

Research Article

Sharp Estimates of m -Linear p -Adic Hardy and Hardy-Littlewood-Pólya Operators

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The sharp estimates of the m -linear p -adic Hardy and Hardy-Littlewood-Pólya operators on Lebesgue spaces with power weights are obtained in this paper.

1. Introduction

In recent years, p -adic numbers are widely used in theoretical and mathematical physics (cf. [1-8]), such as string theory, statistical mechanics, turbulence theory, quantum mechanics, and so forth.

For a prime number p , let \mathbb{Q}_p be the field of p -adic numbers. It is defined as the completion of the field of rational numbers \mathbb{Q} with respect to the non-Archimedean p -adic norm $|\cdot|_p$. This norm is defined as follows: $|0|_p = 0$; If any nonzero rational number x is represented as $x = p^\gamma(m/n)$, where m and n are integers which are not divisible by p and γ is an integer, then $|x|_p = p^{-\gamma}$. It is not difficult to show that the norm satisfies the following properties:

$$|xy|_p = |x|_p |y|_p, \quad |x + y|_p \leq \max\{|x|_p, |y|_p\}. \quad (1.1)$$

From the standard p -adic analysis [6], we see that any nonzero p -adic number $x \in \mathbb{Q}_p$ can be uniquely represented in the canonical series

$$x = p^\gamma \sum_{j=0}^{\infty} a_j p^j, \quad \gamma = \gamma(x) \in \mathbb{Z}, \quad (1.2)$$

where a_j are integers, $0 \leq a_j \leq p-1$, $a_0 \neq 0$. The series (1.2) converges in the p -adic norm because $|a_j p^j|_p = p^{-j}$. Denote by $\mathbb{Q}_p^* = \mathbb{Q}_p \setminus \{0\}$ and $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$.

The space \mathbb{Q}_p^n consists of points $x = (x_1, x_2, \dots, x_n)$, where $x_j \in \mathbb{Q}_p$, $j = 1, 2, \dots, n$. The p -adic norm on \mathbb{Q}_p^n is

$$|x|_p := \max_{1 \leq j \leq n} |x_j|_p, \quad x \in \mathbb{Q}_p^n. \quad (1.3)$$

Denote by

$$B_Y(a) = \{x \in \mathbb{Q}_p^n : |x - a|_p \leq p^Y\}, \quad (1.4)$$

the ball with center at $a \in \mathbb{Q}_p^n$ and radius p^Y , and

$$S_Y(a) := \{x \in \mathbb{Q}_p^n : |x - a|_p = p^Y\} = B_Y(a) \setminus B_{Y-1}(a). \quad (1.5)$$

Since \mathbb{Q}_p^n is a locally compact commutative group under addition, it follows from the standard analysis that there exists a Haar measure dx on \mathbb{Q}_p^n , which is unique up to positive constant multiple and is translation invariant. We normalize the measure dx by the equality

$$\int_{B_0(0)} dx = |B_0(0)|_H = 1, \quad (1.6)$$

where $|E|_H$ denotes the Haar measure of a measurable subset E of \mathbb{Q}_p^n . By simple calculation, we can obtain that

$$|B_Y(a)|_H = p^{Yn}, \quad |S_Y(a)|_H = p^{Yn}(1 - p^{-n}), \quad (1.7)$$

for any $a \in \mathbb{Q}_p^n$. For a more complete introduction to the p -adic field, see [6] or [9].

The space $\underbrace{\mathbb{Q}_p^n \times \mathbb{Q}_p^n \times \dots \times \mathbb{Q}_p^n}_m$ consists of points (y_1, y_2, \dots, y_m) , where $y_i = (y_{i1}, y_{i2}, \dots, y_{in}) \in \mathbb{Q}_p^n$, $i = 1, 2, \dots, m$. The p -adic norm of m -tuple (y_1, y_2, \dots, y_m) is

$$|(y_1, y_2, \dots, y_m)|_p := \max_{1 \leq i \leq m} |y_i|_p. \quad (1.8)$$

Recently, p -adic analysis has received a lot of attention due to its application in mathematical physics. There are numerous papers on p -adic analysis, such as [10, 11] about Riesz potentials, [12–16] about p -adic pseudodifferential equations, and so forth. The harmonic analysis on p -adic field has been drawing more and more concern (cf. [17–21] and references therein).

The well-known Hardy's integral inequality [22] tells us that for $1 < q < \infty$,

$$\|Hf\|_{L^q(\mathbb{R}^+)} \leq \frac{q}{q-1} \|f\|_{L^q(\mathbb{R}^+)}, \quad (1.9)$$

where the classical Hardy operator is defined by

$$Hf(x) := \frac{1}{x} \int_0^x f(t) dt, \quad (1.10)$$

for nonnegative integral function f on \mathbb{R}^+ , and the constant $q/(q-1)$ is the best possible. Thus the norm of Hardy operator on $L^q(\mathbb{R}^+)$ is

$$\|H\|_{L^q(\mathbb{R}^+) \rightarrow L^q(\mathbb{R}^+)} = \frac{q}{q-1}. \quad (1.11)$$

Faris [23] introduced the following n -dimensional Hardy operator, for nonnegative function f on \mathbb{R}^n ,

$$\mathcal{H}f(x) := \frac{1}{\Omega_n |x|^n} \int_{|y| < |x|} f(y) dy, \quad x \in \mathbb{R}^n \setminus \{0\}, \quad (1.12)$$

where Ω_n is the volume of the unit ball in \mathbb{R}^n . Christ and Grafakos [24] obtained that the norm of \mathcal{H} on $L^q(\mathbb{R}^n)$ is

$$\|\mathcal{H}\|_{L^q(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n)} = \frac{q}{q-1}, \quad (1.13)$$

which is the same as that of the 1-dimension Hardy operator. In [25], Fu et al. introduced the m -linear Hardy operator, which is defined by

$$\mathcal{H}^m(f_1, \dots, f_m)(x) = \frac{1}{\Omega_{mn} |x|^{mn}} \int_{|(y_1, \dots, y_m)| < |x|} f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m, \quad (1.14)$$

where $x \in \mathbb{R}^n \setminus \{0\}$ and f_1, \dots, f_m are nonnegative locally integrable functions on \mathbb{R}^n . And they obtained the precise norms of \mathcal{H}^m on Lebesgue spaces with power weight. The authors of [26] also got the best constants of m -linear Hilbert, Hardy and Hardy-Littlewood-Pólya operators on Lebesgue spaces.

The study of multilinear averaging operators in Euclidean spaces is a byproduct of the recent interest in multilinear singular integral operator theory. This subject was established by Coifman and Meyer [27] in 1975. In this article, we consider the sharp estimates of m -linear p -adic Hardy and Hardy-Littlewood-Pólya operators. In contrast with [25], we use a new technique in calculations based on the feature of p -adic field, and Theorem 3.1 is also new. They cannot be obtained immediately by [25]. In [28], we defined the p -adic Hardy operator.

Definition 1.1. For a function f on \mathbb{Q}_p^n , we define the p -adic Hardy operator as follows

$$\mathcal{H}^p f(x) = \frac{1}{|x|_p^n} \int_{B(0, |x|_p)} f(t) dt, \quad x \in \mathbb{Q}_p^n \setminus \{0\}, \quad (1.15)$$

where $B(0, |x|_p)$ is a ball in \mathbb{Q}_p^n with center at $0 \in \mathbb{Q}_p^n$ and radius $|x|_p$.

It is obvious that $|\mathcal{H}^p f| \leq \mathcal{M}^p f$, where \mathcal{M}^p is the Hardy-Littlewood maximal operator [17] defined by

$$\mathcal{M}^p f(x) = \sup_{\gamma \in \mathbb{Z}} \frac{1}{|B_\gamma(x)|_H} \int_{B_\gamma(x)} |f(y)| dy, \quad f \in L^1_{\text{loc}}(\mathbb{Q}_p^n). \quad (1.16)$$

The Hardy-Littlewood maximal operator plays an important role in harmonic analysis. The boundedness of \mathcal{M}^p on $L^q(\mathbb{Q}_p^n)$ has been solved (see, e.g., [9]). But the best estimate of \mathcal{M}^p on $L^q(\mathbb{Q}_p^n)$, $q > 1$, even that of Hardy-Littlewood maximal operator on Euclidean spaces \mathbb{R}^n is very difficult to obtain. Instead, we have obtained the sharp estimates of \mathcal{H}^p (and p -adic Hardy-Littlewood-Pólya operator) elsewhere.

