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Research Article

A Note on Fractional Sumudu Transform

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We propose a new definition of a fractional-order Sumudu transform for fractional differentiable functions. In the development of the definition we use fractional analysis based on the modified Riemann-Liouville derivative that we name the fractional Sumudu transform. We also established a relationship between fractional Laplace and Sumudu duality with complex inversion formula for fractional Sumudu transform and apply new definition to solve fractional differential equations.

1. Introduction

In the literature there are numerous integral transforms that are widely used in physics, astronomy, as well as engineering. In order to solve the differential equations, the integral transforms were extensively used and thus there are several works on the theory and application of integral transforms such as the Laplace, Fourier, Mellin, and Hankel, to name but a few. In the sequence of these transforms in early 90s, Watugala [1] introduced a new integral transforms named the Sumudu transform and further applied it to the solution of ordinary differential equation in control engineering problems. For further detail and properties about Sumudu transforms see [2–7] and many others. Recently Kiliçman et al. applied this transform to solve the system of differential equations; see [8]. The Sumudu transform is defined over the set of the functions

$$A = \left\{ f(t) : \exists M, \tau_1, \tau_2 > 0, \left| f(t) \right| < M e^{t/\tau_j}, \text{ if } t \in (-1)^j \times [0, \infty) \right\} \tag{1.1}$$

by the following formula:

$$G(u) = S\left[f(t); u\right] =: \int_0^\infty f(ut)e^{-t}dt, \quad u \in (-\tau_1, \tau_2). \tag{1.2}$$

The existence and the uniqueness were discussed in [9]; for further details and properties of the Sumudu transform and its derivatives we refer to [2]. In [3], some fundamental properties of the Sumudu transform were established. In [10], this new transform was applied to the one-dimensional neutron transport equation. In fact, one can easily show that there is a strong relationship between double Sumudu and double Laplace transforms; see [9]. Further in [6], the Sumudu transform was extended to the distributions and some of their properties were also studied in [11].

The function $f(t)$ so involved is usually continuous and continuously differentiable. Suppose that the function is continuous but its fractional derivative exists of order $\alpha, 0 < \alpha < 1$, but no derivative, and then (1.2) fails to apply. Thus we have to introduce a new definition of Sumudu transform. For the convenience of the reader, firstly we will give a brief background on the definition of the fractional derivative and basic notations for more details see [12–14] and [15].

1.1. Fractional Derivative via Fractional Difference

Definition 1.1. Let $f : \mathfrak{R} \to \mathfrak{R}$, $t \to f(t)$ denote a continuous (but not necessarily differentiable) function, and let $h > 0$ denote a constant discretization span. Define the forward operator $\mathrm{FW}(h)$ by the equality

$$\mathrm{FW}(h) f(t) := f(t+h). \tag{1.3}$$

Then the fractional difference of order α, $0 < \alpha < 1$ of $f(t)$ is defined by the expression

$$\Delta^\alpha f(t) := (\mathrm{FW} - 1)^\alpha = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f[t + (\alpha - k)h], \tag{1.4}$$

and its fractional derivative of order α is defined by the limit

$$f^{(\alpha)}(t) = \lim_{h \downarrow 0} \frac{\Delta^\alpha f(t)}{h^\alpha}. \tag{1.5}$$

See the details in [13].

1.2. Modified Fractional Riemann-Liouville Derivative

Jumarie proposed an alternative way to the Riemann-Liouville definition of the fractional derivative; see [13].

Definition 1.2. Let $f : \mathfrak{R} \rightarrow \mathfrak{R}$ be a continuous but not necessarily differentiable function. Further, consider the following.

(i) Assume that $f(t)$ is a constant K. Then its fractional derivative of order α is

$$\begin{aligned} D_t^\alpha K &= K\Gamma^{-1}(1-\alpha)t^{-\alpha}, \quad \alpha \leq 0, \\ &= 0, \quad \alpha > 0. \end{aligned} \tag{1.6}$$

(ii) When $f(t)$ is not a constant, then we will set

$$f(t) = f(0) + \left(f(t) - f(0)\right), \tag{1.7}$$

and its fractional derivative will be defined by the expression

$$f^{(\alpha)}(t) = D_t^\alpha f(0) + D_t^\alpha \left(f(t) - f(0)\right), \tag{1.8}$$

in which, for negative α, one has

$$D_t^\alpha \left(f(t) - f(0)\right) := \frac{1}{\Gamma(-\alpha)} \int_o^t (t-\xi)^{-\alpha-1} f(\xi) d\xi, \quad \alpha < 0, \tag{1.9}$$

whilst for positive α, we will set

$$D_t^\alpha \left(f(t) - f(0)\right) = D_t^\alpha f(t) = D_t\left(f^{\alpha-1}(t)\right). \tag{1.10}$$

When $n \leq \alpha < n+1$, we will set

$$f^{(\alpha)}(t) := \left(f^{(\alpha-n)}(t)\right)^{(n)}, \quad n \leq \alpha < n+1, \ n \geq 1. \tag{1.11}$$

We will refer to this fractional derivative as the modified Riemann-Liouville derivative, and it is in order to point out that this definition is strictly equivalent to Definition 1.1, via (1.4).

