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Research Article

A Note on the q-Euler Numbers and Polynomials with Weak Weight α

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We construct a new type of q-Euler numbers and polynomials with weak weight α : $E_{n,q}^{(\alpha)}$, $E_{n,q}^{(\alpha)}(x)$, respectively. Some interesting results and relationships are obtained. Also, we observe the behavior of roots of the q-Euler numbers $E_{n,q}^{(\alpha)}$ and polynomials $E_{n,q}^{(\alpha)}(x)$ with weak weight α. By means of numerical experiments, we demonstrate a remarkably regular structure of the complex roots of q-Euler polynomials $E_{n,q}^{(\alpha)}$ with weak weight α.

1. Introduction

The Euler numbers and polynomials possess many interesting properties are arising in many areas of mathematics and physics. Recently, many mathematicians have studied the area of the q-Euler numbers and polynomials (see [1–19]). In this paper, we construct a new type of q-Euler numbers $E_{n,q}^{(\alpha)}$ and polynomials $E_{n,q}^{(\alpha)}(x)$ with weak weight α. The main purpose of this paper is also to investigate the zeros of the q-Euler polynomials $E_{n,q}^{(\alpha)}(x)$ with weak weight α. Furthermore, we give a table for the zeros of the q-Euler numbers and polynomials $E_{n,q}^{(\alpha)}(x)$ with weak weight α.

Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ denotes the ring of rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{C}$ denotes the set of complex numbers, and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-1/(p-1)}$ so that $q^x = \exp(x \log q)$ for $|x|_p \le 1$. Throughout this paper we use the notation

$$[x]_q = \frac{1-q^x}{1-q}, \qquad [x]_{-q} = \frac{1-(-q)^x}{1+q}. \tag{1.1}$$

(cf. [1–11, 15–18]). Hence, $\lim_{q\to 1}[x]_q = x$ for any x with $|x|_p \le 1$ in the present p-adic case. For

$$g \in UD(\mathbb{Z}_p) = \{g \mid g : \mathbb{Z}_p \longrightarrow \mathbb{C}_p \text{ is uniformly differentiable function}\}, \tag{1.2}$$

the fermionic p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-q}(g) = \int_{\mathbb{Z}_p} g(x)\,d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} g(x)(-q)^x. \tag{1.3}$$

(cf. [3–6]). If we take $g_1(x) = g(x+1)$ in (1.3), then we easily see that

$$qI_{-q}(g_1) + I_{-q}(g) = [2]_q g(0). \tag{1.4}$$

From (1.4), we obtain

$$q^n I_{-q}(g_n) + (-1)^{n-1} I_{-q}(g) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l g(l), \tag{1.5}$$

where $g_n(x) = g(x+n)$ (cf. [3–6]).

As well-known definition, the Euler polynomials are defined by

$$\begin{aligned} F(t) &= \frac{2}{e^t+1} = e^{Et} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \\ F(t,x) &= \frac{2}{e^t+1} e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \end{aligned} \tag{1.6}$$

with the usual convention of replacing $E^n(x)$ by $E_n(x)$. In the special case, $x=0$, $E_n(0) = E_n$ are called the nth Euler numbers (cf. [1–11]).

Our aim in this paper is to define q-Euler numbers $E_{n,q}^{(\alpha)}$ and polynomials $E_{n,q}^{(\alpha)}(x)$ with weak weight α. We investigate some properties which are related to q-Euler numbers $E_{n,q}^{(\alpha)}$ and polynomials $E_{n,q}^{(\alpha)}(x)$ with weak weight α. We also derive the existence of a specific interpolation function which interpolates q-Euler numbers $E_{n,q}^{(\alpha)}$ and polynomials $E_{n,q}^{(\alpha)}(x)$ with weak weight α at negative integers. Finally, we investigate the behavior of roots of the q-Euler polynomials $E_{n,q}^{(\alpha)}$ with weak weight α.

2. Basic Properties for q-Euler Numbers and Polynomials with Weak Weight α

Our primary goal of this section is to define q-Euler numbers $E_{n,q}^{(\alpha)}$ and polynomials $E_{n,q}^{(\alpha)}(x)$ with weak weight α. We also find generating functions of q-Euler numbers $E_{n,q}^{(\alpha)}$ and polynomials $E_{n,q}^{(\alpha)}(x)$ with weak weight α.

For $\alpha \in \mathbb{Z}$ and $q \in \mathbb{C}_p$ with $|1-q|_p \le 1$, q-Euler numbers $E_{n,q}^{(\alpha)}$ are defined by

$$E_{n,q}^{(\alpha)} = \int_{\mathbb{Z}_p} [x]_q^n d\mu_{-q^\alpha}(x). \tag{2.1}$$

By using p-adic q-integral on $\mathbb{Z}_p$, we obtain

$$\begin{aligned}
\int_{\mathbb{Z}_p} [x]_q^n d\mu_{-q^\alpha}(x) &= \lim_{N\to\infty} \frac{1}{[p^N]_{-q^\alpha}} \sum_{x=0}^{p^N-1} [x]_q^n (-q^\alpha)^x \\
&= [2]_{q^\alpha} \left(\frac{1}{1-q}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1+q^{\alpha+l}} \\
&= [2]_{q^\alpha} \sum_{m=0}^{\infty} (-1)^m q^{\alpha m} [m]_q^n.
\end{aligned} \tag{2.2}$$