Definition 1.2. Let m be a positive integer and f_1, \dots, f_m be nonnegative locally integrable functions on \mathbb{Q}_p^n . The m -linear p -adic Hardy operator is defined by

$$\mathcal{H}_m^p(f_1, \dots, f_m)(x) = \frac{1}{|x|_p^{mn}} \int_{|(y_1, \dots, y_m)|_p \leq |x|_p} f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m, \quad (1.17)$$

where $x \in \mathbb{Q}_p^n \setminus \{0\}$.

The Hardy-Littlewood-Pólya's linear operator [26] is defined by

$$Tf(x) = \int_0^\infty \frac{f(y)}{\max(x, y)} dy. \quad (1.18)$$

In [26], the authors obtained that the norm of Hardy-Littlewood-Pólya's operator on $L^q(\mathbb{R}^+)$ (see also [22, page 254]), $1 < q < \infty$, is

$$\|T\|_{L^q(\mathbb{R}^+) \rightarrow L^q(\mathbb{R}^+)} = \frac{q^2}{q-1}. \quad (1.19)$$

We define the p -adic Hardy-Littlewood-Pólya operator as (see [28])

$$T^p(x) = \int_{\mathbb{Q}_p} \frac{f(y)}{\max(|x|_p, |y|_p)} dy, \quad x \in \mathbb{Q}_p^*. \quad (1.20)$$

Definition 1.3. Let m be a positive integer and f_1, \dots, f_m be nonnegative locally integrable functions on \mathbb{Q}_p^n . The m -linear p -adic Hardy-Littlewood-Pólya operator is defined by

$$T_m^p(f_1, \dots, f_m)(x) = \int_{\mathbb{Q}_p} \cdots \int_{\mathbb{Q}_p} \frac{f_1(y_1) \cdots f_m(y_m)}{\left[\max(|x|_p, |y_1|_p, \dots, |y_m|_p) \right]^m} dy_1 \cdots dy_m, \quad x \in \mathbb{Q}_p^*. \quad (1.21)$$

We obtain the sharp estimates of the m -linear p -adic Hardy operator on Lebesgue spaces with power weights in Section 2. In Section 3, we get the best estimate of m -linear p -adic Hardy-Littlewood-Pólya operator on Lebesgue spaces with power weights.

In the following sequel, for $k \in \mathbb{Z}$, we denote $B_k = \{x \in \mathbb{Q}_p^n : |x|_p \leq p^k\}$ and $S_k = \{x \in \mathbb{Q}_p^n : |x|_p = p^k\}$.

2. Sharp Estimates of m -Linear p -Adic Hardy Operator

Theorem 2.1. *Let $m \in \mathbb{Z}^+$, $f_i \in L^{q_i}(|x|_p^{\alpha_i q_i/q} dx)$, $1 < q_i < \infty$, $i = 1, 2, \dots, m$, $1 \leq q < \infty$, $1/q = \sum_{i=1}^m (1/q_i)$, $\alpha_i < qn(1 - (1/q_i))$ and $\alpha = \sum_{i=1}^m \alpha_i$. Then*

$$\|\mathcal{H}^p(f_1, \dots, f_m)\|_{L^q(|x|_p^\alpha dx)} \leq C_{\mathcal{H}} \|f_1\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx)} \cdots \|f_m\|_{L^{q_m}(|x|_p^{\alpha_m q_m/q} dx)}, \tag{2.1}$$

where

$$C_{\mathcal{H}} = \frac{(1 - p^{-n})^m}{\prod_{i=1}^m (1 - p^{n/q_i + (\alpha_i/q) - n})} \tag{2.2}$$

is the best constant.

When $\alpha = 0$, we get the sharp estimates of the m -linear p -adic Hardy operator on Lebesgue spaces.

Corollary 2.2. *Let $m \in \mathbb{Z}^+$, $1 < q_i < \infty$, $i = 1, 2, \dots, m$, $1 \leq q < \infty$ and $1/q = \sum_{i=1}^m 1/q_i$. Then*

$$\|\mathcal{H}_m^p\|_{L^{q_1}(\mathbb{Q}_p^n) \times \dots \times L^{q_m}(\mathbb{Q}_p^n) \rightarrow L^q(\mathbb{Q}_p^n)} = \frac{(1 - p^{-n})^m}{\prod_{i=1}^m (1 - p^{(n/q_i) - n})}. \tag{2.3}$$

Proof of Theorem 2.1. Since the proof of the case when $m = 1$ is similar to and even simpler than that of the case when $m \geq 2$, for simplicity, we will only give the proof of case when $m \geq 2$. To make the proof clearer, we will discuss it in two parts.

(I) *When $m = 2$*

Firstly, we claim that the operator \mathcal{H}^p and its restriction to the functions g satisfying $g(x) = g(|x|_p^{-1})$ have the same operator norm on $L^q(|x|_p^\alpha dx)$. In fact, set

$$g_i(x) = \frac{1}{1 - p^{-n}} \int_{|\xi_i|_p=1} f_i(|x|_p^{-1} \xi_i) d\xi_i, \quad x \in \mathbb{Q}_p^n, \quad i = 1, 2. \tag{2.4}$$

It's clear that $g_i(x) = g_i(|x|_p^{-1})$, $i = 1, 2$, and

$$\begin{aligned}
 \mathcal{A}_2^p(g_1, g_2)(x) &= \frac{1}{|x|_p^{2n}} \int_{|(y_1, y_2)|_p \leq |x|_p} g_1(y_1) g_2(y_2) dy_1 dy_2 \\
 &= \frac{1}{(1-p^{-n})^2} \frac{1}{|x|_p^{2n}} \int_{|(y_1, y_2)|_p \leq |x|_p} \prod_{i=1}^2 \left(\int_{|\xi_i|_p=1} f_i(|y_i|_p^{-1} \xi_i) d\xi_i \right) dy_1 dy_2 \\
 &= \frac{1}{(1-p^{-n})^2} \frac{1}{|x|_p^{2n}} \int_{|(y_1, y_2)|_p \leq |x|_p} \prod_{i=1}^2 \left(\int_{|z_i|_p=|y_i|_p} f_i(z_i) |y_i|_p^{-n} dz_i \right) dy_1 dy_2 \\
 &= \frac{1}{(1-p^{-n})^2} \frac{1}{|x|_p^{2n}} \int_{|(z_1, z_2)|_p \leq |x|_p} \prod_{i=1}^2 \left(\int_{|y_i|_p=|z_i|_p} |y_i|_p^{-n} dy_i \right) f_1(z_1) f_2(z_2) dz_1 dz_2 \\
 &= \frac{1}{|x|_p^{2n}} \int_{|(z_1, z_2)|_p \leq |x|_p} f_1(z_1) f_2(z_2) dz_1 dz_2 \\
 &= \mathcal{A}_2^p(f_1, f_2)(x).
 \end{aligned} \tag{2.5}$$