1.3. Integration with respect to $(dt)^\alpha$

The integral with respect to $(dt)^\alpha$ is defined as the solution of the fractional differential equation

$$dy = f(x)(dx)^\alpha, \quad x \geq 0, \ y(0) = 0, \tag{1.12}$$

which is provided by the following results.

Lemma 1.3. *Let $f(x)$ denote a continuous function; then the solution $y(x)$ with $y(0) = 0$, of (1.12), is defined by the equality*

$$\begin{aligned} y &= \int_o^x f(\xi)(d\xi)^\alpha \\ &= \alpha \int_o^x (x-\xi)^{\alpha-1} f(\xi)d\xi, \quad 0 < \alpha < 1. \end{aligned} \tag{1.13}$$

2. Sumudu Transform of Fractional Order

Definition 2.1. Let $f(t)$ denote a function which vanishes for negative values of t. Its Sumudu's transform of order α (or its fractional Sumudu's transform) is defined by the following expression, when it is finite:

$$\begin{aligned} S_\alpha\{f(t)\} :=: G_\alpha(u) &:= \int_0^\infty E_\alpha(-t^\alpha) f(ut)(dt)^\alpha \\ &:= \lim_{M\uparrow\infty} \int_0^M E_\alpha(-t^\alpha) f(ut)(dt)^\alpha, \end{aligned} \tag{2.1}$$

where $u \in \mathcal{C}$, and $E_\alpha(x)$ is the Mittag-Leffler function $\sum_{k=0}^\infty (x^k/\alpha k!)$.

Recently Tchuenche and Mbare introduced the double Sumudu transform [16]. Analogously, we define the fractional double Sumudu transform in following way.

Definition 2.2. Let $f(x,t)$ denote a function which vanishes for negative values of x and t. Its double Sumudu transform of fractional order (or its fractional double Sumudu transform) is defined as

$$S_\alpha^2\{f(t,x)\} :=: G_\alpha^2(u,v) = \iint_0^\infty E_\alpha\left[-(t+x)^\alpha\right] f(ut,vx)(dt)^\alpha(dx)^\alpha, \tag{2.2}$$

where $u, v \in \mathcal{C}$, and $E_\alpha(x)$ is the Mittag-Leffler function.

2.1. The Laplace-Sumudu Duality of Fractional Order

The following definition was given in [13].

Definition 2.3. Let $f(t)$ denote a function which vanishes for negative values of t. Its Laplace's transform of order α (or its αth fractional Laplace's transform) is defined by the following expression:

$$\begin{aligned} L_\alpha\{f(t)\} :=: F_\alpha(u) &:= \int_0^\infty E_\alpha\left(-(ut)^\alpha\right) f(t)(dt)^\alpha \\ &= \lim_{M\uparrow\infty} \int_0^M E_\alpha\left(-(ut)^\alpha\right) f(t)(dt)^\alpha \end{aligned} \tag{2.3}$$

provided that integral exists.

Theorem 2.4. *If the Laplace transform of fractional order of a function $f(t)$ is $L_\alpha\{f(t)\} = F_\alpha(u)$ and the Sumudu transform of this function is $S_\alpha\{f(t)\} = G_\alpha(u)$, then*

$$G_\alpha(u) = \frac{1}{u^\alpha} F_\alpha\left(\frac{1}{u}\right), \quad 0 < \alpha < 1. \tag{2.4}$$

Proof. By definition of fractional Sumudu transformation,

$$\begin{aligned} G_\alpha(u) = S_\alpha\{f(t)\} &:= \lim_{M\uparrow\infty} \int_0^M E_\alpha(-t^\alpha) f(ut)(dt)^\alpha \\ &= \lim_{M\uparrow\infty} \alpha \int_0^M (M-t)^{\alpha-1} E_\alpha(-t^\alpha) f(ut) dt. \end{aligned} \tag{2.5}$$

