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Research Article

A New System of Generalized Mixed Quasivariational Inclusions with Relaxed Cocoercive Operators and Applications

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A new system of generalized mixed quasivariational inclusions (for short, SGMQVI) with relaxed cocoercive operators, which develop some preexisting variational inequalities, is introduced and investigated in Banach spaces. Next, the existence and uniqueness of solutions to the problem (SGMQVI) are established in real Banach spaces. From fixed point perspective, we propose some new iterative algorithms for solving the system of generalized mixed quasivariational inclusion problem (SGMQVI). Moreover, strong convergence theorems of these iterative sequences generated by the corresponding algorithms are proved under suitable conditions. As an application, the strong convergence theorem for a class of bilevel variational inequalities is derived in Hilbert space. The main results in this paper develop, improve, and unify some well-known results in the literature.

1. Introduction

Generalized mixed quasivariational inclusion problems, which are extensions of variational inequalities introduced by Stampacchia [1] in the early sixties, are among the most interesting and extensively investigated classes of mathematics problems and have many applications in the fields of optimization and control, abstract economics, electrical networks, game theory, auction, engineering science, and transportation equilibria (see, e.g., [2–5] and the references therein). For the past few decades, existence results and iterative algorithms for variational inequality and variational inclusion problems have been obtained (see, e.g., [6–14] and the references cited therein). Recently, some new problems, which are called to be the system of variational inequality and equilibrium problems, received many attentions. Ansari et al. [2] considered a system of vector variational inequalities and obtained its existence results.

In [3], Pang stated that the traffic equilibrium problem, the spatial equilibrium problem, the Nash equilibrium, and the general equilibrium programming problem can be modeled as a system of variational inequalities. Verma [15] and J. K. Kim and D. S. Kim [16] investigated a system of nonlinear variational inequalities. Cho et al. [17] introduced and studied a new system of nonlinear variational inequalities in Hilbert spaces and obtained the existence and uniqueness properties of solutions for the system of nonlinear variational inequalities. In [18], Peng and Zhu introduced a new system of generalized mixed quasivariational inclusions involving (H, η)-monotone operators. Very recently, Qin et al. [19] studied the approximation of solutions to a system of variational inclusions in Banach spaces and established a strong convergence theorem in uniformly convex and 2 uniformly smooth Banach spaces. In [20], Kamraksa and Wangkeeree gave a general iterative method for a general system of variational inclusions and proved a strong convergence theorem in strictly convex and 2 uniformly smooth Banach spaces. Further, Wangkeeree and Kamraksa [21] introduced an iterative algorithm for finding a common element of the set of solutions of a mixed equilibrium problem, the set of fixed points of an infinite family of nonexpansive mappings and the set of solutions of a general system of variational inequalities and then obtained the strong convergence of the iterative in Hilbert spaces. Petrot [22] applied the resolvent operator technique to find the common solutions for a generalized system of relaxed cocoercive mixed variational inequality problems and fixed point problems for Lipschitz mappings in Hilbert spaces. Zhao et al. [23] obtained some existence results for a system of variational inequalities by Brouwer's fixed point theory and proved the convergence of an iterative algorithm in finite Euclidean spaces. Chen and Wan [24] also proved the existence of solutions and convergence analysis for a system of quasivariational inclusions in Banach spaces, proposed some iterative methods for finding the common element of the solutions set for the system of quasivariational inclusions and the fixed point set for Lipschitz mapping, and obtained the convergent rates of corresponding iterative sequences. On the other hand, various bilevel programming problems, bilevel decision problems, and mathematical program problems with equilibrium (variational inequalities) constraints have been wildly investigated (see, e.g., [25, 26]). To the best of our knowledge, there is few results concerning the algorithms and convergence analysis of solutions to bilevel variational inequalities in Hilbert spaces.

The aim of this paper is to introduce and study a new system of generalized mixed quasivariational inclusion problem (SGMQVI) in uniformly smooth Banach spaces which includes some previous variational inequalities as special cases. Furthermore, the existence and uniqueness theorems of solutions for the problem (SGMQVI) are established by using resolvent techniques. Thirdly, we also propose some new iterative algorithms for solving the problem (SGMQVI). Strong convergence of the iterative sequences generated by the corresponding iterative algorithms are proved under suitable conditions. As an application, we study the properties for the lower-level variational inequalities of a class of bilevel variational inequalities (for short, (BVI)) in Hilbert spaces and then suggest a reasonable iterative algorithm for (BVI). Finally, the strong convergence theorem for (BVI) are derived under appropriate assumptions. The results presented in this paper improve, develop, and extend the results of [8, 23, 24, 27].

2. Preliminaries

Throughout this paper, let E be a real q-uniformly Banach space with its dual E^*, $q > 1$, denote the duality between E and E^* by $\langle \cdot, \cdot \rangle$ and the norm of E by $\| \cdot \|$, and let $T : E \to E$ be a nonlinear mapping. If $\{x_n\}$ is a sequence in E, we denote strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$. A Banach space E is called smooth if $\lim_{t\to 0}(\|x+ty\|-\|x\|)/t$ exists for all $x, y \in E$ with $\|x\| = \|y\| = 1$. It is called uniformly smooth if the limit is attained uniformly for $\|x\| = \|y\| = 1$. The function

$$\rho_E(t) = \sup\left\{\frac{\|x+y\|+\|x-y\|}{2} - 1 : \|x\| = 1,\ \|y\| \le t\right\} \tag{2.1}$$

is called the modulus of smoothness of E. E is called q-uniformly smooth if there exists a constant $c_q > 0$ such that $\rho_E(t) \le c_q t^q$.

Example 2.1 (see [4]). All Hilbert spaces, L^p(or l^p), and the Sobolev spaces W_m^p, $(p \ge 2)$ are 2-uniformly smooth, while L^p(or l^p) and W_m^p spaces $(1 < p \le 2)$ are p-uniformly smooth.

The generalized duality mapping $J_q : E \to 2^{E^*}$ defined as follows:

$$J_q(x) = \left\{f^* \in E^* : \langle f^*, x\rangle = \|f^*\|\|x\| = \|x\|^q,\ \|f^*\| = \|x\|^{q-1}\right\}, \tag{2.2}$$

for all $x \in E$. As we know that $J = J_2$ is the usual normalized duality mapping, and $J_q(x) = \|x\|^{q-2}J(x)$ for $x \neq 0$, $J_q(tx) = t^{q-1}J_q(x)$, and $J_q(-x) = -J_q(x)$ for all $x \in E$ and $t \in [0, +\infty)$, and J_q is single-valued if E is smooth (see, e.g., [28]). If E is a Hilbert space, then $J = I$, where I is the identity operator. Let $g_j : E \to E$, let $A_j : E \times E \to E$ be single-valued mappings, and let $M_j : E \to 2^E$ be set-valued mappings for all $j \in \{1, 2, \ldots, n\}$. We consider the system of generalized mixed quasivariational inclusions problem (for short, (SGMQVI)) as follows: find $(x_1^*, x_2^*, \ldots, x_{n-1}^*, x_n^*) \in E^n$ such that

$$\begin{gathered}
0 \in x_1^* - g_1(x_2^*) + \rho_1(A_1(x_2^*, x_1^*) + M_1(x_1^*)),\\
0 \in x_2^* - g_2(x_3^*) + \rho_2(A_2(x_3^*, x_2^*) + M_2(x_2^*)),\\
\vdots\\
0 \in x_{n-1}^* - g_{n-1}(x_n^*) + \rho_{n-1}(A_{n-1}(x_n^*, x_{n-1}^*) + M_{n-1}(x_{n-1}^*)),\\
0 \in x_n^* - g_n(x_1^*) + \rho_n(A_n(x_1^*, x_n^*) + M_n(x_n^*)),
\end{gathered} \tag{2.3}$$

where ρ_i $(i = 1, 2, \ldots, n)$ are positive constants. Denote the set of solutions to (SGMQVI) by Ξ.

Special cases are as follows:

(I) If $g_j(x) = x$ and $A_j(x,y) = T_j(x) + S_j(x)$ for all $x, y \in E$ and $j = 1,2,\ldots,n$, where $T_j, S_j : E \to E$ are single-valued mappings, then the problem (SGMQVI) is equivalent to find $(x_1^*, x_2^*, \ldots, x_{n-1}^*, x_n^*) \in E^n$ such that

$$\begin{gathered}
0 \in x_1^* - x_2^* + \rho_1\left(T_1(x_2^*) + S_1(x_2^*) + M_1(x_1^*)\right), \\
0 \in x_2^* - x_3^* + \rho_2\left(T_2(x_3^*) + S_2(x_3^*) + M_2(x_2^*)\right), \\
\vdots \\
0 \in x_{n-1}^* - x_n^* + \rho_{n-1}\left(T_{n-1}(x_n^*) + S_{n-1}(x_n^*) + M_{n-1}(x_{n-1}^*)\right), \\
0 \in x_n^* - x_1^* + \rho_n\left(T_n(x_1^*) + S_n(x_1^*) + M_n(x_n^*)\right),
\end{gathered} \tag{2.4}$$

where ρ_i $(i = 1,2,\ldots,n)$ are positive constants, which is called the system of generalized nonlinear mixed variational inclusions problem [8].

(II) If $n = 2$, $A_1 = A_2 = A$, $E = H$ is a Hilbert space, $g_1(x) = g_2(x) = x$ and $M_1(x) = M_2(x) = \partial\phi(x)$ for all $x \in E$, where $\phi : E \to R \cup \{+\infty\}$ is a proper, convex, and lower semicontinuous functional, and $\partial\phi$ denotes the subdifferential operator of ϕ, then the problem (SGMQVI) is equivalent to find $(x^*, y^*) \in E \times E$ such that

$$\begin{gathered}
\left\langle \rho_1 A(y^*, x^*) + x^* - y^*,\ x - x^* \right\rangle + \phi(x) - \phi(x^*) \geq 0, \quad \forall x \in E, \\
\left\langle \rho_2 A(x^*, y^*) + y^* - x^*,\ x - y^* \right\rangle + \phi(x) - \phi(y^*) \geq 0, \quad \forall x \in E,
\end{gathered} \tag{2.5}$$

where ρ_i $(i = 1,2)$ are positive constants, which is called the generalized system of relaxed cocoercive mixed variational inequality problem [29].

(III) If $n = 2, E = H$ is a Hilbert space, and K is a closed convex subset of E, and $\phi(x) = \delta_K(x)$ for all $x \in K$, where δ_K is the indicator function of K defined by

$$\phi(x) = \delta_K(x) = \begin{cases} 0, & \text{if } x \in K, \\ +\infty, & \text{otherwise}, \end{cases} \tag{2.6}$$

then the problem (SGMQVI) is equivalent to find $(x^*, y^*) \in K \times K$ such that

$$\begin{gathered}
\left\langle \rho_1 A_1(y^*, x^*) + x^* - g_1(y^*),\ g_1(x) - x^* \right\rangle \geq 0, \quad \forall x \in K, \\
\left\langle \rho_2 A_2(x^*, y^*) + y^* - g_2(x^*),\ g_2(x) - y^* \right\rangle \geq 0, \quad \forall x \in K,
\end{gathered} \tag{2.7}$$

where ρ_i $(i = 1,2)$ are positive constants, which is called the system of general variational inequalities problem [27].

(IV) If $n = 2$, $A_1 = A_2 = A$, $E = H$ is a Hilbert space, and K is a closed convex subset of E, $g_1(y) = g_2(y) = y$, and $\phi(x) = \delta_K(x)$ for all $x \in K$, $y \in E$, where δ_K is the indicator function of K defined by

$$\phi(x) = \delta_K(x) = \begin{cases} 0, & \text{if } x \in K, \\ +\infty, & \text{otherwise,} \end{cases} \tag{2.8}$$

then the problem (SGMQVI) is equivalent to find $(x^*, y^*) \in K \times K$ such that

$$\begin{aligned} &\langle \rho_1 A(y^*, x^*) + x^* - y^*,\ x - x^* \rangle \geq 0, \quad \forall x \in K, \\ &\langle \rho_2 A(x^*, y^*) + y^* - x^*,\ x - y^* \rangle \geq 0, \quad \forall x \in K, \end{aligned} \tag{2.9}$$

where ρ_i $(i = 1, 2)$ are positive constants, which is called the generalized system of relaxed cocoercive variational inequality problem [30].