By Hölder's inequality, we get

$$\begin{aligned}
 \|g_i\|_{L^{q_i}(|x|_p^{\alpha_i q_i/q} dx)} &= \left(\int_{\mathbb{Q}_p^n} \left| \frac{1}{1-p^{-n}} \int_{|\xi_i|_p=1} f_i(|x|_p^{-1} \xi_i) d\xi_i \right|^{q_i} |x|_p^{\alpha_i q_i/q} dx \right)^{1/q_i} \\
 &\leq \left\{ \int_{\mathbb{Q}_p^n} \frac{1}{(1-p^{-n})^{q_i}} \left(\int_{|\xi_i|_p=1} |f_i(|x|_p^{-1} \xi_i)|^{q_i} d\xi_i \right) \left(\int_{|\xi_i|_p=1} d\xi_i \right)^{q_i-1} |x|_p^{\alpha_i q_i/q} dx \right\}^{1/q_i} \\
 &= \left\{ \int_{\mathbb{Q}_p^n} \frac{1}{1-p^{-n}} \left(\int_{|\xi_i|_p=1} |f_i(|x|_p^{-1} \xi_i)|^{q_i} d\xi_i \right) |x|_p^{\alpha_i q_i/q} dx \right\}^{1/q_i} \\
 &= \frac{1}{(1-p^{-n})^{1/q_i}} \left\{ \int_{\mathbb{Q}_p^n} \left(\int_{|z_i|_p=|x|_p} |f_i(z_i)|^{q_i} dz_i \right) |x|_p^{(\alpha_i q_i/q)-n} dx \right\}^{1/q_i} \\
 &= \frac{1}{(1-p^{-n})^{1/q_i}} \left\{ \int_{\mathbb{Q}_p^n} \left(\int_{|x|_p=|z_i|_p} |x|_p^{(\alpha_i q_i/q)-n} dx \right) |f_i(z_i)|^{q_i} dz_i \right\}^{1/q_i} \\
 &= \|f_i\|_{L^{q_i}(|x|_p^{\alpha_i q_i/q} dx)}, \quad i = 1, 2.
 \end{aligned} \tag{2.6}$$

Therefore,

$$\frac{\|\mathcal{A}_2^p(f_1, f_2)\|_{L^q(|x|_p^s dx)}}{\|f_1\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx)} \|f_2\|_{L^{q_2}(|x|_p^{\alpha_2 q_2/q} dx)}} \leq \frac{\|\mathcal{A}_2^p(g_1, g_2)\|_{L^q(|x|_p^s dx)}}{\|g_1\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx)} \|g_2\|_{L^{q_2}(|x|_p^{\alpha_2 q_2/q} dx)}}, \tag{2.7}$$

which implies the claim. In the following, without loss of generality, we may assume that $f_i \in L^{q_i}(|x|_p^{\alpha_i q_i/q} dx)$, $i = 1, 2$, which satisfy that $f_i(x) = f_i(|x|_p^{-1})$, $i = 1, 2$.

By changing of variables $y_i = |x|_p^{-1} z_i$, $i = 1, 2$, we have

$$\begin{aligned} \|\mathcal{H}_2^p(f_1, f_2)\|_{L^q(|x|_p^\alpha dx)} &= \left(\int_{\mathbb{Q}_p^n} \left| \frac{1}{|x|_p^{2n}} \int_{|(y_1, y_2)|_p \leq |x|_p} f_1(y_1) f_2(y_2) dy_1 dy_2 \right|^q |x|_p^\alpha dx \right)^{1/q} \\ &= \left(\int_{\mathbb{Q}_p^n} \left| \int_{|(z_1, z_2)|_p \leq 1} f_1(|x|_p^{-1} z_1) f_2(|x|_p^{-1} z_2) dz_1 dz_2 \right|^q |x|_p^\alpha dx \right)^{1/q} \quad (2.8) \\ &= \left(\int_{\mathbb{Q}_p^n} \left| \int_{|(z_1, z_2)|_p \leq 1} f_1(|z_1|_p^{-1} x) f_2(|z_2|_p^{-1} x) dz_1 dz_2 \right|^q |x|_p^\alpha dx \right)^{1/q}. \end{aligned}$$

Then using Minkowski's integral inequality and Hölder's inequality ($(q/q_1) + (q/q_2) = 1$), we get

$$\begin{aligned} \|\mathcal{H}_2^p(f_1, f_2)\|_{L^q(|x|_p^\alpha dx)} &\leq \int_{|(z_1, z_2)|_p \leq 1} \left(\int_{\mathbb{Q}_p^n} |f_1(|z_1|_p^{-1} x) f_2(|z_2|_p^{-1} x)|^q |x|_p^\alpha dx \right)^{1/q} dz_1 dz_2 \\ &\leq \int_{|(z_1, z_2)|_p \leq 1} \prod_{i=1}^2 \left(\int_{\mathbb{Q}_p^n} |f_i(|z_i|_p^{-1} x)|^{q_i} |x|_p^{\alpha_i q_i/q} dx \right)^{1/q_i} dz_1 dz_2 \quad (2.9) \\ &= \left(\int_{|(z_1, z_2)|_p \leq 1} \prod_{i=1}^2 |z_i|_p^{-(n/q_i) - (\alpha_i/q)} dz_1 dz_2 \right) \prod_{i=1}^2 \|f_i\|_{L^{q_i}(|x|_p^{\alpha_i q_i/q} dx)}. \end{aligned}$$

By calculation, we have

$$\begin{aligned} &\int_{|(z_1, z_2)|_p \leq 1} \prod_{i=1}^2 |z_i|_p^{-(n/q_i) - (\alpha_i/q)} dz_1 dz_2 \\ &= \int_{|z_1|_p \leq 1} \int_{|z_2|_p \leq |z_1|_p} \prod_{i=1}^2 |z_i|_p^{-(n/q_i) - (\alpha_i/q)} dz_1 dz_2 \\ &\quad + \int_{|z_2|_p \leq 1} \int_{|z_1|_p < |z_2|_p} \prod_{i=1}^2 |z_i|_p^{-(n/q_i) - (\alpha_i/q)} dz_1 dz_2 \\ &= \int_{|z_1|_p \leq 1} |z_1|_p^{-(n/q_1) - (\alpha_1/q)} \left(\sum_{k=-\infty}^{\log_p |z_1|_p} \int_{S_k} |z_2|_p^{-(n/q_2) - (\alpha_2/q)} dz_2 \right) dz_1 \\ &\quad + \int_{|z_2|_p \leq 1} |z_2|_p^{-(n/q_2) - (\alpha_2/q)} \left(\sum_{k=-\infty}^{\log_p |z_2|_p - 1} \int_{S_k} |z_1|_p^{-(n/q_1) - (\alpha_1/q)} dz_1 \right) dz_2 \end{aligned}$$

$$\begin{aligned}
&= \frac{(1-p^{-n})}{1-p^{(n/q_2)+(\alpha_2/q)-n}} \int_{|z_1|_p \leq 1} |z_1|_p^{-(n/q)-(\alpha/q)+n} dz_1 \\
&\quad + \frac{(1-p^{-n})p^{(n/q_1)+(\alpha_1/q)-n}}{1-p^{(n/q_1)+(\alpha_1/q)-n}} \int_{|z_2|_p \leq 1} |z_2|_p^{-(n/q)-(\alpha/q)+n} dz_2 \\
&= \frac{(1-p^{-n})^2}{\prod_{i=1}^2 (1-p^{(n/q_i)+(\alpha_i/q)-n})}.
\end{aligned} \tag{2.10}$$

Therefore,

$$\left\| \mathcal{L}_2^p \right\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx) \times L^{q_2}(|x|_p^{\alpha_2 q_2/q} dx) \rightarrow L^q(|x|_p^\alpha dx)} \leq \frac{(1-p^{-n})^2}{\prod_{i=1}^2 (1-p^{(n/q_i)+(\alpha_i/q)-n})}. \tag{2.11}$$

Now let us prove that our estimate is sharp. For $0 < \epsilon < 1$ and $|e|_p > 1$, we take

$$f_i^\epsilon(x_i) = \begin{cases} 0, & |x_i|_p < 1, \\ |x_i|_p^{-(n/q_i)-(\alpha_i/q)-(q_2\epsilon/q_i)}, & |x_i|_p \geq 1, \end{cases} \quad i = 1, 2. \tag{2.12}$$

Then by calculation, we have

$$\|f_1^\epsilon\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx)}^{q_1} = \|f_2^\epsilon\|_{L^{q_2}(|x|_p^{\alpha_2 q_2/q} dx)}^{q_2} = \frac{1-p^{-n}}{1-p^{-\epsilon q_2}}. \tag{2.13}$$