By using the change of variable $ut \leftarrow t'$

$$\begin{aligned} &= \frac{1}{u^\alpha} \lim_{M\uparrow\infty} \alpha \int_0^{Mu} (Mu - t')^{\alpha-1} E_\alpha\left(-\left(\frac{t'}{u}\right)^\alpha\right) f(t') dt' \\ &= \frac{1}{u^\alpha} \int_0^\infty E_\alpha\left(-\left(\frac{t'}{u}\right)^\alpha\right) f(t')(dt')^\alpha \\ &= \frac{1}{u^\alpha} F_\alpha\left(\frac{1}{u}\right). \end{aligned} \tag{2.6}$$

Similarly, on using the definition of fractional Sumudu transform, the following operational formulae can easily be obtained:

$$\begin{aligned} &\text{(i)} \quad S_\alpha\{f(at)\} = G_\alpha(au), \\ &\text{(ii)} \quad S_\alpha\{f(t-b)\} = E_\alpha(-b^\alpha) G_\alpha(u), \\ &\text{(iii)} \quad S_\alpha\{E_\alpha(-c^\alpha t^\alpha) f(t)\} = \frac{1}{(1+cu)^\alpha} G_\alpha\left(\frac{u}{1+cu}\right), \\ &\text{(iv)} \quad S_\alpha\left\{\int_0^t f(t)(dt)^\alpha\right\} = u^\alpha \Gamma(1+\alpha) G_\alpha(u), \\ &\text{(v)} \quad S_\alpha\{f^\alpha(t)\} = \frac{G_\alpha(u) - \Gamma(1+\alpha) f(0)}{u^\alpha}. \end{aligned} \tag{2.7}$$

□

Proof of (i). It can easily be proved by using Definition 2.1. □

Proof of (ii). We start with the following equality by using (2.1):

$$\begin{aligned}\int_0^M E_\alpha(-t^\alpha) f(u(t-b))(dt)^\alpha &= \int_0^M (M-t)^{\alpha-1} E_\alpha(-t^\alpha) f(u(t-b))dt \\ &= \int_0^M (M-t'-b)^{\alpha-1} E_\alpha\left(-(b+t')^\alpha\right) f(ut')dt\end{aligned} \tag{2.8}$$

on using the change of variable $t-b \leftarrow t'$. Then it follows that

$$E_\alpha(x+y)^\alpha = E_\alpha(x^\alpha)E_\alpha(y^\alpha). \tag{2.9}$$

□

Proof of (iii). We start from equality (1.13):

$$\int_0^M E_\alpha(-t^\alpha)E_\alpha(-c^\alpha u^\alpha t^\alpha) f(ut)(dt)^\alpha = \int_0^M (M-t)^{\alpha-1} E_\alpha\left(-(1+cu)^\alpha t^\alpha\right) f(ut)dt \tag{2.10}$$

using the change of variable $(1+cu) \leftarrow t'$

$$= \frac{1}{(1+cu)^\alpha}\int_0^{M(1+cu)} \left(M(1+cu)-t'\right)^{\alpha-1} E_\alpha(-t'^\alpha) f\left(\frac{u}{1+cu}t'\right)dt. \tag{2.11}$$

□

Proof of (iv) and (v). Using fractional Laplace-Sumudu duality and using the result of Jumarie (see [14]), we can easily obtain these results.

Now we will obtain very similar properties for the fractional double Sumudu transform. Since proofs of these properties are straight, due to this reason, we will give only statements of these properties:

$$\begin{aligned}&\text{(vi)} \quad S_\alpha^2\{f(at)g(bx)\} = G_\alpha(au)H_\alpha(bv), \\ &\text{(vii)} \quad S_\alpha^2\{f(at,bx)\} = G_\alpha^2(au,bv), \\ &\text{(viii)} \quad S_\alpha^2\{f(t-a,x-b)\} = E_\alpha\left(-(a+b)^\alpha\right)G_\alpha^2(au,bv), \\ &\text{(ix)} \quad S_\alpha^2\{\partial_t^\alpha f(t,x)\} = \frac{G_\alpha^2(u,v)-\Gamma(1+\alpha)f(0,x)}{u^\alpha},\end{aligned} \tag{2.12}$$

where ∂_t^α is the fractional partial derivative of order $\alpha(0<\alpha<1)$ (see [13]). □

3. The Convolution Theorem and Complex Inversion Formula

Proposition 3.1. *If one defines the convolution of order of the two functions $f(t)$ and $g(t)$ by the expression*

$$\left(f(x)*g(x)\right)_\alpha := \int_0^x f(x-v)g(v)(dv)^\alpha, \tag{3.1}$$

then

$$S_\alpha\{(f(t) * g(t))_\alpha\} = u^\alpha G_\alpha(u) H_\alpha(u), \tag{3.2}$$

where $G_\alpha(u) = S_\alpha\{f(t)\}$ *and* $H_\alpha(u) = S_\alpha\{g(t)\}$.

