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Research Article

Double-Framed Soft Sets with Applications in BCK/BCI-Algebras

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In order to lay a foundation for providing a soft algebraic tool in considering many problems that contain uncertainties, the notion of double-framed soft sets is introduced, and applications in BCK/BCI-algebras are discussed. The notions of double-framed soft algebras in BCK/BCI-algebras are introduced, and related properties are investigated. Characterizations of double-framed soft algebras are considered. Product and int-uni structure of double-framed soft algebras are discussed, and several examples are provided.

1. Introduction

The real world is inherently uncertain, imprecise, and vague. Various problems in system identification involve characteristics which are essentially nonprobabilistic in nature [1]. In response to this situation, Zadeh [2] introduced *fuzzy set theory* as an alternative to probability theory. Uncertainty is an attribute of information. In order to suggest a more general framework, the approach to uncertainty is outlined by Zadeh [3]. To solve a complicated problem in economics, engineering, and environment, we cannot successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties cannot be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [4]. Maji et al. [5] and Molodtsov [4] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [4] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. Worldwide, there has been a rapid growth in interest in soft set theory and its applications in recent years. Evidence of this can be found in the increasing number of high-quality articles on soft sets and related topics that have been published in a variety of international journals, symposia, workshops, and international conferences in recent years. Maji et al. [5] described the application of soft set theory to a decision-making problem. Maji et al. [6] also studied several operations on the theory of soft sets. Aktaş and Çağman [7] studied the basic concepts of soft set theory and compared soft sets to fuzzy and rough sets, providing examples to clarify their differences. They also discussed the notion of soft groups. Jun and Park [8] studied applications of soft sets in ideal theory of BCK/BCI-algebras. Jun et al. [9, 10] introduced the notion of intersectional soft sets and considered its applications to BCK/BCI-algebras. Also, Jun [11] discussed the union soft sets with applications in BCK/BCI-algebras. We refer the reader to the papers [12–24] for further information regarding algebraic structures/properties of soft set theory.

The aim of this paper is to lay a foundation for providing a soft algebraic tool in considering many problems that contain uncertainties. To do this, we introduce the notion of double-framed soft sets and apply it to BCK/BCI-algebras. We discuss double-framed soft algebras in BCK/BCI-algebras and investigate related properties. We consider characterizations of double-framed soft algebras and deal with the product and int-uni structure of double-framed soft algebras. We provide several examples.

2. Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $(X;*,0)$ of type $(2,0)$ is called a BCI-algebra if it satisfies the following conditions:

(i) $(\forall x,y,z\in X)(((x*y)*(x*z))*(z*y)=0)$,

(ii) $(\forall x,y\in X)((x*(x*y))*y=0)$,

(iii) $(\forall x\in X)(x*x=0)$,

(iv) $(\forall x,y\in X)(x*y=0, y*x=0\Rightarrow x=y)$.

If a BCI-algebra X satisfies the following identity:

(v) $(\forall x\in X)(0*x=0)$,

then X is called a BCK-algebra. Any BCK/BCI-algebra X satisfies the following conditions:

$$(\forall x\in X)(x*0=x), \tag{2.1}$$

$$(\forall x,y,z\in X)\big(x\le y\Longrightarrow x*z\le y*z, z*y\le z*x\big), \tag{2.2}$$

$$(\forall x, y, z \in X)((x * y) * z = (x * z) * y), \tag{2.3}$$

$$(\forall x, y, z \in X)((x * z) * (y * z) \leq x * y), \tag{2.4}$$

where $x \leq y$ if and only if $x * y = 0$.

A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$.

We refer the reader to the books [25, 26] for further information regarding BCK/BCI-algebras.

Molodtsov [4] defined the soft set in the following way: let U be an initial universe set, and let E be a set of parameters. Let $\mathcal{P}(U)$ denote the power set of U and $A, B, C, \ldots \subseteq E$.