It is clear that when $|x|_p < 1$, $\mathcal{L}_2^p(f_1^\epsilon, f_2^\epsilon)(x) = 0$. But when $|x|_p \geq 1$,

$$\begin{aligned}
&\mathcal{L}_2^p(f_1^\epsilon, f_2^\epsilon)(x) \\
&= |x|_p^{-(n/q)-(\alpha/q)-(q_2\epsilon/q)} \int_{|(y_1, y_2)|_p \leq 1, |y_1|_p \geq 1/|x|_p, |y_2|_p \geq 1/|x|_p} \prod_{i=1}^2 |y_i|_p^{-(n/q_i)-(\alpha_i/q)-(q_2\epsilon/q_i)} dy_1 dy_2.
\end{aligned} \tag{2.14}$$

Since $|\epsilon|_p > 1$, we get

$$\begin{aligned}
& \left\| \mathcal{H}_2^p(f_1^\epsilon, f_2^\epsilon) \right\|_{L^q(|x|_p^\alpha dx)} \\
&= \left\{ \int_{|x|_p \geq 1} \left(|x|_p^{-(n/q) - (\alpha/q) - (q_2\epsilon/q)} \right. \right. \\
&\quad \left. \left. \times \int_{|(y_1, y_2)|_p \leq 1, |y_1|_p \geq 1/|x|_p, |y_2|_p \geq 1/|x|_p} \prod_{i=1}^2 |y_i|_p^{-(n/q_i) - (\alpha_i/q) - (q_2\epsilon/q_i)} dy_1 dy_2 \right)^q |x|_p^\alpha dx \right\}^{1/q} \\
&\geq \left\{ \int_{|x|_p \geq |\epsilon|_p} \left(|x|_p^{-(n/q) - (\alpha/q) - (q_2\epsilon/q)} \right. \right. \\
&\quad \left. \left. \times \int_{|(y_1, y_2)|_p \leq 1, |y_1|_p \geq 1/|\epsilon|_p, |y_2|_p \geq 1/|\epsilon|_p} \prod_{i=1}^2 |y_i|_p^{-(n/q_i) - (\alpha_i/q) - (q_2\epsilon/q_i)} dy_1 dy_2 \right)^q |x|_p^\alpha dx \right\}^{1/q} \\
&= \left(\int_{|(y_1, y_2)|_p \leq 1, |y_1|_p \geq 1/|\epsilon|_p, |y_2|_p \geq 1/|\epsilon|_p} \prod_{i=1}^2 |y_i|_p^{-(n/q_i) - (\alpha_i/q) - (q_2\epsilon/q_i)} dy_1 dy_2 \right) \left(\int_{|x|_p \geq |\epsilon|_p} |x|_p^{-n - \epsilon q_2} dx \right)^{1/q} \\
&= \left(\int_{|(y_1, y_2)|_p \leq 1, |y_1|_p \geq 1/|\epsilon|_p, |y_2|_p \geq 1/|\epsilon|_p} \prod_{i=1}^2 |y_i|_p^{-(n/q_i) - (\alpha_i/q) - (q_2\epsilon/q_i)} dy_1 dy_2 \right) |\epsilon|_p^{-\epsilon q_2/q} \prod_{i=1}^2 \|f_i^\epsilon\|_{L^{q_i}(|x|_p^{\alpha_i q_i/q} dx)}. \tag{2.15}
\end{aligned}$$

By the same calculation as that in (2.10), we obtain that

$$\begin{aligned}
& \int_{|(y_1, y_2)|_p \leq 1, |y_1|_p \geq 1/|\epsilon|_p, |y_2|_p \geq 1/|\epsilon|_p} \prod_{i=1}^2 |y_i|_p^{-(n/q_i) - (\alpha_i/q) - (q_2\epsilon/q_i)} dy_1 dy_2 \\
&= \frac{(1 - p^{-n})^2 \left[1 - (p|\epsilon|_p)^{(n/q) + (\alpha/q) + (q_2\epsilon/q) - 2n} \right]}{\prod_{i=1}^2 (1 - p^{(n/q_i) + (\alpha_i/q) + (q_2\epsilon/q_i) - n})}. \tag{2.16}
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \frac{|\epsilon|_p^{-\epsilon q_2/q} (1 - p^{-n})^2 \left[1 - (p|\epsilon|_p)^{(n/q) + (\alpha/q) + (q_2\epsilon/q) - 2n} \right]}{\prod_{i=1}^2 (1 - p^{(n/q_i) + (\alpha_i/q) + (q_2\epsilon/q_i) - n})} \\
&\leq \left\| \mathcal{H}_2^p \right\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx) \times L^{q_2}(|x|_p^{\alpha_2 q_2/q} dx) \rightarrow L^q(|x|_p^\alpha dx)}. \tag{2.17}
\end{aligned}$$

Now take $\epsilon = p^{-k}$, $k = 1, 2, 3, \dots$. Then $|\epsilon|_p = p^k > 1$. Letting k approach to ∞ , then ϵ approaches to 0 and $|\epsilon|_p^{-\epsilon q_2/q}$ approaches to 1. Since $\alpha_i < qn(1 - (1/q_i))$, $i = 1, 2$, we have

$$\frac{(1 - p^{-n})^2}{\prod_{i=1}^2 (1 - p^{(n/q_i) + (\alpha_i/q) - n})} \leq \left\| \mathcal{L}_2^p \right\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx) \times L^{q_2}(|x|_p^{\alpha_2 q_2/q} dx) \rightarrow L^q(|x|_p^\alpha dx)}. \quad (2.18)$$

Then (2.11) and (2.18) imply that

$$\left\| \mathcal{L}_2^p \right\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx) \times L^{q_2}(|x|_p^{\alpha_2 q_2/q} dx) \rightarrow L^q(|x|_p^\alpha dx)} = \frac{(1 - p^{-n})^2}{\prod_{i=1}^2 (1 - p^{(n/q_i) + (\alpha_i/q) - n})}. \quad (2.19)$$

(II) When $m \geq 3$

The proof of the upper bound in this case is similar to that of the previous case, and we can obtain that

$$\left\| \mathcal{L}_m^p(f_1, \dots, f_m) \right\|_{L^q(|x|_p^\alpha dx)} \leq C_{\mathcal{L}} \|f_1\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx)} \cdots \|f_m\|_{L^{q_m}(|x|_p^{\alpha_m q_m/q} dx)}, \quad (2.20)$$

where

$$C_{\mathcal{L}} = \int_{|(z_1, \dots, z_m)|_p \leq 1} \prod_{k=1}^m |z_k|_p^{-(n/q_k) - (\alpha_k/q)} dz_1 \cdots dz_m. \quad (2.21)$$

Let

$$\begin{aligned} D_1 &= \left\{ (z_1, \dots, z_m) \in \mathbb{Q}_p^n \times \cdots \times \mathbb{Q}_p^n \mid |z_1|_p \leq 1, |z_k|_p \leq |z_1|_p, 1 < k \leq m \right\}, \\ D_i &= \left\{ (z_1, \dots, z_m) \in \mathbb{Q}_p^n \times \cdots \times \mathbb{Q}_p^n \mid |z_i|_p \leq 1, |z_j|_p < |z_i|_p, |z_k|_p \leq |z_i|_p, 1 \leq j < i < k \leq m \right\}, \\ D_m &= \left\{ (z_1, \dots, z_m) \in \mathbb{Q}_p^n \times \cdots \times \mathbb{Q}_p^n \mid |z_m|_p \leq 1, |z_j|_p < |z_m|_p, 1 \leq j < m \right\}. \end{aligned} \quad (2.22)$$

It is clear that

$$\bigcup_{j=1}^m D_j = \left\{ (z_1, \dots, z_m) \in \mathbb{Q}_p^n \times \cdots \times \mathbb{Q}_p^n \mid |(z_1, \dots, z_m)|_p \leq 1 \right\}, \quad (2.23)$$

and $D_i \cap D_j = \emptyset$. Then

$$C_{\mathcal{L}} = \sum_{j=1}^m \int_{D_j} \prod_{k=1}^m |z_k|_p^{-(n/q_k) - (\alpha_k/q)} dz_1 \cdots dz_m := \sum_{j=1}^m I_j. \quad (2.24)$$