By (2.1), we have

$$\begin{aligned}
E_{n,q}^{(\alpha)} &= [2]_{q^\alpha} \left(\frac{1}{1-q}\right)^n \sum_{l=0}^{n} \binom{n}{1} (-1)^l \frac{1}{1+q^{\alpha+l}} \\
&= [2]_{q^\alpha} \sum_{m=0}^{\infty} (-1)^m q^{\alpha m} [m]_q^n.
\end{aligned} \tag{2.3}$$

We set

$$F_q^{(\alpha)}(t) = \sum_{n=0}^{\infty} E_{n,q}^{(\alpha)} \frac{t^n}{n!}. \tag{2.4}$$

By using above equation and (2.2), we have

$$\begin{aligned}
F_q^{(\alpha)}(t) &= \sum_{n=0}^{\infty} E_{n,q}^{(\alpha)} \frac{t^n}{n!} \\
&= [2]_{q^\alpha} \sum_{n=0}^{\infty} \left(\left(\frac{1}{1-q}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1+q^{\alpha+l}} \right) \frac{t^n}{n!} \\
&= [2]_{q^\alpha} \sum_{m=0}^{\infty} (-1)^m q^{\alpha m} e^{[m]_q t}.
\end{aligned} \tag{2.5}$$

Thus q-Euler numbers with weak weight α, $E_{n,q}^{(\alpha)}$ are defined by means of the generating function

$$F_q^{(\alpha)}(t) = [2]_{q^\alpha} \sum_{m=0}^{\infty} (-1)^m q^{\alpha m} e^{[m]_q t}. \tag{2.6}$$

By using (2.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} E_{n,q}^{(\alpha)} \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} [x]_q^n d\mu_{-q^\alpha}(x) \frac{t^n}{n!} \\ &= \int_{\mathbb{Z}_p} e^{[x]_q t} d\mu_{-q^\alpha}(x).\end{aligned} \tag{2.7}$$

By (2.5), (2.7), we have

$$\int_{\mathbb{Z}_p} e^{[x]_q t} d\mu_{-q^\alpha}(x) = [2]_{q^\alpha} \sum_{m=0}^{\infty} (-1)^m q^{\alpha m} e^{[m]_q t}. \tag{2.8}$$

Next, we introduce q-Euler polynomials $E_{n,q}^{(\alpha)}(x)$ with weak weight α. The q-Euler polynomials $E_{n,q}^{(\alpha)}(x)$ with weak weight α are defined by

$$E_{n,q}^{(\alpha)}(x) = \int_{\mathbb{Z}_p} [x+y]_q^n d\mu_{-q^\alpha}(y). \tag{2.9}$$

By using p-adic q-integral, we obtain

$$E_{n,q}^{(\alpha)}(x) = [2]_{q^\alpha} \left(\frac{1}{1-q}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{xl} \frac{1}{1+q^{\alpha+l}}. \tag{2.10}$$

We set

$$F_q^{(\alpha)}(t,x) = \sum_{n=0}^{\infty} E_{n,q}^{(\alpha)}(x) \frac{t^n}{n!}. \tag{2.11}$$

By using (2.10) and (2.11), we obtain

$$F_q^{(\alpha)}(t,x) = \sum_{n=0}^{\infty} E_{n,q}^{(\alpha)}(x) \frac{t^n}{n!} = [2]_{q^\alpha} \sum_{m=0}^{\infty} (-1)^m q^{\alpha m} e^{[m+x]_q t}. \tag{2.12}$$

Obverse that if $q \to 1$, then $F_q^{(\alpha)}(t,x) \to F(t,x)$ and $F_q^{(\alpha)}(t) \to F(t)$.

Since $[x+y]_q = [x]_q + q^x [y]_q$, we easily obtain that

$$\begin{aligned} E_{n,q}^{(\alpha)}(x) &= \int_{\mathbb{Z}_p} [x+y]_q^n d\mu_{-q^\alpha}(y) \\ &= \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{xl} E_{l,q}^{(\alpha)} \\ &= \left([x]_q + q^x E_q^{(\alpha)}\right)^n \\ &= [2]_{q^\alpha} \sum_{m=0}^{\infty} (-1)^m q^{\alpha m} [x+m]_q^n. \end{aligned} \tag{2.13}$$

Observe that if $q \to 1$, then $E_{n,q}^{(\alpha)} \to E_n$ and $E_{n,q}^{(\alpha)}(x) \to E_n(x)$.
By (2.10), we have the following complement relation.

