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Research Article

Some Refinements of Inequalities for Circular Functions

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We give new lower bound and upper bound for Papenfuss-Bach inequality and improve Ruehr-Shafer inequality by providing a new lower bound.

1. Introduction

Papenfuss [1] proposes an open problem described as follows.

Theorem 1.1. *Let* $0 \leq x < \pi/2$*. Then*

$$x\sec^2 x - \tan x \leq \frac{8\pi^2 x^3}{(\pi^2 - 4x^2)^2}. \tag{1.1}$$

Bach [2] prove Theorem 1.1 and obtain a further result.

Theorem 1.2. *Let* $0 \leq x < \pi/2$*. Then*

$$x\sec^2 x - \tan x \leq \frac{2\pi^2}{3}\frac{x^3}{(\pi^2 - 4x^2)^2}. \tag{1.2}$$

Ge gives a lower bound of the above inequality in [3] as follows.

Theorem 1.3. *Let* $0 < x < \pi/2$*. Then*

$$\frac{64x^3}{(\pi^2-4x^2)^2} < x\sec^2 x - \tan x < \frac{2\pi^4}{3}\frac{x^3}{(\pi^2-4x^2)^2}. \tag{1.3}$$

Furthermore, 64 and $2\pi^4/3$ *are the best constants in* (1.3).

Besides, Bach [2] obtain the improvement of the Papenfuss-Bach inequality as another one.

Theorem 1.4. *Let* $0 \le x < \pi/2$*. Then*

$$x\sec^2 x - \tan x \le \frac{2\pi^2(\tan x - x)}{\pi^2-4x^2}. \tag{1.4}$$

In this note, we firstly obtain better bounds for Papenfuss-Bach inequality as in the following statement.

Theorem 1.5. *Let* $0 < x < \pi/2$*. Then*

$$\begin{aligned}&\frac{(2\pi^4/3)x^3 + ((8\pi^4/15) - 16\pi^2/3)x^5}{(\pi^2-4x^2)^2}\\ &\quad < x\sec^2 x - \tan x < \frac{(2\pi^4/3)x^3 + ((256/\pi^2)\cdot(513/511) - (8\pi^2/3))x^5}{(\pi^2-4x^2)^2}.\end{aligned} \tag{1.5}$$

And then we give the refinement of the Ruehr-Shafer inequality as follows.

Theorem 1.6. *Let* $0 < x < \pi/2$*. Then*

$$\frac{1018\pi^2}{511}\frac{(\tan x - x)}{\pi^2-4x^2} < x\sec^2 x - \tan x < \frac{2\pi^2(\tan x - x)}{\pi^2-4x^2}. \tag{1.6}$$

Remark 1.7. Since $(256/\pi^2)\cdot(513/511)-(8\pi^2/3) \approx -0.2792 < 0$, we know that the upper bound in the inequality (1.5) is better than the one in (1.3). At the same time, we find that the lower estimate in (1.5) is larger than the one in (1.3) on the interval $(0, 1.169880805)$ meanwhile the lower estimate in (1.3) is larger than the one in (1.5) on $(1.169880805, \pi/2)$.

2. Lemmas

Lemma 2.1 (see [4–7]). *Let* B_{2n} *be the even-indexed Bernoulli numbers. Then*

$$\frac{2(2n)!}{(2\pi)^{2n}}\frac{1}{1-2^{-2n}} < |B_{2n}| < \frac{2(2n)!}{(2\pi)^{2n}}\frac{1}{1-2^{1-2n}}, \quad n = 1, 2, 3, \ldots. \tag{2.1}$$

Lemma 2.2 (see [7–9]). *Let $|x| < \pi/2$. Then*

$$\tan x = \sum_{n=1}^{\infty} \frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}(-1)^{n-1}B_{2n}x^{2n-1} = \sum_{n=1}^{\infty} \frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}|B_{2n}|x^{2n-1},$$
$$\tan x - x = \sum_{n=2}^{\infty} \frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}|B_{2n}|x^{2n-1}. \tag{2.2}$$

Lemma 2.3. *Let $F(x) = (\pi^2 - 4x^2)(x\sec^2 x - \tan\ x)$ and $|x| < \pi/2$. Then*

$$F(x) = \frac{2\pi^2}{3}x^3 + \sum_{n=2}^{\infty}\left[2\pi^2\frac{2^{2n+2}\left(2^{2n+2}-1\right)}{(2n+2)!}n|B_{2n+2}| - 4\frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}(2n-2)|B_{2n}|\right]x^{2n+1}. \tag{2.3}$$