Definition 2.1 (see [4]). A pair (α, A) is called a *soft set* over U, where α is a mapping given by

$$\alpha : A \longrightarrow \mathcal{P}(U). \tag{2.5}$$

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\varepsilon \in A$, $\alpha(\varepsilon)$ may be considered as the set of ε-approximate elements of the soft set (α, A). Clearly, a soft set is not a set. For illustration, Molodtsov considered several examples in [4].

3. Double-Framed Soft Algebras

In what follows, we take $E = X$ as a set of parameters, which is a BCK/BCI-algebra and $A, B, C, \ldots$ as subalgebras of E unless otherwise specified.

Definition 3.1. A double-framed pair $\langle(\alpha, \beta); A\rangle$ is called a *double-framed soft set* over U, where α and β are mappings from A to $\mathcal{P}(U)$.

Definition 3.2. A double-framed soft set $\langle(\alpha, \beta); A\rangle$ over U is called a double-framed soft algebra over U if it satisfies

$$(\forall x, y \in A)(\alpha(x * y) \supseteq \alpha(x) \cap \alpha(y), \beta(x * y) \subseteq \beta(x) \cup \beta(y)). \tag{3.1}$$

Example 3.3. Suppose that there are five houses in the initial universe set U given by

$$U = \{h_1, h_2, h_3, h_4, h_5\}. \tag{3.2}$$

Let a set of parameters $E = \{e_0, e_1, e_2, e_3\}$ be a set of status of houses in which

e_0 stands for the parameter "beautiful,"

e_1 stands for the parameter "cheap,"

e_2 stands for the parameter "in good location,"

e_3 stands for the parameter "in green surroundings," with the following binary operation:

$$\begin{array}{c|cccc} * & e_0 & e_1 & e_2 & e_3 \\ \hline e_0 & e_0 & e_0 & e_0 & e_0 \\ e_1 & e_1 & e_0 & e_1 & e_1 \\ e_2 & e_2 & e_2 & e_0 & e_2 \\ e_3 & e_3 & e_3 & e_3 & e_0 \end{array} \tag{3.3}$$

Then $(E, *, e_0)$ is a BCK-algebra. For a subalgebra $A := \{e_0, e_2, e_3\}$ of E, consider a double-framed soft set $\langle (\alpha, \beta); A \rangle$ over U as follows:

$$\begin{aligned} &\alpha : A \longrightarrow \mathcal{P}(U), \qquad x \longmapsto \begin{cases} \{h_1, h_2, h_3, h_4, h_5\} & \text{if } x = e_0, \\ \{h_1, h_3, h_4\} & \text{if } x = e_2, \\ \{h_2, h_3, h_5\} & \text{if } x = e_3, \end{cases} \\ &\beta : A \longrightarrow \mathcal{P}(U), \qquad x \longmapsto \begin{cases} \{h_2\} & \text{if } x = e_0, \\ \{h_1, h_2, h_3\} & \text{if } x = e_2, \\ \{h_4, h_5\} & \text{if } x = e_3. \end{cases} \end{aligned} \tag{3.4}$$

It is routine to verify that $\langle (\alpha, \beta); A \rangle$ is a double-framed soft algebra over U.

Example 3.4. Consider the initial universe set $U = \{h_1, h_2, h_3, h_4, h_5\}$ and a set of parameters $E = \{e_0, e_1, e_2, e_3\}$ which are given in Example 3.3. Define a binary operation $*$ on E by the following Cayley table:

$$\begin{array}{c|cccc} * & e_0 & e_1 & e_2 & e_3 \\ \hline e_0 & e_0 & e_0 & e_0 & e_0 \\ e_1 & e_1 & e_0 & e_0 & e_1 \\ e_2 & e_2 & e_2 & e_0 & e_2 \\ e_3 & e_3 & e_3 & e_3 & e_0 \end{array} \tag{3.5}$$