Now let us calculate I_j , $j = 1, 2, \dots, m$, respectively,

$$\begin{aligned}
 I_1 &= \int_{D_1} \prod_{k=1}^m |z_k|_p^{-(n/q_k) - (\alpha_k/q)} dz_1 \cdots dz_m \\
 &= \int_{|z_1|_p \leq 1} |z_1|_p^{-(n/q_1) - (\alpha_1/q)} \left(\prod_{k=2}^m \int_{|z_k|_p \leq |z_1|_p} |z_k|_p^{-(n/q_k) - (\alpha_k/q)} dz_k \right) dz_1 \\
 &= \int_{|z_1|_p \leq 1} |z_1|_p^{-(n/q_1) - (\alpha_1/q)} \left(\prod_{k=2}^m \left(\sum_{j=-\infty}^{\log_p |z_1|_p} \int_{S_j} |z_k|_p^{-(n/q_k) - (\alpha_k/q)} dz_k \right) \right) dz_1 \quad (2.25) \\
 &= \frac{(1 - p^{-n})^{m-1}}{\prod_{k=2}^m (1 - p^{(n/q_k) + (\alpha_k/q) - n})} \int_{|z_1|_p \leq 1} |z_1|_p^{-(n/q) - (\alpha/q) + (m-1)n} dz_1 \\
 &= \frac{(1 - p^{-n})^m}{(1 + p^{(n/q) + (\alpha/q) - mn}) \prod_{k=2}^m (1 - p^{(n/q_k) + (\alpha_k/q) - n})}.
 \end{aligned}$$

Similarly, for $i = 2, \dots, m-1$, we have

$$\begin{aligned}
 I_i &= \int_{D_i} \prod_{k=1}^m |z_k|_p^{-(n/q_k) - (\alpha_k/q)} dz_1 \cdots dz_m \\
 &= \int_{|z_i|_p \leq 1} |z_i|_p^{-(n/q_i) - (\alpha_i/q)} \left(\prod_{j=1}^{i-1} \int_{|z_j|_p < |z_i|_p} |z_j|_p^{-(n/q_j) - (\alpha_j/q)} dz_j \right) \\
 &\quad \times \left(\prod_{k=i+1}^m \int_{|z_k|_p \leq |z_i|_p} |z_k|_p^{-(n/q_k) - (\alpha_k/q)} dz_k \right) dz_i \\
 &= \frac{(1 - p^{-n})^{m-1} \prod_{j=1}^{i-1} p^{(n/q_j) + (\alpha_j/q) - n}}{\prod_{1 \leq k \leq m, k \neq i} (1 - p^{(n/q_k) + (\alpha_k/q) - n})} \int_{|z_i|_p \leq 1} |z_i|_p^{-(n/q) - (\alpha/q) + (m-1)n} dz_i \\
 &= \frac{(1 - p^{-n})^m \prod_{j=1}^{i-1} p^{(n/q_j) + (\alpha_j/q) - n}}{(1 + p^{(n/q) + (\alpha/q) - mn}) \prod_{1 \leq k \leq m, k \neq i} (1 - p^{(n/q_k) + (\alpha_k/q) - n})}, \\
 I_m &= \int_{|z_m|_p \leq 1} |z_m|_p^{-(n/q_m) - (\alpha_m/q)} \left(\prod_{j=1}^{m-1} \int_{|z_j|_p < |z_m|_p} |z_j|_p^{-(n/q_j) - (\alpha_j/q)} dz_j \right) dz_m \\
 &= \frac{(1 - p^{-n})^m \prod_{j=1}^{m-1} p^{(n/q_j) + (\alpha_j/q) - n}}{(1 + p^{(n/q) + (\alpha/q) - mn}) \prod_{j=1}^{m-1} (1 - p^{(n/q_j) + (\alpha_j/q) - n})}. \quad (2.26)
 \end{aligned}$$

Therefore,

$$\begin{aligned}
C_{\mathcal{L}} &= \frac{(1-p^{-n})^m}{(1+p^{(n/q)+(\alpha/q)-mn}) \prod_{k=2}^m (1-p^{(n/q_k)+(\alpha_k/q)-n})} \\
&+ \sum_{i=2}^{m-1} \frac{(1-p^{-n})^m \prod_{j=1}^{m-1} p^{(n/q_j)+(\alpha_j/q)-n}}{(1+p^{(n/q)+(\alpha/q)-mn}) \prod_{k=1}^{m-1} (1-p^{(n/q_k)+(\alpha_k/q)-n})} \\
&+ \frac{(1-p^{-n})^m \prod_{j=1}^{m-1} p^{(n/q_j)+(\alpha_j/q)-n}}{(1+p^{(n/q)+(\alpha/q)-mn}) \prod_{j=1}^{m-1} (1-p^{(n/q_j)+(\alpha_j/q)-n})} \\
&= \frac{(1-p^{-n})^m}{\prod_{i=1}^m (1-p^{(n/q_i)+(\alpha_i/q)-n})}.
\end{aligned} \tag{2.27}$$

To show that $C_{\mathcal{L}}$ is the best constant, we should prove that it is also the lower bound of the norm of \mathcal{L}_m^p from $L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx) \times \dots \times L^{q_m}(|x|_p^{\alpha_m q_m/q} dx)$ to $L^q(|x|_p^\alpha dx)$. For $0 < \epsilon < 1$ and $|\epsilon|_p > 1$, we take

$$f_i^\epsilon(x_i) = \begin{cases} 0, & |x_i|_p < 1, \\ |x_i|_p^{-(n/q_i)-(\alpha_i/q)-(q_m \epsilon/q_i)}, & |x_i|_p \geq 1, \end{cases} \quad i = 1, 2, \dots, m. \tag{2.28}$$

By simple calculation, we have

$$\|f_1^\epsilon\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx)}^{q_1} = \dots = \|f_m^\epsilon\|_{L^{q_m}(|x|_p^{\alpha_m q_m/q} dx)}^{q_m} = \frac{1-p^{-n}}{1-p^{-\epsilon q_m}}. \tag{2.29}$$

And when $|x|_p < 1$, $\mathcal{L}_m^p(f_1^\epsilon, \dots, f_m^\epsilon)(x) = 0$. But when $|x|_p \geq 1$,

$$\begin{aligned}
&\mathcal{L}_m^p(f_1^\epsilon, \dots, f_m^\epsilon)(x) \\
&= |x|_p^{-(n/q)-(\alpha/q)-(q_m \epsilon/q)} \int_{|(y_1, \dots, y_m)|_p \leq 1, |y_1|_p \geq 1/|x|_p, \dots, |y_m|_p \geq 1/|x|_p} \prod_{i=1}^m |y_i|_p^{-(n/q_i)-(\alpha_i/q)-(q_2 \epsilon/q_i)} dy_1 \dots dy_m.
\end{aligned} \tag{2.30}$$

Then by the similar discussion to that in previous case, we can obtain that

$$\left\| \mathcal{L}_m^p \right\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx) \times \dots \times L^{q_m}(|x|_p^{\alpha_m q_m/q} dx) \rightarrow L^q(|x|_p^\alpha dx)} \geq C_{\mathcal{L}}. \tag{2.31}$$

Theorem 2.1 is established by (2.20), (2.27), and (2.31). \square

3. Sharp Estimate of m -Linear p -Adic Hardy-Littlewood-Pólya Operator

We get the following best estimate of m -linear p -adic Hardy-Littlewood-Pólya operator on Lebesgue spaces with power weights.

Theorem 3.1. *Let $m \in \mathbb{Z}^+$, $f_i \in L_i^q(|x|_p^{\alpha_i q_i/q} dx)$, $1 < q_i < \infty$, $i = 1, 2, \dots, m$, $1 \leq q < \infty$, $1/q = \sum_{i=1}^m 1/q_i$, $\alpha_i < q(1 - (1/q_i))$ and $\alpha = \sum_{i=1}^m \alpha_i$. Then*

$$\|T_m^p(f_1, \dots, f_m)\|_{L^q(|x|_p^\alpha dx)} = C_T \|f_1\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx)} \cdots \|f_m\|_{L^{q_m}(|x|_p^{\alpha_m q_m/q} dx)}, \tag{3.1}$$

where

$$C_T = \frac{(1 - p^{-1})^m (1 - q^{-m})}{(1 - p^{-(1/q) - (\alpha/q)}) \prod_{i=1}^m (1 - p^{(1/q_i) + (\alpha_i/q) - 1})}, \tag{3.2}$$

is the best constant.