Proof. By using Lemma 2.2, we have

$$\sec^2 x = (\tan x)' = \sum_{n=1}^{\infty} \frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}(2n-1)|B_{2n}|x^{2n-2}, \quad |x| < \frac{\pi}{2},$$
$$x\sec^2 x - \tan x = \sum_{n=2}^{\infty} \frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}(2n-2)|B_{2n}|x^{2n-1}, \quad |x| < \frac{\pi}{2}. \tag{2.4}$$

We calculate

$$\begin{aligned}
&\left(\pi^2 - 4x^2\right)\left(x\sec^2 x - \tan x\right)\\
&\quad= \pi^2\sum_{n=2}^{\infty}\frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}(2n-2)|B_{2n}|x^{2n-1} - 4\sum_{n=2}^{\infty}\frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}(2n-2)|B_{2n}|x^{2n+1}\\
&\quad= \frac{2\pi^2}{3}x^3 + \pi^2\sum_{n=3}^{\infty}\frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}(2n-2)|B_{2n}|x^{2n-1} - 4\sum_{n=2}^{\infty}\frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}(2n-2)|B_{2n}|x^{2n+1}\\
&\quad= \frac{2\pi^2}{3}x^3 + \pi^2\sum_{n=2}^{\infty}\frac{2^{2n+2}\left(2^{2n+2}-1\right)}{(2n+2)!}(2n)|B_{2n+2}|x^{2n+1} - 4\sum_{n=2}^{\infty}\frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}(2n-2)|B_{2n}|x^{2n+1}\\
&\quad= \frac{2\pi^2}{3}x^3 + \sum_{n=2}^{\infty}\left[2\pi^2\frac{2^{2n+2}\left(2^{2n+2}-1\right)}{(2n+2)!}n|B_{2n+2}| - 4\frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}(2n-2)|B_{2n}|\right]x^{2n+1}.
\end{aligned} \tag{2.5}$$

The proof of Lemma 2.3 is completed. □

Lemma 2.4. *Let $|x| < \pi/2$. Then*

$$\frac{1}{\pi^2 - 4x^2} = \frac{1}{\pi^2}\sum_{n=0}^{\infty}\left(\frac{2}{\pi}\right)^{2n}x^{2n}. \tag{2.6}$$

3. The Proof of Theorem 1.5

The proof of Theorem 1.5 is completed when proving the truth of the following double inequality:

$$\begin{aligned}
&\frac{(2\pi^4/3)x^3 + (8\pi^4/15 \ - 16\pi^2/3)x^5}{\pi^2 - 4x^2} \\
&\quad < \left(\pi^2 - 4x^2\right)\left(x\sec^2 x - \tan x\right) \\
&\quad < \frac{(2\pi^4/3)x^3 + ((256/\pi^2)\cdot(513/511) - (8\pi^2/3))x^5}{\pi^2 - 4x^2}.
\end{aligned} \tag{3.1}$$

We firstly process the left-hand side of the above inequality. We compute that

$$\begin{aligned}
&\left(\pi^2 - 4x^2\right)\left(x\sec^2 x - \tan x\right) - \frac{(2\pi^4/3)x^3 + (8\pi^4/15 - 16\pi^2/3)x^5}{\pi^2 - 4x^2} \\
&\quad = \frac{2\pi^2}{3}x^3 + \sum_{n=2}^{\infty}\left[2\pi^2\frac{2^{2n+2}\left(2^{2n+2}-1\right)}{(2n+2)!}n\left|B_{2n+2}\right| - 4\frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}(2n-2)\left|B_{2n}\right|\right]x^{2n+1} \\
&\qquad - \left[\frac{2\pi^2}{3}x^3 + \frac{2\pi^2}{3}\sum_{n=2}^{\infty}\frac{2^{2n-2}}{\pi^{2n-2}}x^{2n+1} + \left(\frac{8\pi^2}{15} - \frac{16}{3}\right)x^5 + \left(\frac{8\pi^2}{15} - \frac{16}{3}\right)\sum_{n=3}^{\infty}\frac{2^{2n-4}}{\pi^{2n-4}}x^{2n+1}\right] \\
&\quad = \sum_{n=3}^{\infty}\left[2\pi^2\frac{2^{2n+2}\left(2^{2n+2}-1\right)}{(2n+2)!}n\left|B_{2n+2}\right| - 4\frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}(2n-2)\left|B_{2n}\right|\right]x^{2n+1} \\
&\qquad - \left[\frac{2\pi^2}{3}\sum_{n=3}^{\infty}\frac{2^{2n-2}}{\pi^{2n-2}}x^{2n+1} + \left(\frac{8\pi^2}{15} - \frac{16}{3}\right)\sum_{n=3}^{\infty}\frac{2^{2n-4}}{\pi^{2n-4}}x^{2n+1}\right] \\
&\quad = a_3x^7 + a_4x^9 + \sum_{n=5}^{\infty}a_nx^{2n+1},
\end{aligned} \tag{3.2}$$