Consider a double-framed soft set $\langle (\alpha, \beta); E \rangle$ over U as follows:

$$\begin{aligned} &\alpha : E \longrightarrow \mathcal{P}(U), \qquad x \longmapsto \begin{cases} U & \text{if } x = e_0, \\ \{h_1, h_2, h_5\} & \text{if } x = e_1, \\ \{h_1, h_2, h_3, h_5\} & \text{if } x = e_2, \\ \{h_1, h_4\} & \text{if } x = e_3, \end{cases} \\ &\beta : E \longrightarrow \mathcal{P}(U), \qquad x \longmapsto \begin{cases} \{h_3, h_4\} & \text{if } x = e_0, \\ \{h_2, h_3, h_4\} & \text{if } x = e_1, \\ \{h_2, h_3, h_4, h_5\} & \text{if } x = e_2, \\ \{h_1, h_3, h_4\} & \text{if } x = e_3. \end{cases} \end{aligned} \tag{3.6}$$

Then $\langle (\alpha, \beta); E \rangle$ is a double-framed soft algebra over U.

Example 3.5. There are six women patients in the initial universe set U given by

$$U = \{h_1, h_2, h_3, h_4, h_5, h_6\}. \tag{3.7}$$

Let a set of parameters $E = \{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ be a set of status of patients in which

e_0 stands for the parameter "chest pain,"
e_1 stands for the parameter "headache,"
e_2 stands for the parameter "mental depression,"
e_3 stands for the parameter "migraine,"
e_4 stands for the parameter "neurosis,"
e_5 stands for the parameter "omodynia,"
e_6 stands for the parameter "period pains,"
e_7 stands for the parameter "toothache,"

with the following binary operation:

$$\begin{array}{c|cccccccc}
* & e_0 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
\hline
e_0 & e_0 & e_0 & e_0 & e_0 & e_4 & e_4 & e_4 & e_4 \\
e_1 & e_1 & e_0 & e_0 & e_0 & e_5 & e_4 & e_4 & e_4 \\
e_2 & e_2 & e_2 & e_0 & e_0 & e_6 & e_6 & e_4 & e_4 \\
e_3 & e_3 & e_2 & e_1 & e_0 & e_7 & e_6 & e_5 & e_4 \\
e_4 & e_4 & e_4 & e_4 & e_4 & e_0 & e_0 & e_0 & e_0 \\
e_5 & e_5 & e_4 & e_4 & e_4 & e_1 & e_0 & e_0 & e_0 \\
e_6 & e_6 & e_6 & e_4 & e_4 & e_2 & e_2 & e_0 & e_0 \\
e_7 & e_7 & e_6 & e_5 & e_4 & e_3 & e_2 & e_1 & e_0
\end{array} \tag{3.8}$$

Then $(E, *, e_0)$ is a BCI-algebra. Consider a double-framed soft set $\langle(\alpha, \beta); E\rangle$ over U as follows:

$$\begin{gathered}
\alpha : E \longrightarrow \mathcal{P}(U), \qquad x \longmapsto \begin{cases} \{h_1, h_2, h_3, h_4\} & \text{if } x \in \{e_0, e_1, e_2, e_3\}, \\ \{h_1, h_3, h_4, h_5, h_6\} & \text{if } x \in \{e_4, e_5, e_6, e_7\}, \end{cases} \\
\beta : E \longrightarrow \mathcal{P}(U), \qquad x \longmapsto \begin{cases} \{h_1, h_3, h_5, h_7\} & \text{if } x \in \{e_0, e_1, e_2, e_3\}, \\ \{h_1, h_2, h_4\} & \text{if } x \in \{e_4, e_5, e_6, e_7\}. \end{cases}
\end{gathered} \tag{3.9}$$

Then we have

$$\alpha(e_4) \cap \alpha(e_5) = \{h_1, h_3, h_4, h_5, h_6\} \nsubseteq \{h_1, h_2, h_3, h_4\} = \alpha(e_0) = \alpha(e_4 * e_5) \tag{3.10}$$

and/or

$$\beta(e_5) \cup \beta(e_6) = \{h_1, h_2, h_4\} \nsupseteq \{h_1, h_3, h_5, h_7\} = \beta(e_0) = \beta(e_5 * e_6). \tag{3.11}$$

Hence, $\langle(\alpha, \beta); E\rangle$ is not a double-framed soft algebra over U.