In particular, when $\alpha = 0$, we obtain the norm of the m -linear p -adic Hardy-Littlewood-Pólya operator on Lebesgue spaces.

Corollary 3.2. *Let $m \in \mathbb{Z}^+$, $1 < q_i < \infty$, $i = 1, 2, \dots, m$, $1 \leq q < \infty$ and $1/q = \sum_{i=1}^m 1/q_i$. Then*

$$\|T_m^p\|_{L^{q_1}(\mathbb{Q}_p) \times \dots \times L^{q_m}(\mathbb{Q}_p) \rightarrow L^q(\mathbb{Q}_p)} = \frac{(1 - p^{-1})^m (1 - q^{-m})}{(1 - p^{-1/q}) \prod_{i=1}^m (1 - p^{(1/q_i) - 1})}. \tag{3.3}$$

Proof of Theorem 3.1. Just as the proof of Theorem 2.1, we will only give the proof of case when $m \geq 2$.

(I) *Case $m = 2$*

By definition and the change of variables $y_i = xz_i$, $i = 1, 2$, we have

$$\begin{aligned} \|T_2^p(f_1, f_2)\|_{L^q(|x|_p^\alpha dx)} &= \left(\int_{\mathbb{Q}_p} \left| \iint_{\mathbb{Q}_p} \frac{f_1(y_1) f_2(y_2)}{[\max(|x|_p, |y_1|_p, |y_2|_p)]^2} dy_1 dy_2 \right|^q |x|_p^\alpha dx \right)^{1/q} \\ &\leq \left(\int_{\mathbb{Q}_p} \left(\iint_{\mathbb{Q}_p} \frac{|f_1(y_1) f_2(y_2)|}{[\max(|x|_p, |y_1|_p, |y_2|_p)]^2} dy_1 dy_2 \right)^q |x|_p^\alpha dx \right)^{1/q} \\ &= \left(\int_{\mathbb{Q}_p} \left(\iint_{\mathbb{Q}_p} \frac{|f_1(xz_1) f_2(xz_2)|}{[\max(1, |z_1|_p, |z_2|_p)]^2} dz_1 dz_2 \right)^q |x|_p^\alpha dx \right)^{1/q}. \end{aligned} \tag{3.4}$$

By Minkowski's integral inequality and Hölder's inequality $((q/q_1) + (q/q_2) = 1)$, we get

$$\begin{aligned}
 \|T_2^p(f_1, f_2)\|_{L^q(|x|_p^\alpha dx)} &\leq \iint_{\mathbb{Q}_p} \left(\int_{\mathbb{Q}_p} |f_1(xz_1)f_2(xz_2)|^q |x|_p^\alpha dx \right)^{1/q} \frac{1}{[\max(1, |z_1|_p, |z_2|_p)]^2} dz_1 dz_2 \\
 &\leq \iint_{\mathbb{Q}_p} \prod_{i=1}^2 \left(\int_{\mathbb{Q}_p} |f_i(xz_i)|^{q_i} |x|_p^{\alpha_i q_i/q} dx \right)^{1/q_i} \frac{1}{[\max(1, |z_1|_p, |z_2|_p)]^2} dz_1 dz_2 \\
 &\leq \left(\iint_{\mathbb{Q}_p} \frac{\prod_{i=1}^2 |z_i|_p^{-(1/q_i)-(\alpha_i/q)}}{[\max(1, |z_1|_p, |z_2|_p)]^2} dz_1 dz_2 \right) \prod_{i=1}^2 \|f_i\|_{L^{q_i}(|x|_p^{\alpha_i q_i/q} dx)}.
 \end{aligned} \tag{3.5}$$

By calculation, we have

$$\begin{aligned}
 &\iint_{\mathbb{Q}_p} \frac{\prod_{i=1}^2 |z_i|_p^{-(1/q_i)-(\alpha_i/q)}}{[\max(1, |z_1|_p, |z_2|_p)]^2} dz_1 dz_2 \\
 &= \int_{|z_1|_p \leq 1} \int_{|z_2|_p \leq 1} \prod_{i=1}^2 |z_i|_p^{-(1/q_i)-(\alpha_i/q)} dz_1 dz_2 \\
 &\quad + \int_{|z_1|_p > 1} \int_{|z_2|_p \leq |z_1|_p} |z_1|_p^{-(1/q_1)-(\alpha_1/q)-2} |z_2|_p^{-(1/q_2)-(\alpha_2/q)} dz_1 dz_2 \\
 &\quad + \int_{|z_2|_p > 1} \int_{|z_1|_p < |z_2|_p} |z_1|_p^{-(1/q_1)-(\alpha_1/q)} |z_2|_p^{-(1/q_2)-(\alpha_2/q)-2} dz_1 dz_2 \\
 &:= L_0 + L_1 + L_2.
 \end{aligned} \tag{3.6}$$

By definition,

$$\begin{aligned}
 L_0 &= \int_{|z_1|_p \leq 1} \int_{|z_2|_p \leq 1} \prod_{i=1}^2 |z_i|_p^{-(1/q_i)-(\alpha_i/q)} dz_1 dz_2 \\
 &= \frac{(1-p^{-1})^2}{(1-p^{(1/q_1)+(\alpha_1/q)-1})(1-p^{(1/q_2)+(\alpha_2/q)-1})}, \\
 L_1 &= \int_{|z_1|_p > 1} \int_{|z_2|_p \leq |z_1|_p} |z_1|_p^{-(1/q_1)-(\alpha_1/q)-2} |z_2|_p^{-(1/q_2)-\alpha_2/q} dz_1 dz_2 \\
 &= \frac{(1-p^{-1})}{1-p^{(1/q_2)+(\alpha_2/q)-1}} \int_{|z_1|_p > 1} |z_1|_p^{-(1/q_1)-(\alpha_1/q)-1} dz_1 \\
 &= \frac{(1-p^{-1})^2 p^{-(1/q_1)-(\alpha_1/q)}}{(1-p^{(1/q_2)+(\alpha_2/q)-1})(1-p^{-(1/q_1)-(\alpha_1/q)})}.
 \end{aligned} \tag{3.7}$$

Similarly,

$$\begin{aligned} L_2 &= \int_{|z_2|_p > 1} \int_{|z_1|_p < |z_2|_p} |z_1|_p^{-(1/q_1) - (\alpha_1/q)} |z_2|_p^{-(1/q_2) - (\alpha_2/q) - 2} dz_1 dz_2 \\ &= \frac{(1-p^{-1})^2 p^{(1/q_1) + (\alpha_1/q) - 1} p^{-(1/q) - (\alpha/q)}}{(1-p^{(1/q_1) + (\alpha_1/q) - 1}) (1-p^{-(1/q) - (\alpha/q)})}. \end{aligned} \quad (3.8)$$

Substituting (3.7) and (3.8) into (3.6), we get

$$\iint_{\mathbb{Q}_p} \frac{\prod_{i=1}^2 |z_i|_p^{-(1/q_i) - (\alpha_i/q)}}{\left[\max(1, |z_1|_p, |z_2|_p) \right]^2} dz_1 dz_2 = \frac{(1-p^{-1})^2 (1-p^{-2})}{(1-p^{-(1/q) - (\alpha/q)}) \prod_{i=1}^2 (1-p^{(1/q_i) + (\alpha_i/q) - 1})}. \quad (3.9)$$

Then (3.5) and (3.9) imply that

$$\|T_2^p\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx) \times L^{q_2}(|x|_p^{\alpha_2 q_2/q} dx) \rightarrow L^q(|x|_p^\alpha dx)} \leq \frac{(1-p^{-1})^2 (1-p^{-2})}{(1-p^{-(1/q) - (\alpha/q)}) \prod_{i=1}^2 (1-p^{(1/q_i) + (\alpha_i/q) - 1})}. \quad (3.10)$$