(V) If for each $i \in \{1, 2\}$, $z \in E$, $A_i(x, z) = \Psi_i(x)$, and $g_i(x) = x$ for all $x \in E$, where $\Psi_i : E \to E$, then the problem (SGMQVI) is equivalent to find $(x^*, y^*) \in E \times E$ such that

$$\begin{aligned} &0 \in x^* - y^* + \rho_1(\Psi_1(y^*) + M_1(x^*)), \\ &0 \in y^* - x^* + \rho_2(\Psi_2(x^*) + M_2(y^*)), \end{aligned} \tag{2.10}$$

where ρ_i $(i = 1, 2)$ are positive constants, which is called the system of quasivariational inclusion [19, 20].

(VI) If $n = 2$, for each $i \in \{1, 2\}$, $z \in E$, $A_i(x, z) = \Psi(x)$, and $g_i(x) = x$ for all $x \in E$, where $\Psi : E \to E$ and $M_1(x) = M_2(x) = M$, then the problem (SGMQVI) is equivalent to find $(x^*, y^*) \in E \times E$ such that

$$\begin{aligned} &0 \in x^* - y^* + \rho_1(\Psi(y^*) + M(x^*)), \\ &0 \in y^* - x^* + \rho_2(\Psi(x^*) + M(y^*)), \end{aligned} \tag{2.11}$$

where ρ_i $(i = 1, 2)$ are positive constants, which is called the system of quasivariational inclusion [20].

We first recall some definitions and lemmas which are needed in our main results.

Definition 2.2. Let $M : \mathrm{dom}(M) \subset E \to 2^E$ be a set-valued mapping, where $\mathrm{dom}(M)$ is the effective domain of the mapping M. M is said to be

(i) accretive if, for any $x, y \in \mathrm{dom}(M)$, $u \in M(x)$ and $v \in M(y)$, there exists $j_q(x-y) \in J_q(x - y)$ such that

$$\langle u - v, j_q(x - y) \rangle \geq 0, \tag{2.12}$$

(ii) m-accretive (maximal-accretive) if M is accretive and $(I + \rho M)\,\mathrm{dom}(M) = E$ holds for every $\rho > 0$, where I is the identity operator on E.

Remark 2.3. If E is a Hilbert space, then accretive operator and m-accretive operator are reduced to monotone operator and maximal monotone operator, respectively.

Definition 2.4 (see [24, 31]). Let $T : E \to E$ be a single-valued mapping. T is said to be

(i) γ-Lipschitz continuous mapping if there exists a constant $\gamma > 0$ such that

$$\|Tx - Ty\| \leq \gamma \|x - y\|, \quad \forall x, y \in E, \tag{2.13}$$

(ii) (a, b)-relaxed cocoercive if there exist two constants $a \geq 0$ and $b > 0$ such that

$$\langle T(x) - T(\tilde{x}), J_q(x - \tilde{x}) \rangle \geq (-a)\|T(x) - T(\tilde{x})\|^q + b\|x - \tilde{x}\|^q, \quad \forall x, \tilde{x} \in E. \tag{2.14}$$

Remark 2.5 (see [24]). (1) If $\gamma = 1$, then a γ-Lipschitz continuous mapping reduces to a non-expansive mapping.

(2) If $\gamma \in (0, 1)$, then a γ-Lipschitz continuous mapping reduces to a contractive mapping.

(3) It is easy to see that the identity operator $I : E \to E$ is $(0, 1)$ relaxed cocoercive, where $I(x) = x$ for all $x \in E$.

Definition 2.6 (see [24]). Let $A : E \times E \to E$ be a mapping. A is said to be

(i) τ-Lipschitz continuous in the first variable if there exists a constant $\tau > 0$ such that, for $x, \tilde{x} \in E$,

$$\|A(x, y) - A(\tilde{x}, \tilde{y})\| \leq \tau \|x - \tilde{x}\|, \quad \forall y, \tilde{y} \in E \tag{2.15}$$

(ii) α-strongly accretive if there exists a constant $\alpha > 0$ such that

$$\langle A(x, y) - A(\tilde{x}, \tilde{y}), J_q(x - \tilde{x}) \rangle \geq \alpha \|x - \tilde{x}\|^q, \quad \forall (x, y), (\tilde{x}, \tilde{y}) \in E \times E, \tag{2.16}$$

or equivalently,

$$\langle A(x, y) - A(\tilde{x}, \tilde{y}), J(x - \tilde{x}) \rangle \geq \alpha \|x - \tilde{x}\|, \quad \forall (x, y), (\tilde{x}, \tilde{y}) \in E \times E, \tag{2.17}$$

(iii) (μ, ν) relaxed cocoercive if there exist two constants $\mu \geq 0$ and $\nu > 0$ such that

$$\begin{aligned} \langle A(x, y) - A(\tilde{x}, \tilde{y}), J_q(x - \tilde{x}) \rangle &\geq (-\mu)\|A(x, y) - A(\tilde{x}, \tilde{y})\|^q + \nu \|x - \tilde{x}\|^q, \\ &\qquad \forall (x, y), (\tilde{x}, \tilde{y}) \in E \times E. \end{aligned} \tag{2.18}$$

Remark 2.7. (1) Every α-strongly accretive mapping is a (μ, α) relaxed cocoercive for any positive constant μ. But the converse is not true in general.

(2) The conception of the cocoercivity is applied in several directions, especially for solving variational inequality problems by using the auxiliary problem principle and projection methods [14]. Several classes of relaxed cocoercive variational inequalities have been investigated in [5, 22, 28, 30].

Definition 2.8 (see [9, 32]). Let the set-valued mapping $M : \mathrm{dom}(M) \subset E \to 2^E$ be m-accretive. For any positive number $\rho > 0$, the mapping $R_\rho^M : E \to \mathrm{dom}(M)$ defined by

$$R_\rho^M(x) = (I + \rho M)^{-1}(x), \quad x \in E, \tag{2.19}$$

is called the resolvent operator associated with M and ρ, where I is the identity operator on E.

Remark 2.9. Let $C \subset E$ be a nonempty closed convex set. If E is a Hilbert space and $M = \partial\phi$, the subdifferential of the indicator function ϕ, that is,

$$\phi(x) = \delta_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{otherwise}, \end{cases} \tag{2.20}$$

then $R_{(\rho,M)} = P_C$, the metric projection operator from E onto C.

Lemma 2.10 (see [9, 32]). *Let the set-valued mapping $M : \mathrm{dom}(M) \subset E \to 2^E$ be m-accretive. Then the resolvent operator R_ρ^M is single-valued and nonexpansive for all $\rho > 0$.*

Lemma 2.11 (see [33]). *Let $\{B_n\}, \{C_n\}$, and $\{D_n\}$ be three nonnegative real sequences satisfying the following conditions:*

$$B_{n+1} \le (1 - \lambda_n)B_n + C_n + D_n, \quad \forall n \ge n_0, \tag{2.21}$$

for some $n_0 \in N, \{\lambda_n\} \subset (0,1)$ with $\sum_{n=0}^{\infty} \lambda_n = \infty$, $C_n = 0(\lambda_n)$ and $\sum_{n=0}^{\infty} D_n < +\infty$. Then $\lim_{n\to\infty} B_n = 0$.

Lemma 2.12 (see [34]). *Let E be a real q-uniformly Banach space. Then there exists a constant $c_q > 0$ such that*

$$\|x + y\|^q \le \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q, \quad \forall x, y \in E. \tag{2.22}$$

3. Existence Theorems

In this section, we will investigate the existence and uniqueness of solutions for the problem (SGMQVI) in q-uniformly smooth Banach space under some suitable conditions.

Theorem 3.1. *Let $(x_1^*, x_2^*, \ldots, x_{n-1}^*, x_n^*) \in E^n$, $M_i : E \to 2^E$ $(i = 1, 2, \ldots, n)$ be maximal accretive. Then $(x_1^*, x_2^*, \ldots, x_{n-1}^*, x_n^*)$ is a solution of the problem (SGMQVI) if and only if*

$$\begin{aligned}
x_1^* &= R_{\rho_1}^{M_1}\left(g_1(x_2^*) - \rho_1 A_1(x_2^*, x_1^*)\right), \\
x_2^* &= R_{\rho_2}^{M_2}\left(g_2(x_3^*) - \rho_2 A_2(x_3^*, x_2^*)\right), \\
&\vdots \\
x_{n-1}^* &= R_{\rho_{n-1}}^{M_{n-1}}\left(g_{n-1}(x_n^*) - \rho_{n-1} A_{n-1}(x_n^*, x_{n-1}^*)\right), \\
x_n^* &= R_{\rho_n}^{M_n}\left(g_n(x_1^*) - \rho_n A_n(x_1^*, x_n^*)\right),
\end{aligned} \tag{3.1}$$

where ρ_i $(i = 1, 2, \ldots, n)$ are positive constants.

Proof. Let $(x_1^*, x_2^*, \ldots, x_{n-1}^*, x_n^*) \in E^n$ be a solution of the problem (SGMQVI). Then, for any given positive constants ρ_i $(i = 1, 2, \ldots, n)$, the problem (SGMQVI) is equivalent to

$$\begin{aligned}
g_1(x_2^*) - \rho_1 A_1(x_2^*, x_1^*) &\in x_1^* + \rho_1 M_1(x_1^*), \\
g_2(x_3^*) - \rho_2 A_2(x_3^*, x_2^*) &\in x_2^* + \rho_2 M_2(x_2^*), \\
&\vdots \\
g_{n-1}(x_n^*) - \rho_{n-1} A_{n-1}(x_n^*, x_{n-1}^*) &\in x_{n-1}^* + \rho_{n-1} M_{n-1}(x_{n-1}^*), \\
g_n(x_1^*) - \rho_n A_n(x_1^*, x_n^*) &\in x_n^* + \rho_n M_n(x_n^*).
\end{aligned} \tag{3.2}$$

From Definition 2.8 and Lemma 2.10, it yields that (3.2) is equivalent to (3.1). This completes the proof. □

Theorem 3.2. *Let E be a real q-uniformly smooth Banach space. Let $j \in \{1, 2, \ldots, n\}$, $M_j : E \to 2^E$ be m-accretive mapping, Let $A_j : E \times E \to E$ be (μ_j, ν_j)-relaxed cocoercive and Lipschitz continuous in the first variable with constant τ_j, and Let $g_j : E \to E$ be (a_j, b_j)-relaxed cocoercive and Lipschitz continuous with constant ι_j. Then, for each $j \in \{1, 2, \ldots, n\}, x \in E$, the mapping $R_{\rho_j}^{M_j}(g_j(x) - \rho_j A_j(x, \cdot)) : E \to E$ has at most one fixed point. If*

$$\min\left\{1 + q a_j \iota_j^q + c_q \iota_j^q - q b_j, 1 + q\rho_j \mu_j \tau_j^q + c_q \rho_j^q \tau_j^q - q\rho_j \nu_j\right\} \geq 0, \tag{3.3}$$

then the implicit function $y_j(x)$ determined by

$$y_j(x) = R_{\rho_j}^{M_j}\left(g_j(x) - \rho_j A_j(x, y_j(x))\right) \tag{3.4}$$

is continuous on E.