Proof. First recall that the Laplace transform of fractional order of $(f * g)_\alpha$ is given by

$$L_\alpha\{(f(t) * g(t))_\alpha\} = L_\alpha\{f(t)\} L_\alpha\{g(t)\}. \tag{3.3}$$

Now, by the fractional Laplace-Sumudu duality relation,

$$\begin{aligned} S_\alpha\{(f(t) * g(t))_\alpha\} &= \frac{1}{u^\alpha} L_\alpha\{(f(t) * g(t))_\alpha\} \\ &= \frac{1}{u^\alpha} L_\alpha\{f(t)\} L_\alpha\{g(t)\} \\ &= u^\alpha \frac{L_\alpha\{f(t)\}}{u^\alpha} \frac{L_\alpha\{g(t)\}}{u^\alpha} \\ &= u^\alpha G_\alpha(u) H_\alpha(u). \end{aligned} \tag{3.4}$$

□

Proposition 3.2. *Given Sumudus transforms that one recalls here for convenience:*

$$G_\alpha(u) = \int_0^\infty E_\alpha(-x^\alpha) f(ux) dx, \quad 0 < \alpha < 1, \tag{3.5}$$

one has the inversion formula

$$f(x) = \frac{1}{(M_\alpha)^\alpha} \int_{-i\infty}^{i\infty} \frac{E_\alpha((xu)^\alpha)}{u^\alpha} G\left(\left(\frac{1}{u}\right)^\alpha\right) (du)^\alpha, \tag{3.6}$$

where M_α *is the period of the Mittag-Leffler function.*

Proof. By using complex inversion formula of fractional Laplace transform, see [14], if

$$F_\alpha(u) = \int_0^\infty E_\alpha\left(-(ux)^\alpha\right) f(x) dx, \quad 0 < \alpha < 1, \tag{3.7}$$

then inversion formula is given as

$$f(x) = \frac{1}{(M_\alpha)^\alpha} \int_{-i\infty}^{i\infty} E_\alpha((xu)^\alpha) F_\alpha(u) (du)^\alpha \tag{3.8}$$

According to fractional Sumudu-Laplace duality, we can easily yield the desired result. □

4. An Application of Fractional Sumudu Transform

Example 4.1. Solution of the equation

$$y^{(\alpha)} + y = f(x), \quad y(0) = 0, \quad 0 < \alpha < 1, \tag{4.1}$$

is given by

$$f(x) = \frac{1}{(M_\alpha)^\alpha} \int_{-i\infty}^{i\infty} \frac{E_\alpha((xu)^\alpha)}{u^\alpha} G_\alpha\left(\left(\frac{1}{u}\right)^\alpha\right)(du)^\alpha. \tag{4.2}$$

Proof. Taking Sumudu transform of (4.1) both side, we can easily get

$$y_\alpha(u) = \frac{u^\alpha}{1 + u^\alpha} G_\alpha(u) \tag{4.3}$$

on using $y(0) = 0$. Then by applying the complex inversion formula of fractional Sumudu transforms we get the following result:

$$y(x) = \frac{1}{(M_\alpha)^\alpha} \int_{-i\infty}^{i\infty} \frac{E_\alpha((xu)^\alpha)}{u^\alpha(1 + u^\alpha)} G_\alpha\left(\left(\frac{1}{u}\right)^\alpha\right)(du)^\alpha. \tag{4.4}$$

Now we apply the fractional double Sumudu transform to solve fractional partial differential equation. □

Example 4.2. Consider the linear fractional partial differential equation (see [12])

$$\partial_t^\alpha z(x,t) = c\partial_x^\beta z(x,t), \quad x, t \in \Re^+, \tag{4.5}$$

with the boundary condition

$$z(0,t) = f(t), \qquad z(x,0) = g(x), \tag{4.6}$$

where c is a positive coefficient, and $0 < \alpha, \beta < 1$.

Proof. Taking fractional double Sumudu transform of (4.5) both side, we can easily get

$$\left(\frac{1}{u^\alpha} - \frac{1}{v^\beta}\right) G_\alpha^2(u,v) = \frac{\Gamma(1+\alpha)}{u^\alpha} f(t) - \frac{\Gamma(1+\beta)}{v^\alpha} g(x), \tag{4.7}$$

which gives

$$G_\alpha^2(u,v) = \Gamma(1+\alpha)\left(\frac{v^\beta}{v^\beta - u^\alpha}\right) f(t) - \Gamma(1+\beta)\left(\frac{u^\alpha}{v^\beta - u^\alpha}\right) g(x). \tag{4.8}$$

□

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