Theorem 2.1 (property of complement). *One has*

$$E_{n,q^{-1}}^{(\alpha)}(1-x) = (-1)^n q^n E_{n,q}^{(\alpha)}(x). \tag{2.14}$$

By (2.10), *we have the following distribution relation.*

Theorem 2.2 (distribution relation). *For any positive integer m(=odd), one has*

$$E_{n,q}^{(\alpha)}(x) = \frac{[2]_{q^\alpha}}{[2]_{q^{\alpha m}}} [m]_q^n \sum_{i=0}^{m-1} (-1)^i q^{\alpha i} E_{n,q^m}^{(\alpha)}\left(\frac{i+x}{m}\right), \quad n \in \mathbb{Z}_+. \tag{2.15}$$

By (1.5), (2.1), *and* (2.9), *we easily see that*

$$[2]_{q^\alpha} \sum_{l=0}^{n-1} (-1)^{n-1-l} q^{\alpha l} [l]_q^m = q^{\alpha n} E_{m,q}^{(\alpha)}(n) + (-1)^{n-1} E_{m,q}^{(\alpha)}. \tag{2.16}$$

Hence, we have the following theorem.

Theorem 2.3. *Let $m \in \mathbb{Z}_+$. If $n \equiv 0 (\mathrm{mod}\ 2)$, then*

$$q^{\alpha n} E_{m,q}^{(\alpha)}(n) - E_{m,q}^{(\alpha)} = [2]_{q^\alpha} \sum_{l=0}^{n-1} (-1)^{l+1} q^{\alpha l} [l]_q^m. \tag{2.17}$$

If $n \equiv 1 (\mathrm{mod}\ 2)$, then

$$q^{\alpha n} E_{m,q}^{(\alpha)}(n) + E_{m,q}^{(\alpha)} = [2]_{q^\alpha} \sum_{l=0}^{n-1} (-1)^{l} q^{\alpha l} [l]_q^m. \tag{2.18}$$

From (1.4), *one notes that*

$$
\begin{aligned}
[2]_{q^\alpha} &= q^\alpha \int_{\mathbb{Z}_p} e^{[x+1]_q t} d\mu_{-q^\alpha}(x) + \int_{\mathbb{Z}_p} e^{[x]_q t} d\mu_{-q^\alpha}(x) \\
&= \sum_{n=0}^{\infty} \left(q^\alpha \int_{\mathbb{Z}_p} [x+1]_q^n d\mu_{-q^\alpha}(x) + \int_{\mathbb{Z}_p} [x]_q^n d\mu_{-q^\alpha}(x) \right) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(q^\alpha E_{n,q}^{(\alpha)}(1) + E_{n,q}^{(\alpha)} \right) \frac{t^n}{n!}.
\end{aligned}
\tag{2.19}
$$

Therefore, we obtain the following theorem.

Theorem 2.4. *For $n \in \mathbb{Z}_+$, one has*

$$
q^\alpha E_{n,q}^{(\alpha)}(1) + E_{n,q}^{(\alpha)} = \begin{cases} [2]_{q^\alpha}, & \text{if } n = 0, \\ 0, & \text{if } n > 0. \end{cases}
\tag{2.20}
$$

By Theorem 2.4 and (2.13), *we have the following corollary.*

Corollary 2.5. *For $n \in \mathbb{Z}_+$, one has*

$$
q^\alpha \left(qE_q^{(\alpha)} + 1 \right)^n + E_{n,q}^{(\alpha)} = \begin{cases} [2]_{q^\alpha}, & \text{if } n = 0, \\ 0, & \text{if } n > 0, \end{cases}
\tag{2.21}
$$

with the usual convention of replacing $(E_q^{(\alpha)})^n$ by $E_{n,q}^{(\alpha)}$.

By (2.12), *one has*

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \left(q^\alpha E_{n,q}^{(\alpha)}(x+1) + E_{n,q}^{(\alpha)}(x) \right) \frac{t^n}{n!} \\
&\quad = [2]_{q^\alpha} q^\alpha \sum_{m=0}^{\infty} (-1)^m q^{\alpha m} e^{[m+1+x]_q t} + [2]_{q^\alpha} \sum_{m=0}^{\infty} (-1)^m q^{\alpha m} e^{[m+x]_q t} \\
&\quad = [2]_{q^\alpha} e^{[x]_q t} \\
&\quad = [2]_{q^\alpha} \sum_{n=0}^{\infty} [x]_q^n \frac{t^n}{n!}.
\end{aligned}
\tag{2.22}
$$

Hence we have the following difference equation.

Theorem 2.6 (difference equation). *For $n \in \mathbb{Z}_+$, one has*

$$
q^\alpha E_{n,q}^{(\alpha)}(x+1) + E_{n,q}^{(\alpha)}(x) = [2]_{q^\alpha} [x]_q^n.
\tag{2.23}
$$

Using q-Euler numbers and polynomials with weak weight α, q-Euler zeta function with weak weight α and Hurwitz q-Euler zeta functions with weak weight α are defined. These functions interpolate the q-Euler numbers and q-Euler polynomials with weak weight α, respectively. In this section we assume that $q \in \mathbb{C}$ with $|q| < 1$. From (2.6), we note that

$$\frac{d^k}{dt^k} F_q^{(\alpha)}(t)\bigg|_{t=0} = [2]_{q^\alpha} \sum_{n=1}^{\infty} (-1)^n q^{\alpha n} [n]_q^k, \quad (k \in \mathbb{N}). \tag{2.24}$$

Using the above equation, we are now ready to define q-Euler zeta functions.