Proof. Let $x \in E$. We show by contradiction that $R_{\rho_j}^{M_j}(g_j(x) - \rho_j A_j(x,\cdot)) : E \to E$ has at most one fixed point. Suppose to the contrary that $y, \tilde{y} \in E$ and $y \neq \tilde{y}$ such that

$$\begin{aligned} y &= R_{\rho_j}^{M_j}(g_j(x) - \rho_j A_j(x,y)), \\ \tilde{y} &= R_{\rho_j}^{M_j}(g_j(x) - \rho_j A_j(x,\tilde{y})). \end{aligned} \tag{3.5}$$

Since A_j is Lipschitz continuous in the first variable with constant τ_j, then

$$\begin{aligned} \|y - \tilde{y}\| &= \left\| R_{\rho_j}^{M_j}(g_j(x) - \rho_j A_j(x,y)) - R_{\rho_j}^{M_j}(g_j(x) - \rho_j A_j(x,\tilde{y})) \right\| \\ &\leq \|g_j(x) - \rho_j A_j(x,y) - (g_j(x) - \rho_j A_j(x,\tilde{y}))\| \\ &= \rho_j \|A_j(x,y) - A_j(x,\tilde{y})\| \\ &\leq \rho_j \tau_j \|x - x\| \\ &= 0, \end{aligned} \tag{3.6}$$

which is a contradiction. Therefore, the mapping $R_{\rho_j}^{M_j}(g_j(x) - \rho_j A_j(x,\cdot)) : E \to E$ has at most one fixed point.

Next, we show that the implicit function $y_j(x)$ is continuous on E. For any sequence, $\{x_n\} \subset E, x_0 \in E, x_n \to x_0$ as $n \to \infty$. Since $A_j : E \times E \to E$ is (μ_j, ν_j)-relaxed cocoercive and Lipschitz continuous in the first variable with constant τ_j and $g_j : E \to E$ is (a_j, b_j)-relaxed cocoercive and Lipschitz continuous with constant ι_j, one has

$$\begin{aligned} \tilde{L}_{A_j} &= \|x_n - x_0 - \rho_j[A_j(x_n, y(x_n)) - A_j(x_0, y(x_0))]\|^q \\ &\leq \|x_n - x_0\|^q - q\rho_j \langle A_j(x_n, y(x_n)) - A_j(x_0, y(x_0)), J_q(x_n - x_0)\rangle \\ &\quad + c_q \rho_j^q \|A_j(x_n, y(x_n)) - A_j(x_0, y(x_0))\|^q \\ &\leq q\rho_j(\mu_j \|A_j(x_n, y(x_n)) - A_j(x_0, y(x_0))\|^q - \nu_j \|x_n - x_0\|^q) \\ &\quad + \left(1 + c_q \rho_j^q \tau_j^q\right) \|x_n - x_0\|^q \\ &\leq q\rho_j\left(\mu_j \tau_j^q - \nu_j\right)\|x_n - x_0\|^q + \left(1 + c_q \rho_j^q \tau_j^q\right)\|x_n - x_0\|^q \\ &= \left(1 + q\rho_j \mu_j \tau_j^q + c_q \rho_j^q \tau_j^q - q\rho_j \nu_j\right)\|x_n - x_0\|^q, \end{aligned}$$

$$\begin{aligned}
\widetilde{L}_{g_j} &= \left\| x_n - x_0 - \left(g_j(x_n) - g_j(x_0)\right) \right\|^q \\
&\le \|x_n - x_0\|^q - q\langle g_j(x_n) - g_j(x_0), J_q(x_n - x_0)\rangle + c_q \left\| g_j(x_n) - g_j(x_0) \right\|^q \\
&\le \left(1 + c_q \iota_j^q\right) \|x_n - x_0\|^q + q\left(a_j \left\| g_j(x_n) - g_j(x_0) \right\|^q - b_j \|x_n - x_0\|^q\right) \\
&\le \left(1 + c_q \iota_j^q\right) \|x_n - x_0\|^q + q\left(a_j \iota_j^q - b_j\right) \|x_n - x_0\|^q \\
&= \left(1 + q a_j \iota_j^q + c_q \iota_j^q - q b_j\right) \|x_n - x_0\|^q.
\end{aligned} \tag{3.7}$$

Therefore, from Lemma 2.10, we get

$$\begin{aligned}
\left\| y_j(x_n) - y_j(x_0) \right\| &= \left\| R_{\rho_j}^{M_j}\left(g_j(x_n) - \rho_j A_j(x_n, y_j(x_n))\right) - R_{\rho_j}^{M_j}\left(g_j(x_0) - \rho_j A_j(x_0, y_j(x_0))\right) \right\| \\
&\le \left\| g_j(x_n) - \rho_j A_j(x_n, y_j(x_n)) - \left(g_j(x_0) - \rho_j A_j(x_0, y_j(x_0))\right) \right\| \\
&= \left\| \left(g_j(x_n) - g_j(x_0)\right) - \rho_j\left(A_j(x_n, y_j(x_n)) - A_j(x_0, y_j(x_0))\right) \right\| \\
&\le \left\| x_n - x_0 - \left(g_j(x_n) - g_j(x_0)\right) \right\| \\
&\quad + \left\| x_n - x_0 - \rho_j\left(A_j(x_n, y_j(x_n)) - A_j(x_0, y_j(x_0))\right) \right\| \\
&\le \sqrt[q]{\widetilde{L}_{A_j}} + \sqrt[q]{\widetilde{L}_{g_j}} \\
&\le \left(\sqrt[q]{1 + q\rho_j \mu_j \tau_j^q + c_q \rho_j^q \tau_j^q - q\rho_j \nu_j} + \sqrt[q]{1 + q a_j \iota_j^q + c_q \iota_j^q - q b_j} \right) \|x_n - x_0\|.
\end{aligned} \tag{3.8}$$

From (3.3), it follows that the implicit function $y_j(x)$ is continuous on E. This completes the proof. □

If $j = 2$ and $g(x) = x$ for all $x \in E$, then Theorem 3.2 is reduced to the following result.

Corollary 3.3 (see [24]). *Let E be a real q-uniformly smooth Banach space. Let $M_2 : E \to 2^E$ be m-accretive mapping; Let $A_2 : E \times E \to E$ be (μ_2, ν_2)-relaxed cocoercive and Lipschitz continuous in the first variable with constant τ_2. Then, for each $x \in E$, the mapping $R_{\rho_2}^{M_2}(x - \rho_2 A_2(x, \cdot)) : E \to E$ has at most one fixed point. If*

$$1 + q\rho_2 \mu_2 \tau_2^q + c_q \rho_2^q \tau_2^q - q\rho_2 \nu_2 \ge 0, \tag{3.9}$$

then the implicit function $y(x)$ determined by

$$y(x) = R_{\rho_2}^{M_2}\left(x - \rho_2 A_2(x, y_2(x))\right) \tag{3.10}$$

is continuous on E.

Theorem 3.4. *Let E be a real q-uniformly smooth Banach space. Let $M_j : E \to 2^E$ be m-accretive mapping, Let $A_j : E \times E \to E$ be (μ_j, ν_j)-relaxed cocoercive and Lipschitz continuous in the first variable with constant τ_j, and let $g_j : E \to E$ be (a_j, b_j)-relaxed cocoercive and Lipschitz continuous with constant ι_j for $j \in \{1, 2, \ldots, n\}$. Assume that*

$$\min\left\{1 + qa_j\iota_j^q + c_q\iota_j^q - qb_j, 1 + q\rho_j\mu_j\tau_j^q + c_q\rho_j^q\tau_j^q - q\rho_j\nu_j\right\} \geq 0, \quad j = 1, 2, \ldots, n, \tag{3.11}$$

$$0 \leq \prod_{j=1}^{n}\left(\sqrt[q]{1 + qa_j\iota_j^q + c_q\iota_j^q - qb_j} + \sqrt[q]{1 + q\rho_j\mu_j\tau_j^q + c_q\rho_j^q\tau_j^q - q\rho_j\nu_j}\right) < 1. \tag{3.12}$$

Then the problem (SGMQVI) has a solution. Moreover, the solutions set Ξ of (SGMQVI) is a singleton.

Proof. By Theorem 3.1, the problem (SGMQVI) has a solution if and only if (3.1) holds. For the convenience, we define a mapping $F : E \to E$ by

$$\begin{aligned}
F(x) &= R_{\rho_1}^{M_1}\left(g_1(y_2(x)) - \rho_1 A_1(y_2(x), x)\right),\\
y_2(x) &= R_{\rho_2}^{M_2}\left(g_2(y_3(x)) - \rho_2 A_2(y_3(x), y_2(x))\right),\\
&\vdots\\
y_{n-1}(x) &= R_{\rho_{n-1}}^{M_{n-1}}\left(g_{n-1}(y_n(x)) - \rho_{n-1} A_{n-1}(y_n(x), y_{n-1}(x))\right),\\
y_n(x) &= R_{\rho_n}^{M_n}\left(g_n(x) - \rho_n A_n(x, y_n(x))\right), \quad x \in E.
\end{aligned} \tag{3.13}$$

Since $A_j : E \times E \to E$ are (μ_j, ν_j)-relaxed cocoercive and Lipschitz continuous in the first variable with constant τ_j and $g_j : E \to E$ are (a_j, b_j)-relaxed cocoercive and Lipschitz continuous with constant ι_j for $j \in \{1, 2, \ldots, n\}$, by Theorem 3.2, we know that $F(x)$ and $y_i(x) (i = 2, 3, \ldots, n)$ are continuous on E. For any $x, z \in E$,

$$\begin{aligned}
L_{A_j} &= \left\|y_{j+1}(x) - y_{j+1}(z) - \rho_j\left[A_j(y_{j+1}(x), y_j(x)) - A_j(y_{j+1}(z), y_j(z))\right]\right\|^q\\
&\leq -q\rho_j\left\langle A_j(y_{j+1}(x), y_j(x)) - A_j(y_{j+1}(z), y_j(z)), J_q(y_{j+1}(x) - y_{j+1}(z))\right\rangle\\
&\quad + \left\|y_{j+1}(x) - y_{j+1}(z)\right\|^q + c_q\rho_j^q\left\|A_j(y_{j+1}(x), y_j(x)) - A_j(y_{j+1}(z), y_j(z))\right\|^q\\
&\leq q\rho_j\left(\mu_j\left\|A_j(y_{j+1}(x), y_j(x)) - A_j(y_{j+1}(z), y_j(z))\right\|^q - \nu_j\left\|y_{j+1}(x) - y_{j+1}(z)\right\|^q\right)\\
&\quad + \left(1 + c_q\rho_j^q\tau_j^q\right)\left\|y_{j+1}(x) - y_{j+1}(z)\right\|^q
\end{aligned}$$

$$
\begin{aligned}
&\le \left(1 + q\rho_j\mu_j\tau_j^q + c_q\rho_j^q\tau_j^q - q\rho_j\nu_j\right)\left\|y_{j+1}(x) - y_{j+1}(z)\right\|^q, \quad j = 2, 3, \ldots, n-1, \\
L_{g_j} &= \left\|y_{j+1}(x) - y_{j+1}(z) - \left(g_j\left(y_{j+1}(x)\right) - g_j\left(y_{j+1}(z)\right)\right)\right\|^q \\
&\le \left\|y_{j+1}(x) - y_{j+1}(z)\right\|^q - q\left\langle g_j\left(y_{j+1}(x)\right) - g_j\left(y_{j+1}(z)\right), J_q\left(y_{j+1}(x) - y_{j+1}(z)\right)\right\rangle \\
&\quad + c_q\left\|g_j\left(y_{j+1}(x)\right) - g_j\left(y_{j+1}(z)\right)\right\|^q \\
&\le \left(1 + qa_j\iota_j^q + c_q\iota_j^q - qb_j\right)\left\|y_{j+1}(x) - y_{j+1}(z)\right\|^q, \quad j = 2, 3, \ldots, n-1, \\
L_{A_n} &= \left\|x - z - \rho_n\left[A_n\left(x, y_n(x)\right) - A_n\left(z, y_n(z)\right)\right]\right\|^q \\
&\le \|x - z\|^q - q\rho_n\left\langle A_n\left(x, y_n(x)\right) - A_n\left(z, y_n(z)\right), J_q(x - z)\right\rangle \\
&\quad + c_q\rho_n^q\left\|A_n\left(x, y_n(x)\right) - A_n\left(z, y_n(z)\right)\right\|^q \\
&\le \left(1 + q\rho_n\mu_n\tau_n^q + c_q\rho_n^q\tau_n^q - q\rho_n\nu_n\right)\|x - z\|^q, \\
L_{g_n} &= \left\|x - z - \left(g_n(x) - g_n(z)\right)\right\|^q \\
&\le \|x - z\|^q - q\left\langle g_n(x) - g_n(z), J_q(x - z)\right\rangle + c_q\left\|g_n(x) - g_n(z)\right\|^q \\
&\le \left(1 + qa_n\iota_n^q + c_q\iota_n^q - qb_n\right)\|x - z\|^q, \\
L_{A_1} &= \left\|y_2(x) - y_2(z) - \rho_1\left[A_1\left(y_2(x), x\right) - A_1\left(y_2(z), z\right)\right]\right\|^q \\
&\le \left(1 + q\rho_1\mu_1\tau_1^q + c_q\rho_1^q\tau_1^q - q\rho_1\nu_1\right)\left\|y_2(x) - y_2(z)\right\|^q, \\
L_{g_1} &= \left\|y_2(x) - y_2(z) - \left(g_1\left(y_2(x)\right) - g_1\left(y_2(z)\right)\right)\right\|^q \\
&\le \left(1 + qa_1\iota_1^q + c_q\iota_1^q - qb_1\right)\left\|y_2(x) - y_2(z)\right\|^q.
\end{aligned}
\tag{3.14}
$$

