA)(\mathbf{W}_0)$.

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