Lemma 3.6. *Every double-framed soft algebra $\langle(\alpha,\beta);A\rangle$ over U satisfies the following condition:*

$$(\forall x \in A)\big(\alpha(x) \subseteq \alpha(0), \beta(x) \supseteq \beta(0)\big). \tag{3.12}$$

Proof. It is straightforward. □

Proposition 3.7. *For a double-framed soft algebra $\langle(\alpha,\beta);A\rangle$ over U, the following are equivalent:*

(1) $(\forall x \in A)(\alpha(x) = \alpha(0), \beta(x) = \beta(0))$,

(2) $(\forall x, y \in A)(\alpha(y) \subseteq \alpha(x * y), \beta(y) \supseteq \beta(x * y))$.

Proof. Assume that (2) is valid. Taking $y = 0$ and using (2.1), we have $\alpha(0) \subseteq \alpha(x * 0) = \alpha(x)$ and $\beta(0) \supseteq \beta(x * 0) = \beta(x)$. It follows from Lemma 3.6 that $\alpha(x) = \alpha(0)$ and $\beta(x) = \beta(0)$.

Conversely, suppose that $\alpha(x) = \alpha(0)$ and $\beta(x) = \beta(0)$ for all $x \in A$. Using (3.1), we have

$$\begin{aligned} \alpha(y) &= \alpha(0) \cap \alpha(y) = \alpha(x) \cap \alpha(y) \subseteq \alpha(x * y), \\ \beta(y) &= \beta(0) \cup \beta(y) = \beta(x) \cup \beta(y) \supseteq \beta(x * y), \end{aligned} \tag{3.13}$$

for all $x, y \in A$. This completes the proof. □

Proposition 3.8. *In a BCI-algebra X, every double-framed soft algebra $\langle(\alpha,\beta);A\rangle$ over U satisfies the following condition:*

$$(\forall x, y \in A)\begin{pmatrix} \alpha(x * (0 * y)) \supseteq \alpha(x) \cap \alpha(y), \\ \beta(x * (0 * y)) \subseteq \beta(x) \cup \beta(y) \end{pmatrix}. \tag{3.14}$$

Proof. Using Lemma 3.6, we have

$$\begin{aligned} \alpha(x * (0 * y)) &\supseteq \alpha(x) \cap \alpha(0 * y) \\ &\supseteq \alpha(x) \cap \alpha(0) \cap \alpha(y) = \alpha(x) \cap \alpha(y), \\ \beta(x * (0 * y)) &\subseteq \beta(x) \cup \beta(0 * y) \\ &\subseteq \beta(x) \cup \beta(0) \cup \beta(y) = \beta(x) \cup \beta(y), \end{aligned} \tag{3.15}$$

for all $x, y \in A$. □

Let $\langle(\alpha,\beta);A\rangle$ and $\langle(f,g);B\rangle$ be double-framed soft sets over a common universe U. Then $\langle(\alpha,\beta);A\rangle$ is called a *double-framed soft subset* of $\langle(f,g);B\rangle$, denoted by $\langle(\alpha,\beta);A\rangle \tilde{\subseteq} \langle(f,g);B\rangle$, if

(i) $A \subseteq B$,

(ii) $(\forall e \in A)\begin{pmatrix} \alpha(e) \text{ and } f(e) \text{ are identical approximations} \\ \beta(e) \text{ and } g(e) \text{ are identical approximations} \end{pmatrix}$.

Theorem 3.9. *Let $\langle(\alpha,\beta);A\rangle$ be a double-framed soft subset of a double-framed soft set $\langle(f,g);B\rangle$. If $\langle(f,g);B\rangle$ is a double-framed soft algebra over U, then so is $\langle(\alpha,\beta);A\rangle$.*

Proof. Let $x, y \in A$. Then $x, y \in B$, and so

$$\begin{aligned} \alpha(x) \cap \alpha(y) &= f(x) \cap f(y) \subseteq f(x * y) = \alpha(x * y), \\ \beta(x) \cup \beta(y) &= g(x) \cup g(y) \supseteq g(x * y) = \beta(x * y). \end{aligned} \tag{3.16}$$

Hence, $\langle(\alpha, \beta); A\rangle$ is a double-framed soft algebra over U. □

The converse of Theorem 3.9 is not true as seen in the following example.