From Lemma 2.10, it yields that

$$
\begin{aligned}
&\|F(x) - F(z)\| \\
&\quad = \left\|R_{\rho_1}^{M_1}\left(g_1\left(y_2(x)\right) - \rho_1A_1\left(y_2(x), x\right)\right) - R_{\rho_1}^{M_1}\left(g_1\left(y_2(z)\right) - \rho_1A_1\left(y_2(z), z\right)\right)\right\| \\
&\quad \le \left\|\left(g_1\left(y_2(x)\right) - \rho_1A_1\left(y_2(x), x\right)\right) - \left(g_1\left(y_2(z)\right) - \rho_1A_1\left(y_2(z), z\right)\right)\right\| \\
&\quad = \left\|\left(g_1\left(y_2(x)\right) - g_1\left(y_2(z)\right)\right) - \rho_1\left(A_1\left(y_2(x), x\right) - A_1\left(y_2(z), z\right)\right)\right\| \\
&\quad \le \left\|\left(y_2(x) - y_2(z)\right) - \rho_1\left(A_1\left(y_2(x), x\right) - A_1\left(y_2(z), z\right)\right)\right\| \\
&\qquad + \left\|\left(y_2(x) - y_2(z)\right) - \left(g_1\left(y_2(x)\right) - g_1\left(y_2(z)\right)\right)\right\| \\
&\quad \le \left(\sqrt[q]{1 + qa_1\iota_1^q + c_q\iota_1^q - qb_1} + \sqrt[q]{1 + q\rho_1\mu_1\tau_1^q + c_q\rho_1^q\tau_1^q - q\rho_1\nu_1}\right)\left\|y_2(x) - y_2(z)\right\|.
\end{aligned}
\tag{3.15}
$$

Note that, for each $j \in \{2, 3, \ldots, n-1\}$,

$$\begin{aligned}
&\|y_j(x) - y_j(z)\| \\
&\quad = \left\| R_{\rho_j}^{M_j}\left(g_j(y_{j+1}(x)) - \rho_j A_j(y_{j+1}(x), y_j(x))\right) - R_{\rho_j}^{M_j}\left(g_j(y_{j+1}(z)) - \rho_j A_j(y_{j+1}(z), y_j(z))\right) \right\| \\
&\quad \leq \left\| \left(g_j(y_{j+1}(x)) - \rho_j A_j(y_{j+1}(x), y_j(x))\right) - \left(g_j(y_{j+1}(z)) - \rho_j A_j(y_{j+1}(z), y_j(z))\right) \right\| \\
&\quad = \left\| \left(g_j(y_{j+1}(x)) - g_j(y_{j+1}(z))\right) - \rho_j\left(A_j(y_{j+1}(x), y_j(x)) - A_j(y_{j+1}(z), y_j(z))\right) \right\| \\
&\quad \leq \left\| \left(y_{j+1}(x) - y_{j+1}(z)\right) - \rho_j\left(A_j(y_{j+1}(x), y_j(x)) - A_j(y_{j+1}(z), y_j(z))\right) \right\| \\
&\qquad + \left\| \left(y_{j+1}(x) - y_{j+1}(z)\right) - \left(g_j(y_{j+1}(x)) - g_j(y_{j+1}(z))\right) \right\| \\
&\quad \leq \left(\sqrt[q]{1 + q\rho_j\mu_j\tau_j^q + c_q\rho_j^q\tau_j^q - q\rho_j\nu_j} + \sqrt[q]{1 + qa_j\iota_j^q + c_q\iota_j^q - qb_j} \right) \|y_{j+1}(x) - y_{j+1}(z)\|, \\
&\|y_n(x) - y_n(z)\| \\
&\quad = \left\| R_{\rho_n}^{M_n}\left(g_n(x) - \rho_n A_n(x, y_n(x))\right) - R_{\rho_n}^{M_n}\left(g_n(z) - \rho_n A_n(z, y_n(z))\right) \right\| \\
&\quad \leq \left\| \left(g_n(x) - g_n(z)\right) - \rho_n\left(A_n(x, y_n(x)) - A_n(z, y_n(z))\right) \right\| \\
&\quad \leq \left\| x - z - \rho_n\left(A_n(x, y_n(x)) - A_n(z, y_n(z))\right) \right\| + \left\| x - z - \left(g_n(x) - g_n(z)\right) \right\| \\
&\quad \leq \left(\sqrt[q]{1 + q\rho_n\mu_n\tau_n^q + c_q\rho_n^q\tau_n^q - q\rho_n\nu_n} + \sqrt[q]{1 + qa_n\iota_n^q + c_q\iota_n^q - qb_n} \right) \|x - z\|.
\end{aligned} \tag{3.16}$$

Therefore, we obtain

$$\|F(x) - F(z)\| \leq \prod_{j=1}^{n} \left(\sqrt[q]{1 + q\rho_j\mu_j\tau_j^q + c_q\rho_j^q\tau_j^q - q\rho_j\nu_j} + \sqrt[q]{1 + qa_j\iota_j^q + c_q\iota_j^q - qb_j} \right) \|x - z\|. \tag{3.17}$$

It follows from (3.12) that the mapping F is contractive. By banach's contraction principle, there exists a unique $x_1^* \in E$ such that $F(x_1^*) = x_1^*$. Therefore, by Theorem 3.2, there exists an unique $(x_1^*, x_2^*, \ldots, x_n^*) \in E^n$ such that $(x_1^*, x_2^*, \ldots, x_n^*)$ is a solution of the problem (SGMQVI), where $x_i^* = y_i(x_1^*)$ for $i = 2, 3, \ldots, n$, that is, $\Xi = \{(x_1^*, x_2^*, \ldots, x_n^*)\}$. This completes the proof. □

4. Convergence Analysis

In this section, we introduce several implicit algorithms with errors and explicit algorithms without errors for solving the system of generalized mixed quasivariational inclusions problem (SGMQVI) and then explore the convergence analysis of the iterative sequences generated by the corresponding algorithms.

From Section 3, we know that the system of generalized mixed quasivariational inclusions problem (SGMQVI) is equivalent to the fixed point problem (3.1). This equivalent formulation is crucial from the numerical analysis point of view. As we know, this fixed point formulation has been used to suggest and analyze some iterative methods for solving variational inequalities and related optimization problems. By using the relations between the problem (SGMQVI) and the fixed point problem (3.1), we construct the following iterative algorithms for solving the system of generalized mixed quasivariational inclusions problem (2.3).

Algorithm 4.1. Let ρ_j be positive constants for all $j = 1, 2, \ldots, n$. For any given points $x_{1,0} \in E$, define sequences $\{x_{j,k}\} j = 1, 2, \ldots, n$ in E by the following implicit algorithm:

$$
\begin{aligned}
x_{n,k} &= R_{\rho_n}^{M_n}\left(g_n(x_{1,k}) - \rho_n A_n(x_{1,k}, x_{n,k})\right), \\
x_{n-1,k} &= R_{\rho_{n-1}}^{M_{n-1}}\left(g_{n-1}(x_{n,k}) - \rho_{n-1} A_{n-1}(x_{n,k}, x_{n-1,k})\right), \\
&\ \ \vdots \\
x_{2,k} &= R_{\rho_2}^{M_2}\left(g_2(x_{3,k}) - \rho_2 A_2(x_{3,k}, x_{2,k})\right), \\
x_{1,k+1} &= (1-\alpha_k)x_{1,k} + \alpha_k R_{\rho_1}^{M_1}\left(g_1(x_{2,k}) - \rho_1 A_1(x_{2,k}, x_{1,k})\right) + e_k, \quad k = 0, 1, 2, \ldots,
\end{aligned}
\tag{4.1}
$$

where $\{e_k\} \subset E$ and $\{\alpha_k\}$ is a real sequence in $[0,1]$.

If $e_k \equiv 0$ for all $k \geq 0$, then Algorithm 4.1 is reduced to the following result.

Algorithm 4.2. Let ρ_j be positive constants for all $j = 1, 2, \ldots, n$. For any given points $x_{1,0} \in E$, define sequences $\{x_{j,k}\} j = 1, 2, \ldots, n$ in E by the following implicit algorithm

$$
\begin{aligned}
x_{n,k} &= R_{\rho_n}^{M_n}\left(g_n(x_{1,k}) - \rho_n A_n(x_{1,k}, x_{n,k})\right), \\
x_{n-1,k} &= R_{\rho_{n-1}}^{M_{n-1}}\left(g_{n-1}(x_{n,k}) - \rho_{n-1} A_{n-1}(x_{n,k}, x_{n-1,k})\right), \\
&\ \ \vdots \\
x_{2,k} &= R_{\rho_2}^{M_2}\left(g_2(x_{3,k}) - \rho_2 A_2(x_{3,k}, x_{2,k})\right), \\
x_{1,k+1} &= (1-\alpha_k)x_{1,k} + \alpha_k R_{\rho_1}^{M_1}\left(g_1(x_{2,k}) - \rho_1 A_1(x_{2,k}, x_{1,k})\right), \quad k = 0, 1, 2, \ldots,
\end{aligned}
\tag{4.2}
$$

where $\{\alpha_k\}$ is a real sequence in $[0,1]$.

Now we construct an explicit algorithms for solving the system of generalized mixed quasivariational inclusions problem (SGMQVI).

Algorithm 4.3. Let ρ_j be positive constants for all $j = 1, 2, \ldots, n$. For any given points $(x_{1,0}, x_{2,0}, \ldots, x_{n-1,0}, x_{n,0}) \in E^n$, define sequences $\{x_{j,k}\}(j = 1, 2, \ldots, n)$ in E by the following explicit algorithm

$$
\begin{aligned}
x_{n,k+1} &= R_{\rho_n}^{M_n}\left(g_n(x_{1,k}) - \rho_n A_n(x_{1,k}, x_{n,k})\right), \\
x_{n-1,k+1} &= R_{\rho_{n-1}}^{M_{n-1}}\left(g_{n-1}(x_{n,k+1}) - \rho_{n-1} A_{n-1}(x_{n,k+1}, x_{n-1,k})\right), \\
&\vdots \\
x_{2,k+1} &= R_{\rho_2}^{M_2}\left(g_2(x_{3,k+1}) - \rho_2 A_2(x_{3,k+1}, x_{2,k})\right), \\
x_{1,k+1} &= (1-\alpha_k)x_{1,k} + \alpha_k R_{\rho_1}^{M_1}\left(g_1(x_{2,k+1}) - \rho_1 A_1(x_{2,k+1}, x_{1,k})\right), \quad k = 0, 1, 2, \ldots,
\end{aligned}
\tag{4.3}
$$

where $\{\alpha_k\}$ is a real sequence in $[0, 1]$.