On the other hand, for $0 < \epsilon < 1$ and $|\epsilon|_p > 1$, we take

$$f_i^\epsilon(x_i) = \begin{cases} 0, & |x_i|_p < 1, \\ |x_i|_p^{-(1/q_i) - (\alpha_i/q) - (q_2 \epsilon / q_i)}, & |x_i|_p \geq 1, \end{cases} \quad i = 1, 2. \quad (3.11)$$

Then

$$\|f_1^\epsilon\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx)}^{q_1} = \|f_2^\epsilon\|_{L^{q_2}(|x|_p^{\alpha_2 q_2/q} dx)}^{q_2} = \frac{1-p^{-1}}{1-p^{-\epsilon q_2}}. \quad (3.12)$$

Since $|\epsilon|_p > 1$, we have

$$\begin{aligned} &\|T_2^p(f_1^\epsilon, f_2^\epsilon)\|_{L^q(|x|_p^\alpha dx)} \\ &= \left(\int_{\mathbb{Q}_p} \left| \iint_{\mathbb{Q}_p} \frac{f_1^\epsilon(x_1) f_2^\epsilon(x_2)}{\left[\max(|x|_p, |x_1|_p, |x_2|_p) \right]^2} dx_1 dx_2 \right|^q |x|_p^\alpha dx \right)^{1/q} \\ &\geq \left(\int_{|x|_p \geq 1} \left(\int_{|x_1|_p \geq 1} \int_{|x_2|_p \geq 1} \frac{\prod_{i=1}^2 |x_i|_p^{-(1/q_i) - (\alpha_i/q) - (q_2 \epsilon / q_i)}}{\left[\max(|x|_p, |x_1|_p, |x_2|_p) \right]^2} dx_1 dx_2 \right)^q |x|_p^\alpha dx \right)^{1/q} \\ &= \left(\int_{|x|_p \geq 1} \left(\int_{|y_1|_p \geq 1/|x|_p} \int_{|y_2|_p \geq 1/|x|_p} \frac{\prod_{i=1}^2 |y_i|_p^{-(1/q_i) - (\alpha_i/q) - (q_2 \epsilon / q_i)}}{\left[\max(1, |y_1|_p, |y_2|_p) \right]^2} dy_1 dy_2 \right)^q |x|_p^{-1 - q_2 \epsilon} dx \right)^{1/q} \end{aligned}$$

$$\begin{aligned}
&\geq \left(\int_{|x|_p \geq |\epsilon|_p} \left(\int_{|y_1|_p \geq 1/|\epsilon|_p} \int_{|y_2|_p \geq 1/|\epsilon|_p} \frac{\prod_{i=1}^2 |y_i|_p^{-(1/q_i) - (\alpha_i/q) - (q_2\epsilon/q_i)}}{\left[\max(1, |y_1|_p, |y_2|_p) \right]^2} dy_1 dy_2 \right)^q |x|_p^{-1-q_2\epsilon} dx \right)^{1/q} \\
&= \prod_{i=1}^2 \|f_i^\epsilon\|_{L^{q_i}(|x|_p^{\alpha_i q_i/q} dx)} |\epsilon|_p^{-q_2\epsilon/q} \int_{|y_1|_p \geq 1/|\epsilon|_p} \int_{|y_2|_p \geq 1/|\epsilon|_p} \frac{\prod_{i=1}^2 |y_i|_p^{-(1/q_i) - (\alpha_i/q) - (q_2\epsilon/q_i)}}{\left[\max(1, |y_1|_p, |y_2|_p) \right]^2} dy_1 dy_2.
\end{aligned} \tag{3.13}$$

Therefore,

$$\begin{aligned}
&\|T_2^p\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx) \times L^{q_2}(|x|_p^{\alpha_2 q_2/q} dx) \rightarrow L^q(|x|_p^\alpha dx)} \\
&\geq |\epsilon|_p^{-q_2\epsilon/q} \int_{|y_1|_p \geq 1/|\epsilon|_p} \int_{|y_2|_p \geq 1/|\epsilon|_p} \frac{\prod_{i=1}^2 |y_i|_p^{-(1/q_i) - (\alpha_i/q) - (q_2\epsilon/q_i)}}{\left[\max(1, |y_1|_p, |y_2|_p) \right]^2} dy_1 dy_2.
\end{aligned} \tag{3.14}$$

As the calculation of (3.6)–(3.8), we obtain that

$$\begin{aligned}
&\int_{|y_1|_p \geq 1/|\epsilon|_p} \int_{|y_2|_p \geq 1/|\epsilon|_p} \frac{\prod_{i=1}^2 |y_i|_p^{-(1/q_i) - (\alpha_i/q) - (q_2\epsilon/q_i)}}{\left[\max(1, |y_1|_p, |y_2|_p) \right]^2} dy_1 dy_2 \\
&= \frac{(1-p^{-1})^2 \prod_{i=1}^2 \left[1 - (p|\epsilon|_p)^{(1/q_i) + (\alpha_i/q) + (q_2\epsilon/q_i) - 1} \right]}{\prod_{i=1}^2 (1 - p^{(1/q_i) + (\alpha_i/q) + (q_2\epsilon/q_i) - 1})} \\
&\quad + \frac{(1-p^{-1})^2 p^{-(1/q) - (\alpha/q) - (q_2\epsilon/q)}}{(1 - p^{(1/q_2) + (\alpha_2/q) + \epsilon - 1}) (1 - p^{-(1/q) - (\alpha/q) - (q_2\epsilon/q)})} \\
&\quad + \frac{(1-p^{-1})^2 p^{(1/q_1) + (\alpha_1/q) + (q_2\epsilon/q_1) - 1} p^{-(1/q) - (\alpha/q) - (q_2\epsilon/q)}}{(1 - p^{(1/q_1) + (\alpha_1/q) + (q_2\epsilon/q_1) - 1}) (1 - p^{-(1/q) - (\alpha/q) - (q_2\epsilon/q)})}.
\end{aligned} \tag{3.15}$$

Now take $\epsilon = p^{-k}$, $k \in \mathbb{Z}^+$ and let k approach to ∞ , then by (3.9), (3.14), (3.15), and the fact that $\alpha_i < qn(1 - (1/q_i))$, $i = 1, 2$, we have

$$\begin{aligned}
&\|T_2^p\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx) \times L^{q_2}(|x|_p^{\alpha_2 q_2/q} dx) \rightarrow L^q(|x|_p^\alpha dx)} \\
&\geq \frac{(1-p^{-1})^2 (1-q^{-2})}{(1-p^{-(1/q) - (\alpha/q)}) \prod_{i=1}^2 (1 - p^{(1/q_i) + (\alpha_i/q) - 1})}.
\end{aligned} \tag{3.16}$$

Then by (3.10) and (3.16), we get

$$\|T_2^p\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx) \times L^{q_2}(|x|_p^{\alpha_2 q_2/q} dx) \rightarrow L^q(|x|_p^\alpha dx)} = \frac{(1-p^{-1})^2 (1-q^{-2})}{(1-p^{-(1/q) - (\alpha/q)}) \prod_{i=1}^2 (1 - p^{(1/q_i) + (\alpha_i/q) - 1})}. \tag{3.17}$$

(II) Case $m \geq 3$

The upper bound estimate for the norm can be obtained by the same way as that when $m = 2$, and we can obtain that

$$\|T_m^p(f_1, \dots, f_m)\|_{L^q(|x|_p^\alpha dx)} \leq C_T \prod_{i=1}^m \|f_i\|_{L^{q_i}(|x|_p^{\alpha_i q_i/q} dx)}, \quad (3.18)$$

where

$$C_T = \int_{|(z_1, \dots, z_m)|_p \leq 1} \frac{\prod_{j=1}^m |z_j|_p^{-(1/q_j) - (\alpha_j/q)}}{[\max(1, |z_1|_p, \dots, |z_m|_p)]^m} dz_1 \cdots dz_m. \quad (3.19)$$