Definition 2.7. Let $s \in \mathbb{C}$.

$$\zeta_q^{(\alpha)}(s) = [2]_{q^\alpha} \sum_{n=1}^{\infty} \frac{(-1)^n q^{\alpha n}}{[n]_q^s}. \tag{2.25}$$

Note that $\zeta_q^{(\alpha)}(s)$ is a meromorphic function on $\mathbb{C}$. Note that, if $q \to 1$, then $\zeta_q^{(\alpha)}(s) = \zeta(s)$ which is the Euler zeta functions. Relation between $\zeta_q^{(\alpha)}(s)$ and $E_{k,q}^{(\alpha)}$ is given by the following theorem.

Theorem 2.8. *For $k \in \mathbb{N}$, one has*

$$\zeta_q^{(\alpha)}(-k) = E_{k,q}^{(\alpha)}. \tag{2.26}$$

Observe that $\zeta_q^{(\alpha)}(s)$ function interpolates $E_{k,q}^{(\alpha)}$ numbers at nonnegative integers. By using (2.12), *we note that*

$$\frac{d^k}{dt^k} F_q^{(\alpha)}(t,x)\bigg|_{t=0} = [2]_{q^\alpha} \sum_{n=0}^{\infty} (-1)^n q^{\alpha n} [n+x]_q^k, \quad (k \in \mathbb{N}), \tag{2.27}$$

$$\left(\frac{d}{dt}\right)^k \left(\sum_{n=0}^{\infty} E_{n,q}^{(\alpha)}(x) \frac{t^n}{n!} \right)\bigg|_{t=0} = E_{k,q}^{(\alpha)}(x), \quad \textit{for } k \in \mathbb{N}. \tag{2.28}$$

By (2.27) *and* (2.28), *we are now ready to define the Hurwitz q-Euler zeta functions.*

Definition 2.9. Let $s \in \mathbb{C}$. Then, one has

$$\zeta_q^{(\alpha)}(s,x) = [2]_{q^\alpha} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\alpha n}}{[n+x]_q^s}. \tag{2.29}$$

Note that $\zeta_q^{(\alpha)}(s,x)$ is a meromorphic function on $\mathbb{C}$. Obverse that, if $q \to 1$, then $\zeta_q^{(\alpha)}(s,x) = \zeta(s,x)$ which is the Hurwitz Euler zeta functions. Relation between $\zeta_q^{(\alpha)}(s,x)$ and $E_{k,q}^{(\alpha)}(x)$ is given by the following theorem.

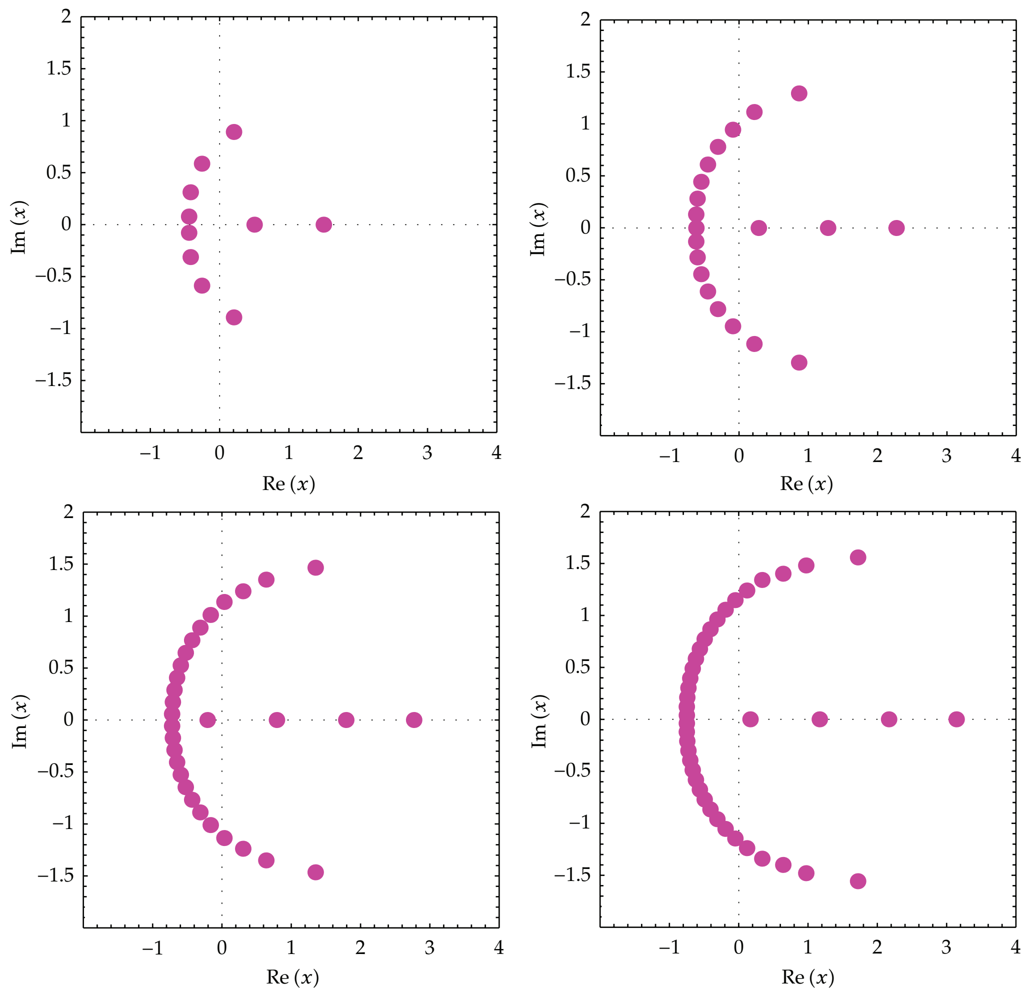

Figure 1: Zeros of $E_{n,1/2}^{(3)}(x)$.