Remark 4.4. If $n = 2, E = H$ is a Hilbert space, and K is a closed convex subset of E, $\phi(x) = \delta_K(x)$ for all $x \in K$, and $M_1(x) = M_2(x) = \partial\phi(x)$ for all $x \in E$, where δ_K is the indicator function of K, and $\partial\phi$ denotes the subdifferential operator of ϕ, then, from Remark 2.9, Algorithms 4.1 and 4.3 are reduced to the Algorithms 4.5 and 4.6 for solving the system of general variational inequalities problem (2.7).

Algorithm 4.5. Let ρ_j be positive constants for all $j = 1, 2$. For any given points $x_{1,0} \in E$, define sequences $\{x_{j,k}\}j = 1, 2$ in E by the following implicit algorithm:

$$
\begin{aligned}
x_{2,k} &= P_K\left(g_2(x_{1,k}) - \rho_2 A_2(x_{1,k}, x_{2,k})\right), \\
x_{1,k+1} &= (1-\alpha_k)x_{1,k} + \alpha_k P_K\left(g_1(x_{2,k}) - \rho_1 A_1(x_{2,k}, x_{1,k})\right) + e_k, \quad k = 0, 1, 2, \ldots,
\end{aligned}
\tag{4.4}
$$

where $\{e_k\} \subset E$ and $\{\alpha_k\}$ is a real sequence in $[0, 1]$.

Algorithm 4.6. Let ρ_j be positive constants for all $j = 1, 2$. For any given points $(x_{1,0}, x_{2,0}) \in E^2$, define sequences $\{x_{j,k}\}(j = 1, 2)$ in E by the following explicit algorithm:

$$
\begin{aligned}
x_{2,k+1} &= P_K\left(g_2(x_{1,k}) - \rho_2 A_2(x_{1,k}, x_{2,k})\right), \\
x_{1,k+1} &= (1-\alpha_k)x_{1,k} + \alpha_k P_K\left(g_1(x_{2,k+1}) - \rho_1 A_1(x_{2,k+1}, x_{1,k})\right), \quad k = 0, 1, 2, \ldots,
\end{aligned}
\tag{4.5}
$$

where $\{\alpha_k\}$ is a real sequence in $[0, 1]$.

Theorem 4.7. *Let E be a real q-uniformly smooth Banach space, and let M_j, A_j, and g_j ($j = 1, 2, \ldots, n$) be the same as in Theorem 3.4. Assume that $\{\alpha_n\}$ is a real sequence in $(0, 1]$ and satisfy the following conditions:*

(i) $\sum_{k=0}^{\infty} \alpha_k = \infty$;

(ii) $\sum_{k=0}^{\infty} \|e_k\| < +\infty$;

(iii) $\min\{1 + qa_j\iota_j^q + c_q\iota_j^q - qb_j, 1 + q\rho_j\mu_j\tau_j^q + c_q\rho_j^q\tau_j^q - q\rho_j\nu_j\} \geq 0,\ j = 1, 2, \ldots, n$;

(iv) $0 < \prod_{j=1}^{n}(\sqrt[q]{1 + qa_j\iota_j^q + c_q\iota_j^q - qb_j} + \sqrt[q]{1 + q\rho_j\mu_j\tau_j^q + c_q\rho_j^q\tau_j^q - q\rho_j\nu_j}) < 1$.

Then the sequences $\{x_{j,k}\}(j = 1, 2, \ldots, n)$ *generated by Algorithm 4.1 converge strongly to* x_j^* $(j = 1, 2, \ldots, n)$, *respectively, such that* $(x_1^*, x_2^*, \ldots, x_n^*) \in \Xi$.

Proof. By Theorem 3.4, we know that there exist an unique point $(x_1^*, x_2^*, \ldots, x_n^*) \in E^n$ such that $(x_1^*, x_2^*, \ldots, x_n^*) \in \Xi$. Then, from Theorem 3.1, one has

$$\begin{aligned}
x_1^* &= R_{\rho_1}^{M_1}\left(g_1(x_2^*) - \rho_1 A_1(x_2^*, x_1^*)\right), \\
x_2^* &= R_{\rho_2}^{M_2}\left(g_2(x_3^*) - \rho_2 A_2(x_3^*, x_2^*)\right), \\
&\vdots \\
x_{n-1}^* &= R_{\rho_{n-1}}^{M_{n-1}}\left(g_{n-1}(x_n^*) - \rho_{n-1} A_{n-1}(x_n^*, x_{n-1}^*)\right), \\
x_n^* &= R_{\rho_n}^{M_n}\left(g_n(x_1^*) - \rho_n A_n(x_1^*, x_n^*)\right).
\end{aligned} \tag{4.6}$$

Therefore, from both (4.1) and (4.6), we have

$$\begin{aligned}
\left\|x_{1,k+1} - x_1^*\right\| &= \left\|(1-\alpha_k)x_{1,k} + \alpha_k R_{\rho_1}^{M_1}\left(g_1(x_{2,k}) - \rho_1 A_1(x_{2,k}, x_{1,k})\right) + e_k - x_1^*\right\| \\
&= \left\|(1-\alpha_k)\left(x_{1,k} - x_1^*\right) + \alpha_k\left(R_{\rho_1}^{M_1}\left(g_1(x_{2,k}) - \rho_1 A_1(x_{2,k}, x_{1,k})\right) - x_1^*\right) + e_k\right\| \\
&= \Big\|\alpha_k\left(R_{\rho_1}^{M_1}\left(g_1(x_{2,k}) - \rho_1 A_1(x_{2,k}, x_{1,k})\right) - R_{\rho_1}^{M_1}\left(g_1(x_2^*) - \rho_1 A_1(x_2^*, x_1^*)\right)\right) \\
&\qquad +(1-\alpha_k)\left(x_{1,k} - x_1^*\right) + e_k\Big\| \\
&\leq \alpha_k\left\|R_{\rho_1}^{M_1}\left(g_1(x_{2,k}) - \rho_1 A_1(x_{2,k}, x_{1,k})\right) - R_{\rho_1}^{M_1}\left(g_1(x_2^*) - \rho_1 A_1(x_2^*, x_1^*)\right)\right\| \\
&\quad + (1-\alpha_k)\left\|x_{1,k} - x_1^*\right\| + \|e_k\| \\
&\leq \alpha_k\left\|\left(g_1(x_{2,k}) - \rho_1 A_1(x_{2,k}, x_{1,k})\right) - \left(g_1(x_2^*) - \rho_1 A_1(x_2^*, x_1^*)\right)\right\| \\
&\quad + (1-\alpha_k)\left\|x_{1,k} - x_1^*\right\| + \|e_k\| \\
&= \alpha_k\left\|\left(g_1(x_{2,k}) - g_1(x_2^*)\right) - \left(\rho_1 A_1(x_{2,k}, x_{1,k}) - \rho_1 A_1(x_2^*, x_1^*)\right)\right\| \\
&\quad + (1-\alpha_k)\left\|x_{1,k} - x_1^*\right\| + \|e_k\| \\
&\leq (1-\alpha_k)\left\|x_{1,k} - x_1^*\right\| + \|e_k\| + \alpha_k\left\|\left(x_{2,k} - x_2^*\right) - \left(g_1(x_{2,k}) - g_1(x_2^*)\right)\right\| \\
&\quad + \alpha_k\left\|\left(x_{2,k} - x_2^*\right) - \rho_1\left(A_1(x_{2,k}, x_{1,k}) - A_1(x_2^*, x_1^*)\right)\right\|.
\end{aligned} \tag{4.7}$$

Since $A_j : E \times E \to E$ are (μ_j, ν_j)-relaxed cocoercive and Lipschitz continuous in the first variable with constant τ_j and $g_j : E \to E$ are (a_j, b_j)-relaxed cocoercive and Lipschitz continuous with constant ι_j for $j \in \{1, 2, \ldots, n\}$, we can conclude

$$
\begin{aligned}
L^*_{A_i} &= \left\| x_{i+1,k} - x^*_{i+1} - \rho_i \left[A_i(x_{i+1,k}, x_{i,k}) - A_i\left(x^*_{i+1}, x^*_i\right) \right] \right\|^q \\
&\le -q\rho_i \left\langle A_i(x_{i+1,k}, x_{i,k}) - A_i\left(x^*_{i+1}, x^*_i\right), J_q\left(x_{i+1,k} - x^*_{i+1}\right) \right\rangle \\
&\quad + \left\| x_{i+1,k} - x^*_{i+1} \right\|^q + c_q \rho_i^q \left\| A_i(x_{i+1,k}, x_{i,k}) - A_i\left(x^*_{i+1}, x^*_i\right) \right\|^q \\
&\le q\rho_i \left(\mu_i \left\| A_i(x_{i+1,k}, x_{i,k}) - A_i\left(x^*_{i+1}, x^*_i\right) \right\|^q - \nu_i \left\| x_{i+1,k} - x^*_{i+1} \right\|^q \right) \\
&\quad + \left(1 + c_q \rho_i^q \tau_i^q \right) \left\| x_{i+1,k} - x^*_{i+1} \right\|^q \\
&\le \left(1 + q\rho_i \mu_i \tau_i^q + c_q \rho_i^q \tau_i^q - q\rho_i \nu_i \right) \left\| x_{i+1,k} - x^*_{i+1} \right\|^q, \quad i = 1, 2, \ldots, n-1, \\
L^*_{g_i} &= \left\| x_{i+1,k} - x^*_{i+1} - \left(g_i(x_{i+1,k}) - g_i\left(x^*_{i+1}\right) \right) \right\|^q \\
&\le -q \left\langle g_i(x_{i+1,k}) - g_i\left(x^*_{i+1}\right), J_q\left(x_{i+1,k} - x^*_{i+1}\right) \right\rangle \\
&\quad + \left\| x_{i+1,k} - x^*_{i+1} \right\|^q + c_q \left\| g_i(x_{i+1,k}) - g_i\left(x^*_{i+1}\right) \right\|^q \\
&\le \left(1 + q a_i \iota_i^q + c_q \iota_i^q - q b_i \right) \left\| x_{i+1,k} - x^*_{i+1} \right\|^q, \quad i = 1, 2, \ldots, n-1, \\
L^*_{A_n} &= \left\| x_{1,k} - x^*_1 - \rho_n \left[A_n(x_{1,k}, x_{n,k}) - A_n\left(x^*_1, x^*_n\right) \right] \right\|^q \\
&\le \left\| x_{1,k} - x^*_1 \right\|^q - q\rho_n \left\langle A_n(x_{1,k}, x_{n,k}) - A_n\left(x^*_1, x^*_n\right), J_q\left(x_{1,k} - x^*_1\right) \right\rangle \\
&\quad + c_q \rho_n^q \left\| A_n(x_{1,k}, x_{n,k}) - A_n\left(x^*_1, x^*_n\right) \right\|^q \\
&\le \left(1 + q\rho_n \mu_n \tau_n^q + c_q \rho_n^q \tau_n^q - q\rho_n \nu_n \right) \left\| x_{1,k} - x^*_1 \right\|^q, \\
L^*_{g_n} &= \left\| x_{1,k} - x^*_1 - \left(g_n(x_{1,k}) - g_n\left(x^*_1\right) \right) \right\|^q \\
&\le \left\| x_{1,k} - x^*_1 \right\|^q - q \left\langle g_n(x_{1,k}) - g_n\left(x^*_1\right), J_q\left(x_{1,k} - x^*_1\right) \right\rangle + c_q \left\| g_n(x_{1,k}) - g_n\left(x^*_1\right) \right\|^q \\
&\le \left(1 + q a_n \iota_n^q + c_q \iota_n^q - q b_n \right) \left\| x_{1,k} - x^*_1 \right\|^q.
\end{aligned}
\tag{4.8}
$$