Example 3.10. Suppose that there are six houses in the initial universe set U given by

$$U = \{h_1, h_2, h_3, h_4, h_5, h_6\}. \tag{3.17}$$

Let a set of parameters $E = \{e_0, e_1, e_2, e_3, e_4\}$ be a set of status of houses in which

e_0 stands for the parameter "beautiful,"

e_1 stands for the parameter "cheap,"

e_2 stands for the parameter "in good location,"

e_3 stands for the parameter "in green surroundings,"

e_4 stands for the parameter "luxury,"

with the following binary operation:

$$\begin{array}{c|ccccc} * & e_0 & e_1 & e_2 & e_3 & e_4 \\ \hline e_0 & e_0 & e_0 & e_0 & e_3 & e_3 \\ e_1 & e_1 & e_0 & e_1 & e_4 & e_3 \\ e_2 & e_2 & e_2 & e_0 & e_3 & e_3 \\ e_3 & e_3 & e_3 & e_3 & e_0 & e_0 \\ e_4 & e_4 & e_3 & e_4 & e_1 & e_0 \end{array} \tag{3.18}$$

Then $(E, *, e_0)$ is a BCI-algebra. For a subalgebra $A = \{e_0, e_1, e_2\}$, define a double-framed soft set $\langle(\alpha, \beta); A\rangle$ over U as follows:

$$\begin{aligned} \alpha : A \longrightarrow \wp(U), \qquad x &\longmapsto \begin{cases} U & \text{if } x = e_0, \\ \{h_1, h_3, h_5\} & \text{if } x = e_1, \\ \{h_2, h_4, h_6\} & \text{if } x = e_2, \end{cases} \\ \beta : A \longrightarrow \wp(U), \qquad x &\longmapsto \begin{cases} \{h_6\} & \text{if } x = e_0, \\ \{h_4, h_6\} & \text{if } x = e_1, \\ \{h_2, h_4, h_6\} & \text{if } x = e_2. \end{cases} \end{aligned} \tag{3.19}$$

It is routine to verify that $\langle(\alpha,\beta);E\rangle$ is a double-framed soft algebra over U. Take $B=E$, and define a double-framed soft set $\langle(f,g);B\rangle$ over U as follows:

$$f : B \longrightarrow \mathcal{P}(U), \qquad x \longmapsto \begin{cases} U & \text{if } x = e_0, \\ \{h_1, h_3, h_5\} & \text{if } x = e_1, \\ \{h_2, h_4, h_6\} & \text{if } x = e_2, \\ \{h_1, h_3\} & \text{if } x = e_3, \\ \{h_2, h_4\} & \text{if } x = e_4, \end{cases}$$
$$g : B \longrightarrow \mathcal{P}(U), \qquad x \longmapsto \begin{cases} \{h_6\} & \text{if } x = e_0, \\ \{h_4, h_6\} & \text{if } x = e_1, \\ \{h_2, h_4, h_6\} & \text{if } x = e_2, \\ \{h_2, h_3\} & \text{if } x = e_3, \\ \{h_2, h_5\} & \text{if } x = e_4. \end{cases} \tag{3.20}$$

Then $\langle(f,g);B\rangle$ is not a double-framed soft algebra over U since

$$f(e_2 * e_4) = f(e_3) = \{h_1, h_3\} \not\supseteq \{h_2, h_4\} = f(e_2) \cap f(e_4) \tag{3.21}$$

and/or

$$g(e_4 * e_3) = g(e_1) = \{h_4, h_6\} \not\subseteq \{h_2, h_3, h_5\} = g(e_4) \cup g(e_3). \tag{3.22}$$