where

$$a_n = 2\pi^2\frac{2^{2n+2}\left(2^{2n+2}-1\right)}{(2n+2)!}n\left|B_{2n+2}\right| - 4\frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}(2n-2)\left|B_{2n}\right| - \frac{2\pi^2}{3}\frac{2^{2n-2}}{\pi^{2n-2}} - \left(\frac{8\pi^2}{15} - \frac{16}{3}\right)\frac{2^{2n-4}}{\pi^{2n-4}} \tag{3.3}$$

for $n \geq 3$ and $n \in N^+$.

We calculate $a_3 \approx 0.0100 > 0$ and $a_4 \approx 0.0049 > 0$. For $n \geq 5$, by using Lemma 2.1, we can estimate a_n as follows:

$$\begin{aligned}
a_n &> 2\pi^2 \frac{2^{2n+2}\left(2^{2n+2}-1\right)}{(2n+2)!} n \frac{2(2n+2)!}{(2\pi)^{2n+2}\left(1-2^{-2n-2}\right)} \\
&\quad - 4\frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}(2n-2)\frac{2(2n)!}{(2\pi)^{2n}\left(1-2^{1-2n}\right)} - \frac{2\pi^2}{3}\frac{2^{2n-2}}{\pi^{2n-2}} - \left(\frac{8\pi^2}{15}-\frac{16}{3}\right)\frac{2^{2n-4}}{\pi^{2n-4}} \\
&= \frac{4n\cdot 2^{2n+2}}{\pi^{2n}} - \frac{(16n-16)2^{2n}}{\pi^{2n}}\cdot\frac{2^{2n}-1}{2^{2n}-2} - \frac{2\pi^2}{3}\frac{2^{2n-2}}{\pi^{2n-2}} - \left(\frac{8\pi^2}{15}-\frac{16}{3}\right)\frac{2^{2n-4}}{\pi^{2n-4}} \\
&= \frac{2^{2n-4}}{\pi^{2n-4}}\left[\frac{4n\cdot 2^6}{\pi^4} - \frac{2^4(16n-16)}{\pi^4}\left(1+\frac{1}{2^{2n}-2}\right) - \frac{2\pi^2}{3}\frac{2^2}{\pi^2} - \frac{8\pi^2}{15} + \frac{16}{3}\right] \\
&= \frac{2^{2n-4}}{\pi^{2n-4}}\left[\frac{256}{\pi^4} - \frac{256(n-1)}{\pi^4\left(2^{2n}-2\right)} - \frac{8\pi^2}{15} + \frac{8}{3}\right].
\end{aligned} \tag{3.4}$$

Since, for $n = 5, 6, \ldots$, we have

$$\frac{256}{\pi^4} - \frac{256(n-1)}{\pi^4(2^{2n}-2)} - \frac{8\pi^2}{15} + \frac{8}{3} \geq \frac{256}{\pi^4} - \frac{256(5-1)}{\pi^4\left(2^{10}-2\right)} - \frac{8\pi^2}{15} + \frac{8}{3} \approx 0.0207 > 0, \tag{3.5}$$

so $a_n > 0$ for $n \geq 5$ and $n \in N^+$. Combining with the results of a_3 and a_4, we finish the proof of the left-hand side of inequality (3.1).