Noticing that, for each $i \in \{1, 2, \ldots, n-2\}$,

$$
\begin{aligned}
\left\| x_{i+1,k} - x^*_{i+1} \right\| &= \Big\| R^{M_{i+1}}_{\rho_{i+1}} \left(g_{i+1}(x_{i+2,k}) - \rho_{i+1} A_{i+1}(x_{i+2,k}, x_{i+1,k}) \right) \\
&\qquad - R^{M_{i+1}}_{\rho_{i+1}} \left(g_{i+1}\left(x^*_{i+2}\right) - \rho_{i+1} A_{i+1}\left(x^*_{i+2}, x^*_{i+1}\right) \right) \Big\| \\
&\le \Big\| \left(g_{i+1}(x_{i+2,k}) - \rho_{i+1} A_{i+1}(x_{i+2,k}, x_{i+1,k}) \right) \\
&\qquad - \left(g_{i+1}\left(x^*_{i+2}\right) - \rho_{i+1} A_{i+1}\left(x^*_{i+2}, x^*_{i+1}\right) \right) \Big\|
\end{aligned}
$$

$$
\begin{aligned}
&= \left\| \left(g_{i+1}(x_{i+2,k}) - g_{i+1}(x^*_{i+2}) \right) - \rho_{i+1}\left(A_{i+1}(x_{i+2,k}, x_{i+1,k}) - A_{i+1}(x^*_{i+2}, x^*_{i+1}) \right) \right\| \\
&\leq \left\| \left(x_{i+2,k} - x^*_{i+2} \right) - \rho_{i+1}\left(A_{i+1}(x_{i+2,k}, x_{i+1,k}) - A_{i+1}(x^*_{i+2}, x^*_{i+1}) \right) \right\| \\
&\quad + \left\| \left(x_{i+2,k} - x^*_{i+2} \right) - \left(g_{i+1}(x_{i+2,k}) - g_{i+1}(x^*_{i+2}) \right) \right\| \\
&\leq \Bigg(\sqrt[q]{1 + q\rho_{i+1}\mu_{i+1}\tau^q_{i+1} + c_q\rho^q_{i+1}\tau^q_{i+1} - q\rho_{i+1}\nu_{i+1}} \\
&\quad + \sqrt[q]{1 + qa_{i+1}\iota^q_{i+1} + c_q\iota^q_{i+1} - qb_{i+1}} \Bigg) \left\| x_{i+2,k} - x^*_{i+2} \right\|, \\
\|x_{n,k} - x^*_n\| &= \left\| R^{M_n}_{\rho_n}\left(g_n(x_{1,k}) - \rho_n A_n(x_{1,k}, x_{n,k}) \right) - R^{M_n}_{\rho_n}\left(g_n(x^*_1) - \rho_n A_n(x^*_1, x^*_n) \right) \right\| \\
&\leq \left\| \left(g_n(x_{1,k}) - \rho_n A_n(x_{1,k}, x_{n,k}) \right) - \left(g_n(x^*_1) - \rho_n A_n(x^*_1, x^*_n) \right) \right\| \\
&= \left\| \left(g_n(x_{1,k}) - g_n(x^*_1) \right) - \rho_n\left(A_n(x_{1,k}, x_{n,k}) - A_n(x^*_1, x^*_n) \right) \right\| \\
&\leq \left\| \left(x_{1,k} - x^*_1 \right) - \left(\rho_n A_n(x_{1,k}, x_{n,k}) - \rho_n A_n(x^*_1, x^*_n) \right) \right\| \\
&\quad + \left\| \left(x_{1,k} - x^*_1 \right) - \left(g_n(x_{1,k}) - g_n(x^*_1) \right) \right\| \\
&\leq \left(\sqrt[q]{1 + q\rho_n\mu_n\tau^q_n + c_q\rho^q_n\tau^q_n - q\rho_n\nu_n} + \sqrt[q]{1 + qa_n\iota^q_n + c_q\iota^q_n - qb_n} \right) \\
&\quad \times \left\| x_{1,k} - x^*_1 \right\|.
\end{aligned}
\tag{4.9}
$$

As a consequence, we have

$$
\begin{aligned}
\|x_{1,k+1} - x^*_1\| &\leq \alpha_k \prod_{j=1}^{n} \left(\sqrt[q]{1 + q\rho_j\mu_j\tau^q_j + c_q\rho^q_j\tau^q_j - q\rho_j\nu_j} + \sqrt[q]{1 + qa_j\iota^q_j + c_q\iota^q_j - qb_j} \right) \left\| x_{1,k} - x^*_1 \right\| \\
&\quad + (1 - \alpha_k)\left\| x_{1,k} - x^*_1 \right\| + \|e_k\| \\
&= \left[1 - \alpha_k \left(1 - \prod_{j=1}^{n} \left(\sqrt[q]{1 + q\rho_j\mu_j\tau^q_j + c_q\rho^q_j\tau^q_j - q\rho_j\nu_j} + \sqrt[q]{1 + qa_j\iota^q_j + c_q\iota^q_j - qb_j} \right) \right) \right] \\
&\quad \times \left\| x_{1,k} - x^*_1 \right\| + \|e_k\|.
\end{aligned}
\tag{4.10}
$$

Putting $\lambda_k = \alpha_k(1 - \prod_{j=1}^{n}(\sqrt[q]{1 + q\rho_j\mu_j\tau^q_j + c_q\rho^q_j\tau^q_j - q\rho_j\nu_j} + \sqrt[q]{1 + qa_j\iota^q_j + c_q\iota^q_j - qb_j}))$, $C_k = 0$, $B_k = \|x_{1,k} - x^*_1\|$, and $D_k = \|e_k\|$. Then $B_{k+1} \leq (1 - \lambda_k)B_k + C_k + D_k$. From the conditions (i)–(iv), it follows that

$$
\sum_{k=0}^{\infty} \lambda_k = \infty, \quad C_k = 0(\lambda_k), \quad \sum_{k=0}^{\infty} D_k < \infty, \quad 0 < \lambda_k < 1, \ \forall k \in N. \tag{4.11}
$$

Therefore, by Lemma 2.11 and (4.11), one has

$$\lim_{k\to\infty} B_k = 0, \tag{4.12}$$

that is, $\lim_{k\to\infty} x_{1,k} = x_1^*$. Again from (iii), this shows that

$$\sqrt[q]{1+q\rho_j\mu_j\tau_j^q + c_q\rho_j^q\tau_j^q - q\rho_j\nu_j} + \sqrt[q]{1+qa_j\iota_j^q + c_q\iota_j^q - qb_j} \geq 0, \quad j = 2,3,\ldots,n, \tag{4.13}$$

and so,

$$\lim_{k\to\infty} \left\| x_{j,k} - x_j^* \right\| = 0, \tag{4.14}$$

That is, $x_{j,k} \to x_j^*$ as $k \to \infty$ for $j = 2,3,\ldots,n$. Thus, $(x_{1,k}, x_{2,k},\ldots,x_{n,k})$ converges strongly to $(x_1^*, x_2^*,\ldots,x_n^*)$. This completes the proof. □

Theorem 4.8. *Let E be a real q-uniformly smooth Banach space, and let M_j, A_j and g_j $(j = 1,2,\ldots, n)$ be the same as in Theorem 3.4. Assume that $\{\alpha_n\}$ is a real sequence in $(0,1]$ and satisfies the following conditions:*

(i) $\sum_{k=0}^{\infty} \alpha_k = \infty$;

(ii) $\min\{1+qa_j\iota_j^q + c_q\iota_j^q - qb_j, 1+q\rho_j\mu_j\tau_j^q + c_q\rho_j^q\tau_j^q - q\rho_j\nu_j\} \geq 0$, $j = 1,2,\ldots,n$;

(iii) $0 < \prod_{j=1}^{n}(\sqrt[q]{1+qa_j\iota_j^q + c_q\iota_j^q - qb_j} + \sqrt[q]{1+q\rho_j\mu_j\tau_j^q + c_q\rho_j^q\tau_j^q - q\rho_j\nu_j}) < 1$.

Then the sequences $\{x_{j,k}\}$ $(j = 1,2,\ldots,n)$ generated by Algorithm 4.2 converge strongly to $x_j^*(j = 1,2,\ldots,n)$, respectively, such that $(x_1^*, x_2^*,\ldots,x_n^*) \in \Xi$.

Proof. It directly follows from Theorem 4.7, and so the proof is omitted. This completes the proof. □

Theorem 4.9. *Let E be a real q-uniformly smooth Banach space, and let M_j, A_j and $g_j(j = 1,2,\ldots, n)$ be the same as in Theorem 3.4. Assume that $\{\alpha_n\}$ is a real sequence in $(0,1]$ and satisfy the following conditions:*

(i) $\sum_{k=0}^{\infty} \alpha_k = \infty$;

(ii) $\min\{1+qa_j\iota_j^q + c_q\iota_j^q - qb_j, 1+q\rho_j\mu_j\tau_j^q + c_q\rho_j^q\tau_j^q - q\rho_j\nu_j\} \geq 0$, $j = 1,2,\ldots,n$;

(iii) $0 < \prod_{j=1}^{n}(\sqrt[q]{1+qa_j\iota_j^q + c_q\iota_j^q - qb_j} + \sqrt[q]{1+q\rho_j\mu_j\tau_j^q + c_q\rho_j^q\tau_j^q - q\rho_j\nu_j}) < 1$.

Then the sequences $\{x_{j,k}\}(j = 1, 2, \ldots, n)$ *generated by Algorithm 4.3 converge strongly to* $x_j^*(j = 1, 2, \ldots, n)$*, respectively, such that* $(x_1^*, x_2^*, \ldots, x_n^*) \in \Xi$.

Proof. As in the proof of Theorem 4.7, we know that there exists an unique point $(x_1^*, x_2^*, \ldots, x_n^*) \in E^n$ such that $(x_1^*, x_2^*, \ldots, x_n^*) \in \Xi$, and so

$$\begin{aligned}
x_1^* &= R_{\rho_1}^{M_1}\left(g_1\left(x_2^*\right) - \rho_1 A_1\left(x_2^*, x_1^*\right)\right), \\
x_2^* &= R_{\rho_2}^{M_2}\left(g_2\left(x_3^*\right) - \rho_2 A_2\left(x_3^*, x_2^*\right)\right), \\
&\vdots \\
x_{n-1}^* &= R_{\rho_{n-1}}^{M_{n-1}}\left(g_{n-1}(x_n^*) - \rho_{n-1} A_{n-1}\left(x_n^*, x_{n-1}^*\right)\right), \\
x_n^* &= R_{\rho_n}^{M_n}\left(g_n\left(x_1^*\right) - \rho_n A_n\left(x_1^*, x_n^*\right)\right),
\end{aligned} \tag{4.15}$$