For two double-framed soft sets $\langle(\alpha,\beta);A\rangle$ and $\langle(f,g);A\rangle$ over U, the *double-framed soft int-uni set* of $\langle(\alpha,\beta);A\rangle$ and $\langle(f,g);A\rangle$ is defined to be a double-framed soft set $\langle(\alpha \tilde{\cap} f, \beta \tilde{\cup} g);A\rangle$ where

$$\begin{aligned} \alpha \tilde{\cap} f : A \longrightarrow \mathcal{P}(U), &\qquad x \longmapsto \alpha(x) \cap f(x), \\ \beta \tilde{\cup} g : A \longrightarrow \mathcal{P}(U), &\qquad x \longmapsto \beta(x) \cup g(x). \end{aligned} \tag{3.23}$$

It is denoted by $\langle(\alpha,\beta);A\rangle \sqcap \langle(f,g);A\rangle = \langle(\alpha \tilde{\cap} f, \beta \tilde{\cup} g);A\rangle$.

Theorem 3.11. *The double-framed soft int-uni set of two double-framed soft algebras $\langle(\alpha,\beta);A\rangle$ and $\langle(f,g);A\rangle$ over U is a double-framed soft algebra over U.*

Proof. For any $x, y \in A$, we have

$$\begin{aligned} (\alpha \tilde{\cap} f)(x * y) &= \alpha(x * y) \cap f(x * y) \supseteq (\alpha(x) \cap \alpha(y)) \cap (f(x) \cap f(y)) \\ &= (\alpha(x) \cap f(x)) \cap (\alpha(y) \cap f(y)) = (\alpha \tilde{\cap} f)(x) \cap (\alpha \tilde{\cap} f)(y), \\ (\beta \tilde{\cup} g)(x * y) &= \beta(x * y) \cup g(x * y) \subseteq (\beta(x) \cup \beta(y)) \cup (g(x) \cup g(y)) \\ &= (\beta(x) \cup g(x)) \cup (\beta(y) \cup g(y)) = (\beta \tilde{\cup} f)(x) \cup (\beta \tilde{\cup} f)(y). \end{aligned} \tag{3.24}$$

Therefore, $\langle(\alpha,\beta);A\rangle \sqcap \langle(f,g);A\rangle$ is a double-framed soft algebra over U. □

For a double-framed soft set $\langle(\alpha,\beta);A\rangle$ over U and two subsets γ and δ of U, the γ-*inclusive set* and the δ-*exclusive set* of $\langle(\alpha,\beta);A\rangle$, denoted by $i_A(\alpha;\gamma)$ and $e_A(\beta;\delta)$, respectively, are defined as follows:

$$i_A(\alpha;\gamma) := \{x \in A \mid \gamma \subseteq \alpha(x)\}, \tag{3.25}$$

$$e_A(\beta;\delta) := \{x \in A \mid \delta \supseteq \beta(x)\}, \tag{3.26}$$

respectively. The set

$$DF_A(\alpha,\beta)_{(\gamma,\delta)} := \{x \in A \mid \gamma \subseteq \alpha(x), \delta \supseteq \beta(x)\} \tag{3.27}$$

is called a *double-framed including set* of $\langle(\alpha,\beta);A\rangle$. It is clear that

$$DF_A(\alpha,\beta)_{(\gamma,\delta)} = i_A(\alpha;\gamma) \cap e_A(\beta;\delta). \tag{3.28}$$

Theorem 3.12. *For a double-framed soft set $\langle(\alpha,\beta);E\rangle$ over U, the following are equivalent:*

(1) *$\langle(\alpha,\beta);E\rangle$ is a double-framed soft algebra over U,*

(2) *for every subsets γ and δ of U with $\gamma \in \mathrm{Im}(\alpha)$ and $\delta \in \mathrm{Im}(\beta)$, the γ-inclusive set and the δ-exclusive set of $\langle(\alpha,\beta);E\rangle$ are subalgebras of E.*