Theorem 2.10. *For $k \in \mathbb{N}$, one has*

$$\zeta_q^{(\alpha)}(-k, x) = E_{k,q}^{(\alpha)}(x). \tag{2.30}$$

Observe that $\zeta_q^{(\alpha)}(-k, x)$ function interpolates $E_{k,q}^{(\alpha)}(x)$ numbers at nonnegative integers.

3. Distribution and Structure of the Zeros

In this section, we assume that $\alpha \in \mathbb{N}$ and $q \in \mathbb{C}$, with $|q| < 1$. We observe the behavior of roots of the q-Euler polynomials $E_{n,q}^{(\alpha)}(x)$. We display the shapes of the q-Euler polynomials $E_{n,q}^{(\alpha)}(x)$, and we investigate the zeros of the q-Euler polynomials $E_{n,q}^{(\alpha)}(x)$. We plot the zeros of the q-Euler polynomials $E_{n,q}^{(\alpha)}(x)$ for $n = 10, 20, 30, 40$ and $x \in \mathbb{C}$ (Figure 1). In Figure 1 (top-left), we

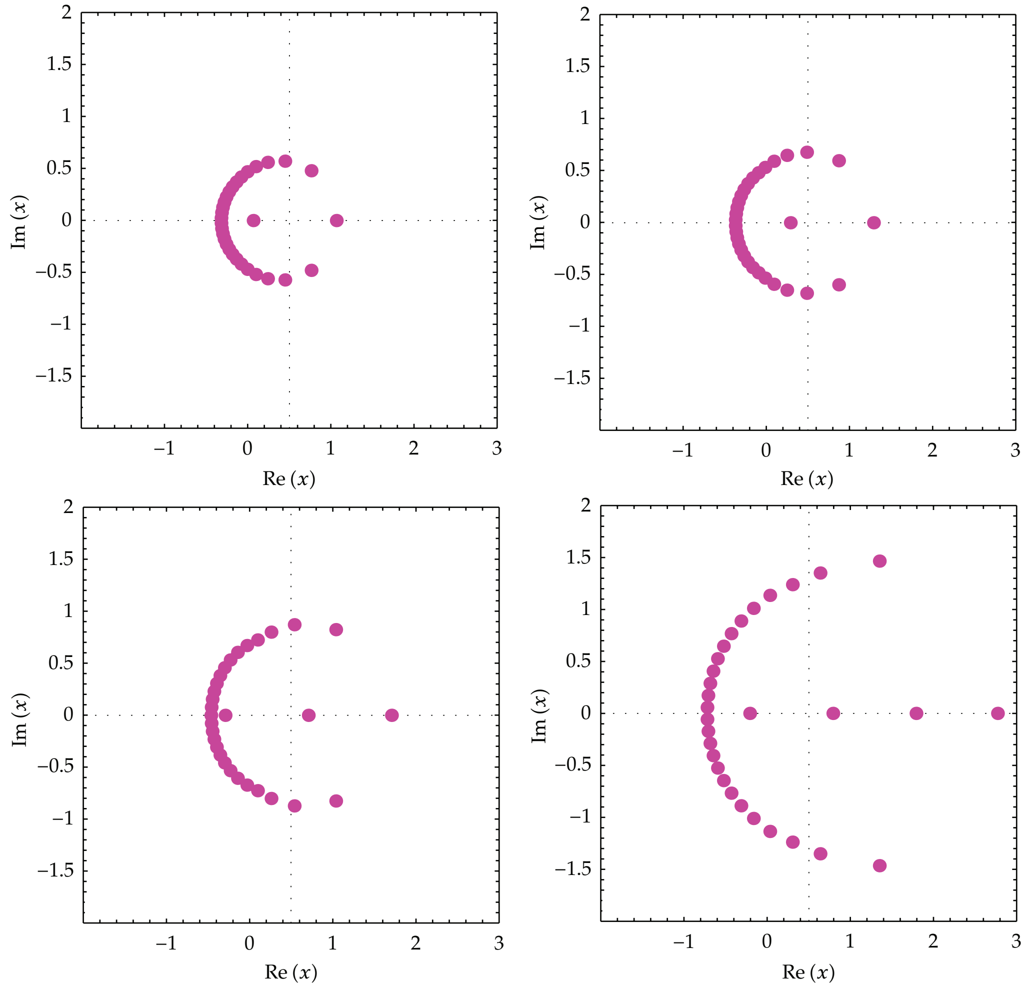

Figure 2: Zeros of $E_{n,q}^{(\alpha)}(x)$.

choose $n = 10, q = 1/2$, and $\alpha = 3$. In Figure 1 (top-right), we choose $n = 20, q = 1/2$, and $\alpha = 3$. In Figure 1 (bottom-left), we choose $n = 30, q = 1/2$, and $\alpha = 3$. In Figure 1 (bottom-right), we choose $n = 40, q = 1/2$, and $\alpha = 3$.