Note that

$$\begin{aligned}
\left\|x_{1,k+1} - x_1^*\right\| &= \left\|(1-\alpha_k)x_{1,k} + \alpha_k R_{\rho_1}^{M_1}\left(g_1(x_{2,k+1}) - \rho_1 A_1(x_{2,k+1}, x_{1,k})\right) - x_1^*\right\| \\
&= \left\|(1-\alpha_k)\left(x_{1,k} - x_1^*\right) + \alpha_k\left[R_{\rho_1}^{M_1}\left(g_1(x_{2,k+1}) - \rho_1 A_1(x_{2,k+1}, x_{1,k})\right) - x_1^*\right]\right\| \\
&= \Big\|\alpha_k\left[R_{\rho_1}^{M_1}\left(g_1(x_{2,k+1}) - \rho_1 A_1(x_{2,k+1}, x_{1,k})\right) - R_{\rho_1}^{M_1}\left(g_1\left(x_2^*\right) - \rho_1 A_1\left(x_2^*, x_1^*\right)\right)\right] \\
&\qquad + (1-\alpha_k)\left(x_{1,k} - x_1^*\right)\Big\| \\
&\le \alpha_k\left\|R_{\rho_1}^{M_1}\left(g_1(x_{2,k+1}) - \rho_1 A_1(x_{2,k+1}, x_{1,k})\right) - R_{\rho_1}^{M_1}\left(g_1\left(x_2^*\right) - \rho_1 A_1\left(x_2^*, x_1^*\right)\right)\right\| \\
&\quad + (1-\alpha_k)\left\|x_{1,k} - x_1^*\right\| \\
&\le \alpha_k\left\|\left(g_1(x_{2,k+1}) - \rho_1 A_1(x_{2,k+1}, x_{1,k})\right) - \left(g_1\left(x_2^*\right) - \rho_1 A_1\left(x_2^*, x_1^*\right)\right)\right\| \\
&\quad + (1-\alpha_k)\left\|x_{1,k} - x_1^*\right\| \\
&= \alpha_k\left\|\left(g_1(x_{2,k+1}) - g_1\left(x_2^*\right)\right) - \rho_1\left(A_1(x_{2,k+1}, x_{1,k}) - A_1\left(x_2^*, x_1^*\right)\right)\right\| \\
&\quad + (1-\alpha_k)\left\|x_{1,k} - x_1^*\right\| \\
&\le (1-\alpha_k)\left\|x_{1,k} - x_1^*\right\| + \alpha_k\left\|\left(x_{2,k+1} - x_2^*\right) - \left(g_1(x_{2,k}) - g_1\left(x_2^*\right)\right)\right\| \\
&\quad + \alpha_k\left\|\left(x_{2,k+1} - x_2^*\right) - \rho_1\left(A_1(x_{2,k+1}, x_{1,k}) - A_1\left(x_2^*, x_1^*\right)\right)\right\|.
\end{aligned} \tag{4.16}$$

Since $A_j : E \times E \to E$ are (μ_j, ν_j)-relaxed cocoercive and Lipschitz continuous in the first variable with constant τ_j and $g_j : E \to E$ are (a_j, b_j)-relaxed cocoercive and Lipschitz continuous with constant ι_j for $j \in \{1, 2, \dots, n\}$, we can conclude that

$$\begin{aligned}
\widehat{L}^*_{A_i} &= \left\| x_{i+1,k+1} - x^*_{i+1} - \rho_i\left[A_i(x_{i+1,k+1}, x_{i,k}) - A_i\left(x^*_{i+1}, x^*_i\right)\right]\right\|^q \\
&\leq -q\rho_i\left\langle A_i(x_{i+1,k+1}, x_{i,k}) - A_i\left(x^*_{i+1}, x^*_i\right), J_q\left(x_{i+1,k+1} - x^*_{i+1}\right)\right\rangle \\
&\quad + \left\|x_{i+1,k+1} - x^*_{i+1}\right\|^q + c_q\rho_i^q\left\|A_i(x_{i+1,k+1}, x_{i,k}) - A_i\left(x^*_{i+1}, x^*_i\right)\right\|^q \\
&\leq q\rho_i\left(\mu_i\left\|A_i(x_{i+1,k+1}, x_{i,k}) - A_i\left(x^*_{i+1}, x^*_i\right)\right\|^q - \nu_i\left\|x_{i+1,k+1} - x^*_{i+1}\right\|^q\right) \\
&\quad + \left(1 + c_q\rho_i^q\tau_i^q\right)\left\|x_{i+1,k+1} - x^*_{i+1}\right\|^q \\
&\leq \left(1 + q\rho_i\mu_i\tau_i^q + c_q\rho_i^q\tau_i^q - q\rho_i\nu_i\right)\left\|x_{i+1,k+1} - x^*_{i+1}\right\|^q, \quad i = 1, 2, \dots, n-1, \\
\widehat{L}^*_{g_i} &= \left\|x_{i+1,k+1} - x^*_{i+1} - \left(g_i(x_{i+1,k+1}) - g_i\left(x^*_{i+1}\right)\right)\right\|^q \\
&\leq -q\left\langle g_i(x_{i+1,k+1}) - g_i\left(x^*_{i+1}\right), J_q\left(x_{i+1,k+1} - x^*_{i+1}\right)\right\rangle \\
&\quad + \left\|x_{i+1,k+1} - x^*_{i+1}\right\|^q + c_q\left\|g_i(x_{i+1,k+1}) - g_i\left(x^*_{i+1}\right)\right\|^q \\
&\leq \left(1 + qa_i\iota_i^q + c_q\iota_i^q - qb_i\right)\left\|x_{i+1,k+1} - x^*_{i+1}\right\|^q, \quad i = 1, 2, \dots, n-1, \\
\widehat{L}^*_{A_n} &= \left\|x_{1,k} - x^*_1 - \rho_n\left[A_n(x_{1,k}, x_{n,k}) - A_n\left(x^*_1, x^*_n\right)\right]\right\|^q \\
&\leq \left\|x_{1,k} - x^*_1\right\|^q - q\rho_n\left\langle A_n(x_{1,k}, x_{n,k}) - A_n\left(x^*_1, x^*_n\right), J_q\left(x_{1,k} - x^*_1\right)\right\rangle \\
&\quad + c_q\rho_n^q\left\|A_n(x_{1,k}, x_{n,k}) - A_n\left(x^*_1, x^*_n\right)\right\|^q \\
&\leq \left(1 + q\rho_n\mu_n\tau_n^q + c_q\rho_n^q\tau_n^q - q\rho_n\nu_n\right)\left\|x_{1,k} - x^*_1\right\|^q, \\
\widehat{L}^*_{g_n} &= \left\|x_{1,k} - x^*_1 - \left(g_n(x_{1,k}) - g_n\left(x^*_1\right)\right)\right\|^q \\
&\leq \left\|x_{1,k} - x^*_1\right\|^q - q\left\langle g_n(x_{1,k}) - g_n\left(x^*_1\right), J_q\left(x_{1,k} - x^*_1\right)\right\rangle + c_q\left\|g_n(x_{1,k}) - g_n\left(x^*_1\right)\right\|^q \\
&\leq \left(1 + qa_n\iota_n^q + c_q\iota_n^q - qb_n\right)\left\|x_{1,k} - x^*_1\right\|^q.
\end{aligned} \tag{4.17}$$

Noticing that, for each $i \in \{1, 2, \dots, n-2\}$,

$$\begin{aligned}
\left\|x_{i+1,k+1} - x^*_{i+1}\right\| &= \Big\| R^{M_{i+1}}_{\rho_{i+1}}\left(g_{i+1}(x_{i+2,k+1}) - \rho_{i+1}A_{i+1}(x_{i+2,k+1}, x_{i+1,k})\right) \\
&\qquad - R^{M_{i+1}}_{\rho_{i+1}}\left(g_{i+1}\left(x^*_{i+2}\right) - \rho_{i+1}A_{i+1}\left(x^*_{i+2}, x^*_{i+1}\right)\right)\Big\| \\
&\leq \Big\|\left(g_{i+1}(x_{i+2,k+1}) - \rho_{i+1}A_{i+1}(x_{i+2,k+1}, x_{i+1,k})\right) \\
&\qquad - \left(g_{i+1}\left(x^*_{i+2}\right) - \rho_{i+1}A_{i+1}\left(x^*_{i+2}, x^*_{i+1}\right)\right)\Big\|
\end{aligned}$$

$$
\begin{aligned}
&= \left\| \left(g_{i+1}(x_{i+2,k+1}) - g_{i+1}\left(x_{i+2}^*\right)\right) - \rho_{i+1} \right. \\
&\qquad \left. \times \left(A_{i+1}(x_{i+2,k+1}, x_{i+1,k}) - A_{i+1}\left(x_{i+2}^*, x_{i+1}^*\right)\right) \right\| \\
&\leq \left\| \left(x_{i+2,k+1} - x_{i+2}^*\right) - \rho_{i+1}\left(A_{i+1}(x_{i+2,k+1}, x_{i+1,k}) - A_{i+1}\left(x_{i+2}^*, x_{i+1}^*\right)\right) \right\| \\
&\quad + \left\| \left(x_{i+2,k+1} - x_{i+2}^*\right) - \left(g_{i+1}(x_{i+2,k+1}) - g_{i+1}\left(x_{i+2}^*\right)\right) \right\| \\
&\leq \left(\sqrt[q]{1 + q\rho_{i+1}\mu_{i+1}\tau_{i+1}^q + c_q\rho_{i+1}^q\tau_{i+1}^q - q\rho_{i+1}\nu_{i+1}} \right. \\
&\qquad \left. + \sqrt[q]{1 + qa_{i+1}\iota_{i+1}^q + c_q\iota_{i+1}^q - qb_{i+1}} \right) \left\| x_{i+2,k+1} - x_{i+2}^* \right\|, \\
\left\| x_{n,k+1} - x_n^* \right\| &= \left\| R_{\rho_n}^{M_n}\left(g_n(x_{1,k}) - \rho_n A_n(x_{1,k}, x_{n,k})\right) - R_{\rho_n}^{M_n}\left(g_n\left(x_1^*\right) - \rho_n A_n\left(x_1^*, x_n^*\right)\right) \right\| \\
&\leq \left\| \left(g_n(x_{1,k}) - \rho_n A_n(x_{1,k}, x_{n,k})\right) - \left(g_n\left(x_1^*\right) - \rho_n A_n\left(x_1^*, x_n^*\right)\right) \right\| \\
&= \left\| \left(g_n(x_{1,k}) - g_n\left(x_1^*\right)\right) - \rho_n\left(A_n(x_{1,k}, x_{n,k}) - A_n\left(x_1^*, x_n^*\right)\right) \right\| \\
&\leq \left\| \left(x_{1,k} - x_1^*\right) - \left(\rho_n A_n(x_{1,k}, x_{n,k}) - \rho_n A_n\left(x_1^*, x_n^*\right)\right) \right\| \\
&\quad + \left\| \left(x_{1,k} - x_1^*\right) - \left(g_n(x_{1,k}) - g_n\left(x_1^*\right)\right) \right\| \\
&\leq \left(\sqrt[q]{1 + q\rho_n\mu_n\tau_n^q + c_q\rho_n^q\tau_n^q - q\rho_n\nu_n} + \sqrt[q]{1 + qa_n\iota_n^q + c_q\iota_n^q - qb_n} \right) \\
&\quad \times \left\| x_{1,k} - x_1^* \right\|.
\end{aligned}
\tag{4.18}
$$

Consequently, we have

$$
\begin{aligned}
&\left\| x_{1,k+1} - x^* \right\| \\
&\leq \alpha_k \prod_{j=1}^{n} \left(\sqrt[q]{1 + q\rho_j\mu_j\tau_j^q + c_q\rho_j^q\tau_j^q - q\rho_j\nu_j} + \sqrt[q]{1 + qa_j\iota_j^q + c_q\iota_j^q - qb_j} \right) \left\| x_{1,k} - x_1^* \right\| \\
&\quad + (1-\alpha_k)\left\| x_{1,k} - x^* \right\| \\
&= \left[1 - \alpha_k \left(1 - \prod_{j=1}^{n} \left(\sqrt[q]{1 + q\rho_j\mu_j\tau_j^q + c_q\rho_j^q\tau_j^q - q\rho_j\nu_j} + \sqrt[q]{1 + qa_j\iota_j^q + c_q\iota_j^q - qb_j} \right) \right) \right] \left\| x_{1,k} - x_1^* \right\|.
\end{aligned}
\tag{4.19}
$$

Taking $\lambda_k = \alpha_k(1-\prod_{j=1}^{n}(\sqrt[q]{1 + q\rho_j\mu_j\tau_j^q + c_q\rho_j^q\tau_j^q - q\rho_j\nu_j}+\sqrt[q]{1 + qa_j\iota_j^q + c_q\iota_j^q - qb_j}))$, $C_k = D_k = 0$, and $B_k = \|x_{1,k} - x_1^*\|$, then $B_{k+1} \leq (1-\lambda_k)B_k + C_k + D_k$. It follows from the conditions (i)–(iii) that

$$
\sum_{k=0}^{\infty} \lambda_k = \infty, \quad C_k = 0(\lambda_k), \quad \sum_{k=0}^{\infty} D_k < \infty, \quad 0 < \lambda_k < 1, \ \forall k \in N. \tag{4.20}
$$

Therefore, by Lemma 2.11 and (4.20), one has $\lim_{k\to\infty} B_k = 0$. By the same argument of Theorem 4.7, we get

$$\lim_{k\to\infty} \left\| x_{j,k} - x_j^* \right\| = 0, \quad j = 1, 2, \ldots, n, \tag{4.21}$$

that is, $x_{j,k} \to x_j^*$ as $k \to \infty$ for $j = 1, 2, \ldots, n$. Thus, $(x_{1,k}, x_{2,k}, \ldots, x_{n,k})$ converges strongly to $(x_1^*, x_2^*, \ldots, x_n^*)$. This completes the proof. □

By Remark 4.4, we have the following strong convergence theorems for the system of general variational inequalities problem (2.7).