Now, Let's discuss the right-hand side of inequality (3.1) by taking the same method. We compute that

$$\begin{aligned}
&\left(\pi^2-4x^2\right)\left(x\sec^2 x-\tan x\right) - \frac{\left(2\pi^4/3\right)x^3 + \left(\left(256/\pi^2\right)\cdot(513/511) - 8\pi^2/3\right)x^5}{\pi^2-4x^2} \\
&\quad= \frac{2\pi^2}{3}x^3 + \sum_{n=2}^{\infty}\left[2\pi^2\frac{2^{2n+2}\left(2^{2n+2}-1\right)}{(2n+2)!}n|B_{2n+2}| - 4\frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}(2n-2)|B_{2n}|\right]x^{2n+1} \\
&\qquad - \left[\frac{2\pi^2}{3}x^3 + \frac{2\pi^2}{3}\sum_{n=2}^{\infty}\frac{2^{2n-2}}{\pi^{2n-2}}x^{2n+1} + \left(\frac{256}{\pi^4}\cdot\frac{513}{511}-\frac{8}{3}\right)x^5 + \left(\frac{256}{\pi^4}\cdot\frac{513}{511}-\frac{8}{3}\right)\sum_{n=3}^{\infty}\frac{2^{2n-4}}{\pi^{2n-4}}x^{2n+1}\right] \\
&\quad= b_2x^5 + \sum_{n=3}^{\infty}\left[2\pi^2\frac{2^{2n+2}\left(2^{2n+2}-1\right)}{(2n+2)!}n|B_{2n+2}| - 4\frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}(2n-2)|B_{2n}|\right]x^{2n+1} \\
&\qquad - \left[\frac{2\pi^2}{3}\sum_{n=3}^{\infty}\frac{2^{2n-2}}{\pi^{2n-2}}x^{2n+1} + \left(\frac{256}{\pi^4}\cdot\frac{513}{511}-\frac{8}{3}\right)\sum_{n=3}^{\infty}\frac{2^{2n-4}}{\pi^{2n-4}}x^{2n+1}\right] \\
&\quad= b_2x^5 + b_3x^7 + \sum_{n=4}^{\infty}b_nx^{2n+1},
\end{aligned} \tag{3.6}$$

where

$$b_2 = \left(\frac{8\pi^2}{15} - \frac{16}{3}\right) - \left(\frac{256}{\pi^4} \cdot \frac{513}{511} - \frac{8}{3}\right) \approx -0.0413 < 0,$$
$$b_3 = \left[2\pi^2 \frac{2^8(2^8-1)3}{8! \cdot 30} - 4\frac{2^6(2^6-1)4}{6! \cdot 42} - \frac{2\pi^2}{3}\frac{2^4}{\pi^4} - \left(\frac{256}{\pi^4} \cdot \frac{513}{511} - \frac{8}{3}\right)\frac{2^2}{\pi^2}\right] \approx -0.0068 < 0, \tag{3.7}$$

and, for $n \geq 4$

$$\begin{aligned} b_n = {} & 2\pi^2 \frac{2^{2n+2}(2^{2n+2}-1)}{(2n+2)!} n|B_{2n+2}| - 4\frac{2^{2n}(2^{2n}-1)}{(2n)!}(2n-2)|B_{2n}| \\ & - \frac{2\pi^2}{3}\frac{2^{2n-2}}{\pi^{2n-2}} - \left(\frac{256}{\pi^4} \cdot \frac{513}{511} - \frac{8}{3}\right)\frac{2^{2n-4}}{\pi^{2n-4}}. \end{aligned} \tag{3.8}$$