In order to understand zeros behavior better, we present Figures 2 and 3. We plot the zeros of $E_{n,q}^{(\alpha)}(x)$ (Figure 2).

In Figure 2 (top-left), we choose $n = 30, q = 1/5$, and $\alpha = 3$. In Figure 2 (top-right), we choose $n = 30, q = 1/4$, and $\alpha = 3$. In Figure 2 (bottom-left), we choose $n = 30, q = 1/3$, and $\alpha = 3$. In Figure 2 (bottom-right), we choose $n = 30, q = 1/2$, and $\alpha = 3$.

We plot the zeros of the q-Euler polynomials $E_{n,q}^{(\alpha)}(x)$ for $n = 30, q = 1/2, \alpha = 5, 7, 9, 11$ and $x \in \mathbb{C}$ (Figure 3).

In Figure 3 (top-left), we choose $n = 30, q = 1/2$, and $\alpha = 5$. In Figure 3 (top-right), we choose $n = 30, q = 1/2$, and $\alpha = 7$. In Figure 3 (bottom-left), we choose $n = 30, q = 1/2$, and $\alpha = 9$. In Figure 3 (bottom-right), we choose $n = 30, q = 1/2$, and $\alpha = 11$.

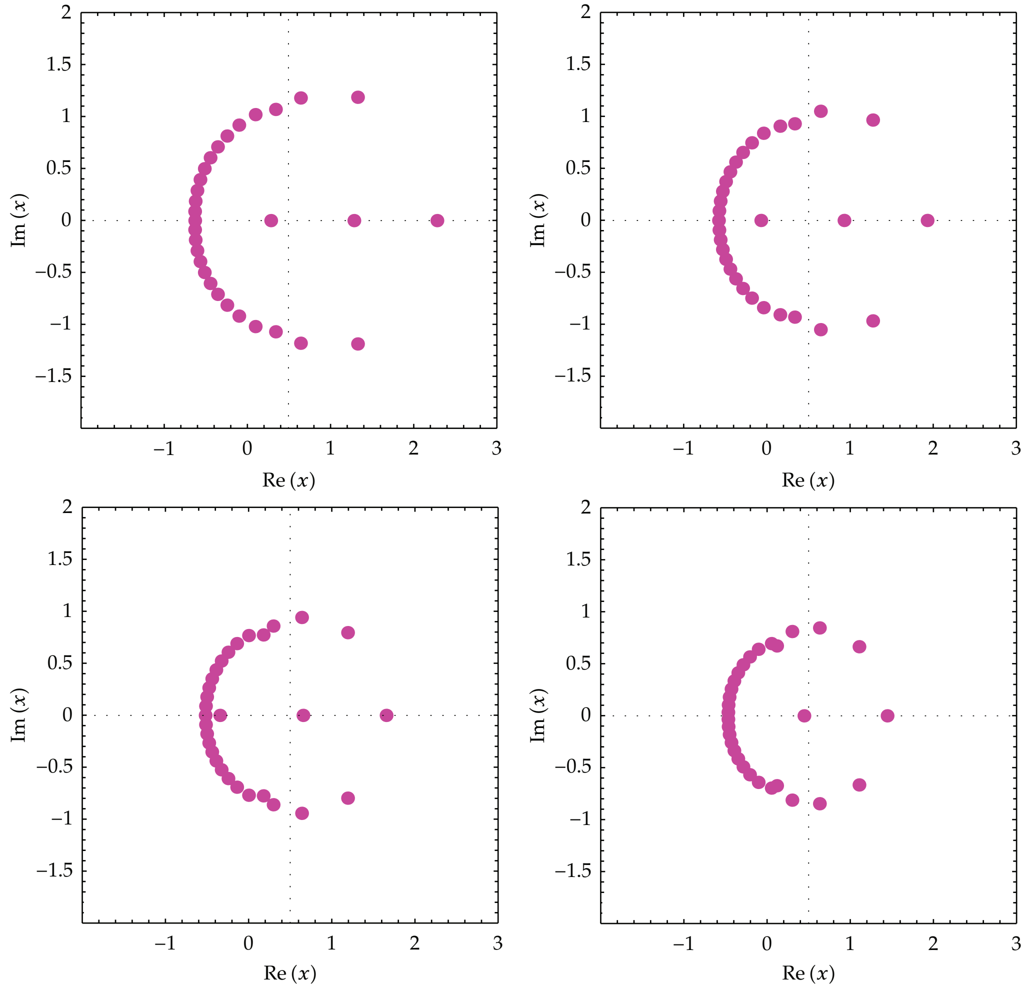

Figure 3: Zeros of $E_{30,1/2}(x)$ for $\alpha = 5, 7, 9, 11$.

Our numerical results for approximate solutions of real zeros of the q-Euler polynomials $E_{n,q}^{(\alpha)}(x), q = 1/2$, are displayed (Tables 1 and 2).

Next, we calculated an approximate solution satisfying the q-Euler polynomials $E_{n,q}^{(\alpha)}(x)$. The results are given in Table 2.