Corollary 4.10. *Let K be a closed convex subset of a real Hilbert space E, and let A_j and g_j $(j = 1, 2)$ be the same as in Theorem 3.4. Assume that $\{\alpha_n\}$ is a real sequence in $(0, 1]$ and satisfies the following conditions:*

(i) $\sum_{k=0}^{\infty} \alpha_k = \infty$;

(ii) $\sum_{k=0}^{\infty} \|e_k\| < +\infty$;

(iii) $\min\{1 + 2a_j\iota_j^2 + \iota_j^2 - 2b_j, 1 + 2\rho_j\mu_j\tau_j^2 + \rho_j^2\tau_j^2 - 2\rho_j\nu_j\} \geq 0,\ j = 1, 2$;

(iv) $0 < \prod_{j=1}^{n}\left(\sqrt{1 + 2a_j\iota_j^2 + \iota_j^2 - 2b_j} + \sqrt{1 + 2\rho_j\mu_j\tau_j^2 + \rho_j^2\tau_j^2 - 2\rho_j\nu_j}\right) < 1$.

Then the sequences $\{x_{j,k}\}$ $(j = 1, 2)$ generated by Algorithm 4.5 converge strongly to x_j^* $(j = 1, 2)$, respectively, such that (x_1^*, x_2^*) is the unique solution of the system of general variational inequalities problem (2.7).

Corollary 4.11. *Let K be a closed convex subset of a real Hilbert space E, and let A_j and g_j $(j = 1, 2)$ be the same as in Theorem 3.4. Assume that $\{\alpha_n\}$ is a real sequence in $(0, 1]$ and satisfies the following conditions:*

(i) $\sum_{k=0}^{\infty} \alpha_k = \infty$;

(ii) $\min\{1 + 2a_j\iota_j^2 + \iota_j^2 - 2b_j, 1 + 2\rho_j\mu_j\tau_j^2 + \rho_j^2\tau_j^2 - 2\rho_j\nu_j\} \geq 0, j = 1, 2$;

(iii) $0 < \prod_{j=1}^{n}\left(\sqrt{1 + 2a_j\iota_j^2 + \iota_j^2 - 2b_j} + \sqrt{1 + 2\rho_j\mu_j\tau_j^2 + \rho_j^2\tau_j^2 - 2\rho_j\nu_j}\right) < 1$.

Then the sequences $\{x_{j,k}\}$ $(j = 1, 2)$ generated by Algorithm 4.6 converge strongly to x_j^* $(j = 1, 2)$, respectively, such that (x_1^*, x_2^*) is the unique solution of the system of general variational inequalities problem (2.7).

5. An Application

In this section, we applied the obtained results to study a class of bilevel variational inequalities in Hilbert space, which includes some bilevel programming as special cases and widely used in many practical problems. Moreover, an iterative algorithm and convergence theorem for solutions to the bilevel variational inequalities are given in Hilbert space.

Let K_1 and K_2 be nonempty closed convex subsets of a Hilbert space E, and let $g, h : E \to E$ and $A : E \times E \to E$ be single-valued mappings. We consider the following bilevel variational inequalities (for short, (BVI)): find $(x^*, y^*) \in K_1 \times K_2$ such that

$$\left\langle x^* + h\left(y^*\right), h(x) - x^* \right\rangle \geq 0, \quad \forall x \in K_1,\ y^* \in \Psi(x^*), \tag{5.1}$$

where $\Psi(x^*)$ is the solutions set of the following parametric variational inequalities with respect to the parametric variable x^*:

$$\langle \rho A(x^*, y^*) + y^* - g(x^*), g(y) - y^* \rangle \geq 0, \quad \forall y \in K_2, \tag{5.2}$$

where ρ is a positive constant. Equations (5.1) and (5.2) are called the upper-level variation inequality (for short, (UVI)) and the lower-level variation inequality (for short, LVI), respectively. Denote the set of solutions to the (BVI) by Θ. An important question for the (BVI) is how to solve the bilevel variational inequalities.

From Remark 2.9 and Theorem 3.1, one can easily conclude the following result.

Lemma 5.1. *Let $(x^*, y^*) \in K_1 \times K_2$. Then (x^*, y^*) is a solution of the problem (BVI) if and only if $x^* = P_{K_1}(-h(y^*))$, where $y^* = P_{K_2}(g(x^*) - \rho A(x^*, y^*))$ and ρ is a positive constant.*

Lemma 5.2. *Let K_1 and K_2 be nonempty closed convex subsets of a Hilbert space E. Let $A : E \times E \to E$ be (μ, ν)-relaxed cocoercive and Lipschitz continuous in the first variable with constant τ, and let $g : E \to E$ be (a_2, b_2)-relaxed cocoercive and Lipschitz continuous with constant c_2. Assume that $\{1 + 2a_2c_2^2 + c_2^2 - 2b_2,\ 1 + 2\rho\mu\tau^2 + \rho^2\tau^2 - 2\rho\nu\} \geq 0$ and*

$$0 \leq \sqrt{1 + 2a_2c_2^2 + c_2^2 - 2b_2} + \sqrt{1 + 2\rho\mu\tau^2 + \rho^2\tau^2 - 2\rho\nu} < 1. \tag{5.3}$$

Then, for each $x \in K_1$, the parametric variational inequalities (5.2) *have a uniquely solution. Further, the solution mapping $y(x)$ of the parametric variational inequalities* (5.2) *is continuous on K_1.*

Proof. It directly follows from Theorems 3.2 and 3.4. This completes the proof. □

Theorem 5.3. *Let K_1 and K_2 be nonempty closed convex subsets of a Hilbert space E. Let $A : E \times E \to E$ be (μ, ν)-relaxed cocoercive and Lipschitz continuous in the first variable with constant τ, $h : E \to E$ a (a_1, b_1)-relaxed cocoercive and Lipschitz continuous with constant c_1, and let $g : E \to E$ be a (a_2, b_2)-relaxed cocoercive and Lipschitz continuous with constant c_2. Assume that $\{\alpha_k\}$ is a real sequence in $(0, 1]$ and satisfies the following conditions:*

(i) $\sum_{k=0}^{\infty} \alpha_k = \infty$;

(ii) $\min\{1 + 2a_jc_j^2 + c_j^2 - 2b_j, 1 + 2\rho\mu\tau^2 + \rho^2\tau^2 - 2\rho\nu,\ j = 1, 2\} \geq 0$;

(iii) $0 < \prod_{j=1}^{n} (\sqrt[q]{1 + qa_j\iota_j^q + c_q\iota_j^q - qb_j} + \sqrt[q]{1 + q\rho_j\mu_j\tau_j^q + c_q\rho_j^q\tau_j^q - q\rho_j\nu_j}) < 1$.

The sequences $\{x_k\}$ and $\{y_k\}$ generated by the following algorithm:

$$\begin{gathered} x_0 \in K_1, \\ y_k = P_{K_2}(g(x_k) - \rho A(x_k, y_k)), \\ x_{k+1} = (1 - \alpha_k)x_k + \alpha_k P_{K_1}(-h(y_k)), \quad k = 0, 1, 2, \ldots, \end{gathered} \tag{5.4}$$

where ρ is a positive constant. Then the sequences $\{x_k\}$ and $\{y_k\}$ converge strongly to x^* and y^*, respectively, such that (x^*, y^*) is a solution of the (BVI).

Proof. The proof is similar to Remark 2.9 and Theorem 4.7, and so the proof is omitted. This completes the proof. □

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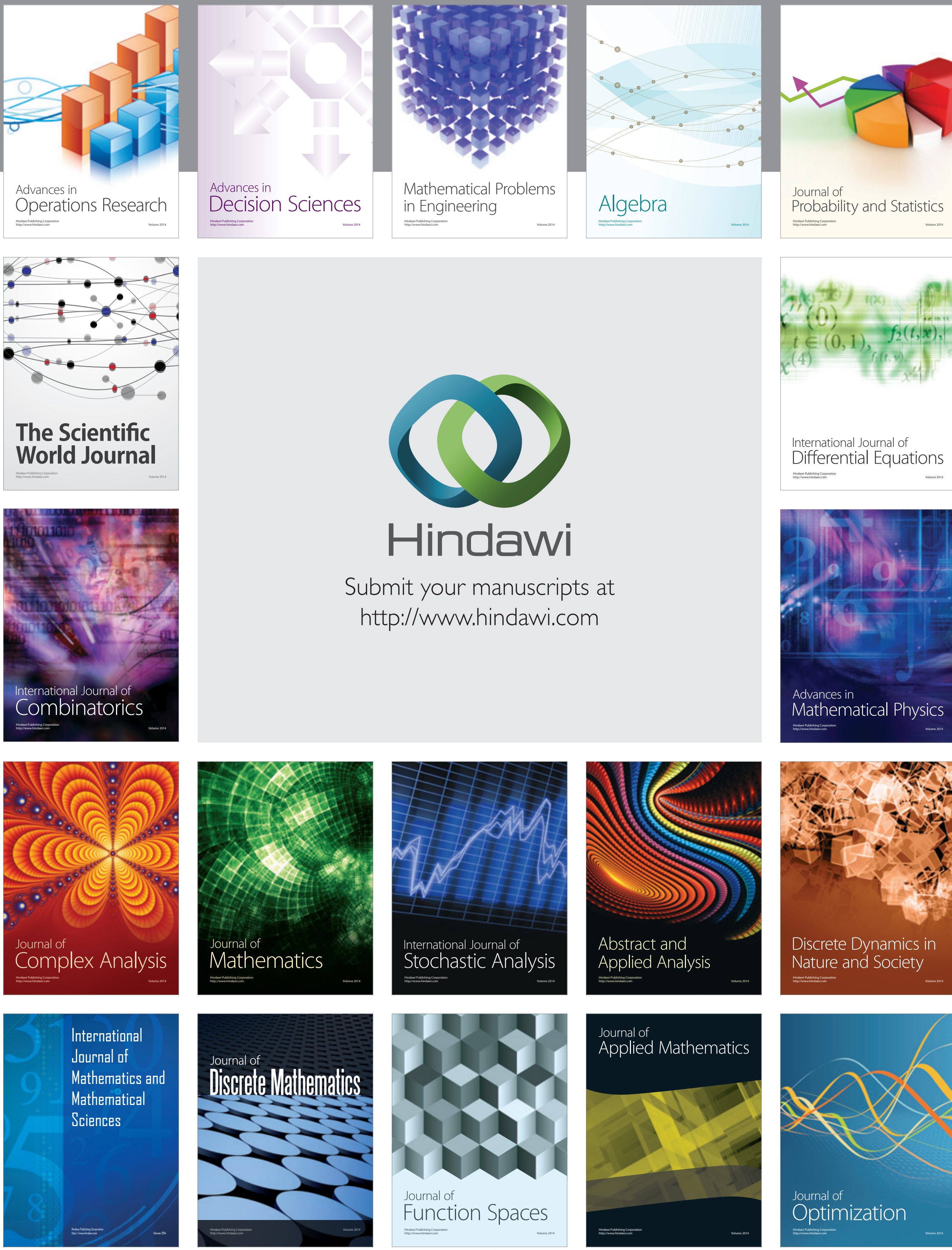
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