We can reckon that, for $n = 4, 5, \ldots$

$$\begin{aligned} b_n < {} & 2\pi^2 \frac{2^{2n+2}(2^{2n+2}-1)}{(2n+2)!} n \frac{2(2n+2)!}{(2\pi)^{2n+2}(1-2^{1-2n-2})} - 4\frac{2^{2n}(2^{2n}-1)}{(2n)!}(2n-2)\frac{2(2n)!}{(2\pi)^{2n}(1-2^{-2n})} \\ & - \frac{2\pi^2}{3}\frac{2^{2n-2}}{\pi^{2n-2}} - \left(\frac{256}{\pi^4} \cdot \frac{513}{511} - \frac{8}{3}\right)\frac{2^{2n-4}}{\pi^{2n-4}} \\ = {} & \frac{4\pi^2 n \cdot 2^{2n+2}(2^{2n+2}-1)}{\pi^{2n}(2^{2n+2}-2)} - \frac{(16n-16)2^{2n}}{\pi^{2n}} - \frac{2\pi^2}{3}\frac{2^{2n-2}}{\pi^{2n-2}} - \left(\frac{256}{\pi^4} \cdot \frac{513}{511} - \frac{8}{3}\right)\frac{2^{2n-4}}{\pi^{2n-4}} \\ = {} & \frac{2^{2n-4}}{\pi^{2n-4}}\left(\frac{256n}{\pi^4}\left(1 + \frac{1}{2^{2n+2}-2}\right) - \frac{2^4(16n-16)}{\pi^4} - \frac{2\pi^2}{3}\frac{2^2}{\pi^2} - \frac{256}{\pi^4} \cdot \frac{513}{511} + \frac{8}{3}\right) \\ = {} & \frac{2^{2n-4}}{\pi^{2n-4}}\left(\frac{256n}{\pi^4}\frac{1}{2^{2n+2}-2} + \frac{256}{\pi^4} - \frac{8}{3} - \frac{256}{\pi^4} \cdot \frac{513}{511} + \frac{8}{3}\right) \\ = {} & \frac{2^{2n-4}}{\pi^{2n-4}}\left(\frac{256n}{\pi^4}\frac{1}{2^{2n+2}-2} - \frac{256}{\pi^4} \cdot \frac{2}{511}\right). \end{aligned} \tag{3.9}$$

As we can see,

$$\frac{256n}{\pi^4}\frac{1}{2^{2n+2}-2} \leq \frac{256 \cdot 4}{\pi^4}\frac{1}{2^{2\cdot 4+2}-2} = \frac{256}{\pi^4}\frac{2}{511} \tag{3.10}$$

holds for $n = 4, 5, \ldots$, so we conclude that $b_n < 0$ for $n \geq 4$ and $n \in N^+$. Observing the value of b_2 and b_3, we complete the proof of the right-hand side of inequality (3.1).

4. The Proof of Theorem 1.6

Now we prove the left-hand side of the inequality in Theorem 1.6, which is equivalent to

$$\left(\pi^2 - 4x^2\right)\left(x\sec^2 x - \tan x\right) - \frac{1018\pi^2}{511}(\tan x - x) > 0. \tag{4.1}$$

By using the power series expansions of $(x\sec^2 x - \tan x)$ and $(\tan x - x)$, we can rewrite the above inequality as follows:

$$\pi^2 \sum_{n=2}^{\infty} \frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}\left(2n - 2 - \frac{1018}{511}\right)|B_{2n}|x^{2n-1} - 4\sum_{n=2}^{\infty}\frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}(2n-2)|B_{2n}|x^{2n+1} > 0. \tag{4.2}$$

Now we simplify the left expression and prove that it is positive:

$$\begin{aligned}
&\pi^2 \sum_{n=2}^{\infty} \frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}\left(2n - 2 - \frac{1018}{511}\right)|B_{2n}|x^{2n-1} - 4\sum_{n=2}^{\infty}\frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}(2n-2)|B_{2n}|x^{2n+1} \\
&\quad = \pi^2 \sum_{n=1}^{\infty} \frac{2^{2n+2}\left(2^{2n+2}-1\right)}{(2n+2)!}\left(2n - \frac{1018}{511}\right)|B_{2n+2}|x^{2n+1} - 4\sum_{n=2}^{\infty}\frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}(2n-2)|B_{2n}|x^{2n+1} \\
&\quad = \frac{4\pi^2}{3\cdot 511}x^3 + c_2x^5 + c_3x^7 + c_4x^9 + c_5x^{11} + \sum_{n=6}^{\infty} c_n x^{2n+1},
\end{aligned} \tag{4.3}$$

where

$$c_n = \pi^2 \frac{2^{2n+2}\left(2^{2n+2}-1\right)}{(2n+2)!}\left(2n - \frac{1018}{511}\right)|B_{2n+2}| - 4\frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}(2n-2)|B_{2n}| \tag{4.4}$$