We observe a remarkably regular structure of the complex roots of the q-Euler polynomials $E_{n,q}^{(\alpha)}(x)$. We hope to verify a remarkably regular structure of the complex roots of the q-Euler polynomials $E_{n,q}^{(\alpha)}(x)$ (Table 1). This numerical investigation is especially exciting because we can obtain an interesting phenomenon of scattering of the zeros of the q-Euler polynomials $E_{n,q}^{(\alpha)}(x)$. These results are used not only in pure mathematics and applied mathematics, but also in mathematical physics and other areas.

Stacks of zeros of $E_{n,q}^{(3)}(x)$ for $q = 1/2, 1 \leq n \leq 30$ from a 3D structure are presented (Figure 4).

Table 1: Numbers of real and complex zeros of $E_{n,q}^{(\alpha)}(x)$.

	$\alpha = 3$		$\alpha = 5$	
Degree n	Real zeros	Complex zeros	Real zeros	Complex zeros
1	1	0	1	0
2	2	0	2	0
3	1	2	1	2
4	2	2	2	2
5	3	2	1	4
6	2	4	2	4
7	3	4	3	4
8	2	6	2	6
9	3	6	3	6
10	2	8	2	8
11	3	8	3	8
12	4	8	2	10
13	3	10	3	10

Table 2: Approximate solutions of $E_{n,q}^{(3)}(x) = 0, q = 1/2, x \in \mathbb{R}$.

Degree n	x
1	0.0824622
2	−0.176174, 0.301704
3	0.513012
4	−0.220226, 0.701301
5	−0.306596, −0.132473, 0.868839
6	0.0191767, 1.01918
7	−0.41178, 0.155365, 1.15534
8	0.279948, 1.27971
⋮	⋮

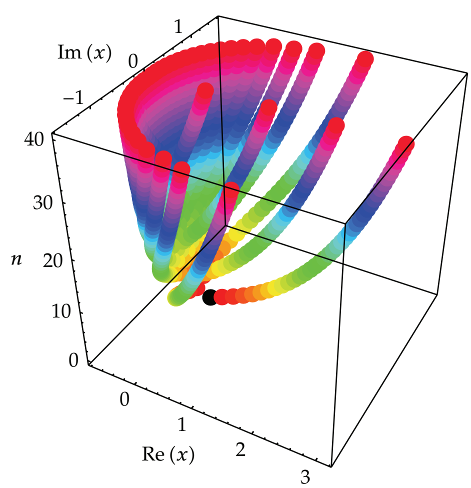

Figure 4: Stacks of zeros of $E_{n,q}^{(3)}(x), 1 \leq n \leq 40$.

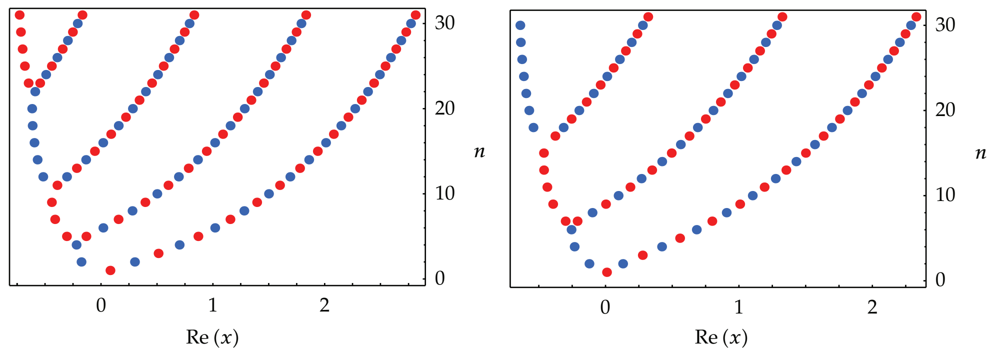

Figure 5: Zeros of $E_{n,30}^{(3)}(x)$.