Let

$$\begin{aligned} E_0 &= \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p, \\ E_1 &= \{(z_1, \dots, z_m) \in \mathbb{Q}_p \times \cdots \times \mathbb{Q}_p \mid |z_1|_p > 1, |z_k|_p \leq |z_1|_p, 1 < k \leq m\}, \\ E_i &= \{(z_1, \dots, z_m) \in \mathbb{Q}_p \times \cdots \times \mathbb{Q}_p \mid |z_i|_p > 1, |z_j|_p < |z_i|_p, |z_k|_p \leq |z_i|_p, 1 \leq j < i < k \leq m\}, \\ E_m &= \{(z_1, \dots, z_m) \in \mathbb{Q}_p \times \cdots \times \mathbb{Q}_p \mid |z_m|_p > 1, |z_j|_p < |z_m|_p, 1 \leq j < m\}. \end{aligned} \quad (3.20)$$

Obviously,

$$\bigcup_{k=1}^m E_k = \mathbb{Q}_p \times \cdots \times \mathbb{Q}_p, \quad E_i \cap E_j = \emptyset, \quad i \neq j, \quad 1 \leq i, j \leq m. \quad (3.21)$$

Then

$$C_T = \sum_{k=0}^m \int_{E_k} \frac{\prod_{j=1}^m |z_j|_p^{-(1/q_j) - (\alpha_j/q)}}{[\max(1, |z_1|_p, \dots, |z_m|_p)]^m} dz_1 \cdots dz_m := \sum_{k=0}^m J_k. \quad (3.22)$$

Now let us calculate $J_k, k = 0, 1, \dots, m$, respectively,

$$\begin{aligned} J_0 &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \prod_{j=1}^m |z_j|_p^{-(1/q_j) - (\alpha_j/q)} dz_1 \cdots dz_m \\ &= \prod_{j=1}^m \int_{\mathbb{Z}_p} |z_j|_p^{-(1/q_j) - (\alpha_j/q)} dz_j = \prod_{j=1}^m \left(\sum_{k=-\infty}^0 \int_{S_k} |z_j|_p^{-(1/q_j) - (\alpha_j/q)} dz_j \right) \\ &= \frac{(1-p^{-1})^m}{\prod_{j=1}^m (1-p^{(1/q_j) + (\alpha_j/q) - 1})}, \end{aligned}$$

$$\begin{aligned}
J_1 &= \int_{|z_1|_p > 1} \int_{|z_2|_p \leq |z_1|_p} \cdots \int_{|z_m|_p \leq |z_1|_p} |z_1|_p^{-(1/q_1) - (\alpha_1/q) - m} \prod_{j=2}^m |z_j|_p^{-(1/q_j) - (\alpha_j/q)} dz_1 \cdots dz_m \\
&= \int_{|z_1|_p > 1} |z_1|_p^{-(1/q_1) - (\alpha_1/q) - m} \left(\prod_{j=2}^m \int_{|z_j|_p \leq |z_1|_p} |z_j|_p^{-(1/q_j) - (\alpha_j/q)} dz_j \right) dz_1 \\
&= \int_{|z_1|_p > 1} |z_1|_p^{-(1/q_1) - (\alpha_1/q) - m} \prod_{j=2}^m \left(\frac{(1-p^{-1})|z_1|_p^{-(1/q_j) - (\alpha_j/q) + 1}}{1 - p^{(1/q_j) + (\alpha_j/q) - 1}} \right) dz_1 \\
&= \frac{(1-p^{-1})^{m-1}}{\prod_{j=2}^m (1 - p^{(1/q_j) + (\alpha_j/q) - 1})} \int_{|z_1|_p > 1} |z_1|_p^{-(1/q) - (\alpha/q) - 1} dz_1 \\
&= \frac{(1-p^{-1})^{m-1} p^{-(1/q) - (\alpha/q)}}{\prod_{j=2}^m (1 - p^{(1/q_j) + (\alpha_j/q) - 1}) (1 - p^{-(1/q) - (\alpha/q)})}.
\end{aligned} \tag{3.23}$$

Similar to J_1 , for $1 < i < m$, it is true that

$$\begin{aligned}
J_i &= \int_{|z_i|_p > 1} |z_i|_p^{-(1/q_i) - (\alpha_i/q) - m} \left(\prod_{j=1}^{i-1} \int_{|z_j|_p < |z_i|_p} |z_j|_p^{-(1/q_j) - (\alpha_j/q)} dz_j \right) \\
&\quad \times \left(\prod_{k=i+1}^m \int_{|z_k|_p \leq |z_i|_p} |z_k|_p^{-(1/q_k) - (\alpha_k/q)} dz_k \right) dz_i \\
&= \frac{(1-p^{-1})^{m-1} \prod_{j=1}^{i-1} p^{(1/q_j) + (\alpha_j/q) - 1}}{\prod_{1 \leq k \leq m, k \neq i} (1 - p^{(1/q_k) + (\alpha_k/q) - 1})} \int_{|z_i|_p > 1} |z_i|_p^{-(1/q) - (\alpha/q) - 1} dz_i \\
&= \frac{(1-p^{-1})^m \left(\prod_{j=1}^{i-1} p^{(1/q_j) + (\alpha_j/q) - 1} \right) p^{-(1/q) - (\alpha/q)}}{(1 - p^{-(1/q) - (\alpha/q)}) \prod_{1 \leq k \leq m, k \neq i} (1 - p^{(1/q_k) + (\alpha_k/q) - 1})},
\end{aligned} \tag{3.24}$$

$$\begin{aligned}
J_m &= \int_{|z_m|_p > 1} |z_m|_p^{-(1/q_m) - (\alpha_m/q) - m} \left(\prod_{j=1}^{m-1} \int_{|z_j|_p < |z_m|_p} |z_j|_p^{-(1/q_j) - (\alpha_j/q)} dz_j \right) dz_m \\
&= \frac{(1-p^{-1})^{m-1} \prod_{j=1}^{m-1} p^{(1/q_j) + (\alpha_j/q) - 1}}{\prod_{j=1}^{m-1} (1 - p^{(1/q_j) + (\alpha_j/q) - 1})} \int_{|z_m|_p > 1} |z_m|_p^{-(1/q) - (\alpha/q) - 1} dz_m \\
&= \frac{(1-p^{-1})^m \left(\prod_{j=1}^{m-1} p^{(1/q_j) + (\alpha_j/q) - 1} \right) p^{-(1/q) - (\alpha/q)}}{(1 - p^{-(1/q) - (\alpha/q)}) \prod_{j=1}^{m-1} (1 - p^{(1/q_j) + (\alpha_j/q) - 1})}.
\end{aligned}$$

Consequently, we have

$$C_T = \sum_{k=0}^m J_k = \frac{(1-p^{-1})^m (1-q^{-m})}{(1-p^{-(1/q)-(\alpha/q)}) \prod_{i=1}^m (1-p^{(1/q_i)+(\alpha_i/q)-1})}. \quad (3.25)$$

To obtain that C_T is also the lower bound, for $0 < \epsilon < 1$ and $|\epsilon|_p > 1$, we define

$$f_i^\epsilon = \begin{cases} 0, & |x_i|_p < 1, \\ |x_i|_p^{-(1/q_i)-(\alpha_i/q)-(q_2\epsilon/q_i)}, & |x_i|_p \geq 1, \end{cases} \quad i = 1, 2, \dots, m. \quad (3.26)$$

By the similar discussion to that in Case $m = 2$, we can also get that

$$\|T_m^p\|_{L^{q_1}(|x|_p^{\alpha_1/q_1} dx) \times \dots \times L^{q_m}(|x|_p^{\alpha_m/q_m} dx) \rightarrow L^q(|x|_p^\alpha dx)} \geq C_T. \quad (3.27)$$

Combining (3.18) with (3.27), we complete the proof of Theorem 3.1. \square

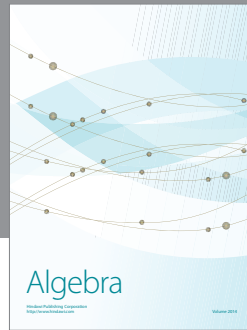
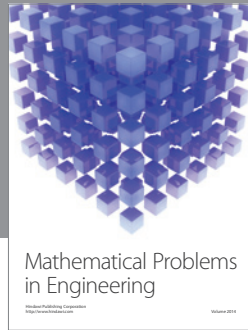
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