for $n = 2, 3, \ldots$. Particularly, $c_2 \approx -0.02447$, $c_3 \approx 0.00142$, and $c_4 \approx 0.00151$, $c_5 \approx 6.74143 \times 10^{-4}$. We also use Lemma 2.1 in order to give the lower bound of c_n for $n = 6, 7, \ldots$:

$$\begin{aligned}
c_n &= \pi^2 \frac{2^{2n+2}\left(2^{2n+2}-1\right)}{(2n+2)!}\left(2n - \frac{1018}{511}\right)|B_{2n+2}| - 4\frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}(2n-2)|B_{2n}| \\
&> \pi^2 \frac{2^{2n+2}\left(2^{2n+2}-1\right)}{(2n+2)!}\left(2n - \frac{1018}{511}\right)\frac{2(2n+2)!}{(2\pi)^{2n+2}\left(1-2^{-2n-2}\right)}
\end{aligned}$$

$$
\begin{aligned}
&\quad - 4\frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}(2n-2)\frac{2(2n)!}{(2\pi)^{2n}\left(1-2^{1-2n}\right)} \\
&= \frac{2\cdot 2^{2n+2}(2n-1018/511)}{\pi^{2n}} - \frac{(16n-16)2^{2n}\left(2^{2n}-1\right)}{\pi^{2n}\left(2^{2n}-2\right)} \\
&= \frac{2^{2n+3}}{\pi^{2n}}\left[2n-\frac{1018}{511}-(2n-2)\left(1+\frac{1}{2^{2n}-2}\right)\right] \\
&= \frac{2^{2n+3}}{\pi^{2n}}\left(\frac{4}{511}-\frac{2n-2}{2^{2n}-2}\right) > \frac{2^{2n+3}}{\pi^{2n}}\left(\frac{4}{511}-\frac{2\cdot 6-2}{2^{2\cdot 6}-2}\right) > 0.
\end{aligned}
\tag{4.5}
$$

We denote that

$$
\begin{aligned}
G(x) &\equiv \frac{4\pi^2}{3\cdot 511}x^3 + c_2x^5 + c_3x^7 + c_4x^9 + c_5x^{11} \\
&= x^3\left(\frac{4\pi^2}{3\cdot 511} + c_2x^2 + c_3x^4 + c_4x^6 + c_5x^8\right) \\
&\equiv x^3H(x),
\end{aligned}
\tag{4.6}
$$

where

$$
H(x) = \frac{4\pi^2}{3\cdot 511} + c_2x^2 + c_3x^4 + c_4x^6 + c_5x^8. \tag{4.7}
$$

Let $t = x^2$, and

$$
P(t) \equiv \frac{4\pi^2}{3\cdot 511} + c_2t + c_3t^2 + c_4t^3 + c_5t^4. \tag{4.8}
$$

We compute that

$$
\begin{aligned}
P'(t) &= c_2 + 2c_3t + 3c_4t^2 + 4c_5t^3, \quad 0 < t < \frac{\pi^2}{4}, \\
P''(t) &= 2 + c_3 + 6c_4t + 12c_5t^2 > 0, \quad 0 < t < \frac{\pi^2}{4}.
\end{aligned}
\tag{4.9}
$$

So $P'(t)$ is increasing on $(0, \pi^2/4)$. Since $P'(0) < 0$ and $P'(\pi^2/4) > 0$, we have that $P(t)$ is decreasing firstly and then increasing. Let t_0 be only one point of minimum of the function $P(t)$. Then $t_0 \approx 1.5262621$ and $P(t_0) \approx 7.33921\times 10^{-4} > 0$; this implies that $P(t) > 0$, so $H(x) > 0$ and $G(x) > 0$ for $0 < x < \pi/2$.

Combining with $c_n > 0$ for $n \geq 6$, we have proved Theorem 1.6.

5. Open Problem

In the last section we pose a problem as follows: Let $0 < x < \pi/2$. Then

$$\frac{(2\pi^4/3)x^3 + (8\pi^4/15 - 16\pi^2/3)x^5}{(\pi^2 - 4x^2)^2} < x\sec^2 x - \tan x < \frac{(2\pi^4/3)x^3 + (256/\pi^2 - 8\pi^2/3)x^5}{(\pi^2 - 4x^2)^2} \tag{5.1}$$

hold, where $(8\pi^4/15 - 16\pi^2/3)$ and $(256/\pi^2 - 8\pi^2/3)$ are the best constants in (5.1).

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