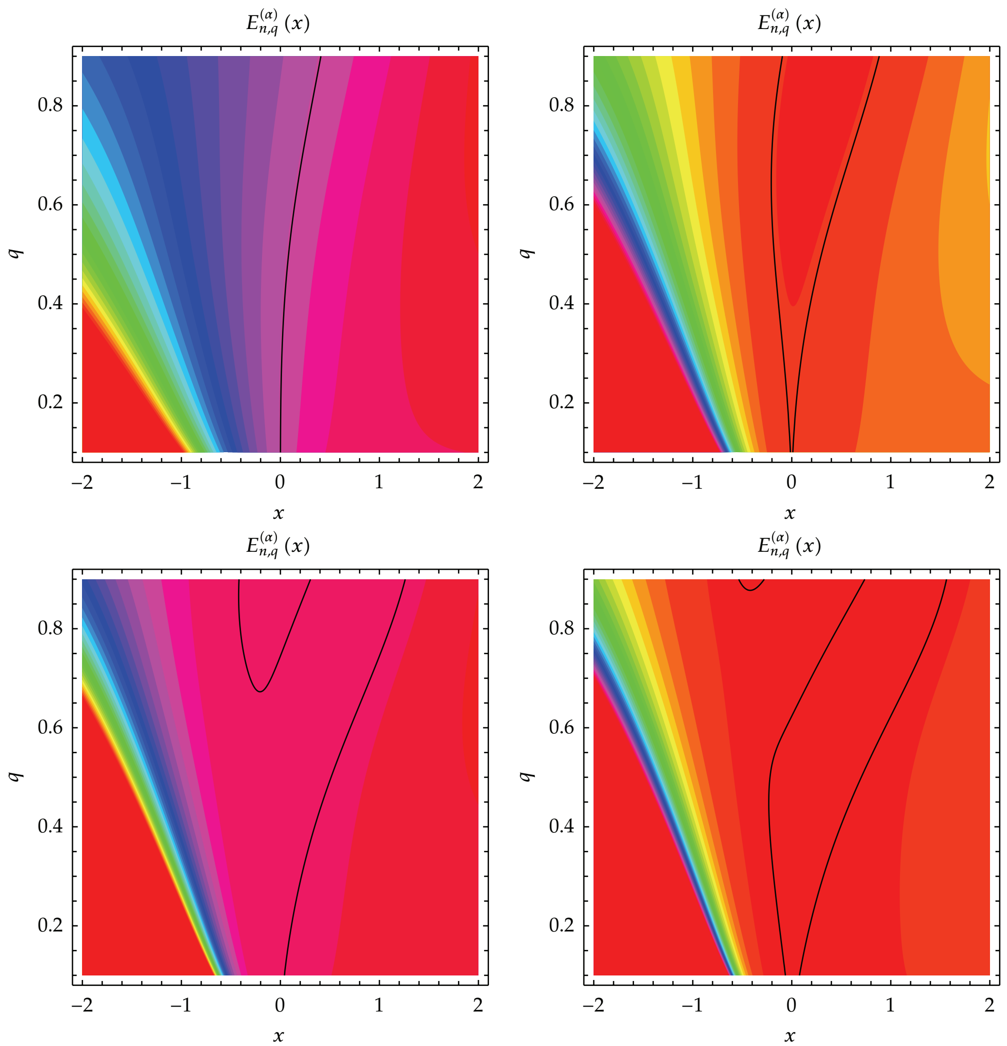

Figure 6: Zero contour of $E_{n,q}^{(\alpha)}(x)$.

We present the distribution of real zeros of the q-Euler polynomials $E_{n,q}^{(\alpha)}(x)$ for $q = 1/2, 1 \leq n \leq 30$ (Figure 5).

In Figure 5 (left), we choose $\alpha = 3$. In Figure 3 (right), we choose $\alpha = 5$.

The plot above shows $E_{n,q}^{(\alpha)}(x)$ for real $1/10 \leq q \leq 9/10$ and $-2 \leq x \leq 2$, with the zero contour indicated in black (Figure 6). In Figure 6 (top-left), we choose $n = 1$ and $\alpha = 3$. In Figure 6 (top-right), we choose $n = 2$ and $\alpha = 3$. In Figure 6 (bottom-left), we choose $n = 3$ and $\alpha = 3$. In Figure 6 (bottom-right), we choose $n = 4$ and $\alpha = 3$.

4. Direction for Further Research

We observe the behavior of complex roots of the q-Euler polynomials $E_{n,q}^{(\alpha)}(x)$, using numerical investigation. How many roots does $E_{n,q}^{(\alpha)}(x)$ have in general? This is an open problem. Prove or disprove: $E_{n,q}^{(\alpha)}(x)$ has n distinct solutions, that is, all the zeros are nondegenerate. Find the numbers of complex zeros $C_{E_{n,q}^{(\alpha)}(x)}$ of $E_{n,q}^{(\alpha)}(x), \mathrm{Im}(x) \neq 0$. Since n is the degree of the polynomial $E_{n,q}^{(\alpha)}(x)$, the number of real zeros $R_{E_{n,q}^{(\alpha)}(x)}$ lying on the real plane $\mathrm{Im}(x) = 0$ is then $R_{E_{n,q}^{(\alpha)}(x)} = n - C_{E_{n,q}^{(\alpha)}(x)}$, where $C_{E_{n,q}^{(\alpha)}(x)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{E_{n,q}^{(\alpha)}(x)}$ and $C_{E_{n,q}^{(\alpha)}(x)}$. We prove that $E_{n,q}^{(\alpha)}(x), x \in \mathbb{C}$, has $\mathrm{Im}(x) = 0$ reflection symmetry analytic complex functions. If $E_{n,q}^{(\alpha)}(x) = 0$, then $E_{n,q}^{(h)}(x^*) = 0$, where $*$ denotes complex conjugate (see Figures 1, 2, and 3). The theoretical prediction on the zeros of $E_{n,q}^{(\alpha)}(x)$ requires further study. In order to study the q-Euler polynomials $E_{n,q}^{(\alpha)}(x)$, we must understand the structure of the q-Euler polynomials $E_{n,q}^{(\alpha)}(x)$. Therefore, using computer, in a realistic study for the q-Euler polynomials $E_{n,q}^{(\alpha)}(x)$ play an important part. The authors have no doubt that investigation along this line will lead to a new approach employing numerical method in the field of research of the q-Euler polynomials $E_{n,q}^{(\alpha)}(x)$ to appear in mathematics and physics. For related topics the interested reader is referred to [16].

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