Proof. Assume that $\langle(\alpha,\beta);E\rangle$ is a double-framed soft algebra over U. Let $x,y \in E$ be such that $x,y \in i_E(\alpha;\gamma)$ and $x,y \in e_E(\beta;\delta)$ for every subsets γ and δ of U with $\gamma \in \mathrm{Im}(\alpha)$ and $\delta \in \mathrm{Im}(\beta)$. It follows from (3.1) that

$$\alpha(x*y) \supseteq \alpha(x) \cap \alpha(y) \supseteq \gamma, \qquad \beta(x*y) \subseteq \beta(x) \cup \beta(y) \subseteq \delta. \tag{3.29}$$

Hence, $x*y \in i_E(\alpha;\gamma)$ and $x*y \in e_E(\beta;\delta)$, and therefore, $i_E(\alpha;\gamma)$ and $e_E(\beta;\delta)$ are subalgebras of E.

Conversely suppose that (2) is valid. Let $x,y \in E$ be such that $\alpha(x) = \gamma_x$, $\alpha(y) = \gamma_y$, $\beta(x) = \delta_x$ and $\beta(y) = \delta_y$. Taking $\gamma = \gamma_x \cap \gamma_y$ and $\delta = \delta_x \cup \delta_y$ implies that $x,y \in i_E(\alpha;\gamma)$ and $x,y \in e_E(\beta;\delta)$. Hence, $x*y \in i_E(\alpha;\gamma)$ and $x*y \in e_E(\beta;\delta)$, which imply that

$$\begin{aligned} \alpha(x*y) &\supseteq \gamma = \gamma_x \cap \gamma_y = \alpha(x) \cap \alpha(y), \\ \beta(x*y) &\subseteq \delta = \delta_x \cup \delta_y = \beta(x) \cup \beta(y). \end{aligned} \tag{3.30}$$

Therefore, $\langle(\alpha,\beta);E\rangle$ is a double-framed soft algebra over U. □

Corollary 3.13. *If $\langle(\alpha,\beta);E\rangle$ is a double-framed soft algebra over U, then the double-framed including set of $\langle(\alpha,\beta);E\rangle$ is a subalgebra X.*

For any double-framed soft set $\langle(\alpha,\beta);E\rangle$ over U, let $\langle(\alpha^*,\beta^*);E\rangle$ be a double-framed soft set over U defined by

$$\begin{aligned}
&\alpha^* : E \longrightarrow \mathcal{P}(U), \qquad x \longmapsto \begin{cases} \alpha(x) & \text{if } x \in i_E(\alpha;\gamma), \\ \eta & \text{otherwise}, \end{cases} \\
&\beta^* : E \longrightarrow \mathcal{P}(U), \qquad x \longmapsto \begin{cases} \beta(x) & \text{if } x \in e_E(\beta;\delta), \\ \rho & \text{otherwise}, \end{cases}
\end{aligned} \tag{3.31}$$

where γ, δ, η, and ρ are subsets of U with $\eta \subsetneq \alpha(x)$ and $\rho \supsetneq \beta(x)$.

Theorem 3.14. *If $\langle(\alpha,\beta);E\rangle$ is a double-framed soft algebra over U, then so is $\langle(\alpha^*,\beta^*);E\rangle$.*

Proof. Assume that $\langle(\alpha,\beta);E\rangle$ is a double-framed soft algebra over U. Then $i_E(\alpha;\gamma)$ and $e_E(\beta;\delta)$ are subalgebras of E for every subsets γ and δ of U with $\gamma \in \operatorname{Im}(\alpha)$ and $\delta \in \operatorname{Im}(\beta)$. Let $x, y \in E$. If $x, y \in i_E(\alpha;\gamma)$, then $x * y \in i_E(\alpha;\gamma)$. Thus,

$$\alpha^*(x * y) = \alpha(x * y) \supseteq \alpha(x) \cap \alpha(y) = \alpha^*(x) \cap \alpha^*(y). \tag{3.32}$$

If $x \notin i_E(\alpha;\gamma)$ or $y \notin i_E(\alpha;\gamma)$, then $\alpha^*(x) = \eta$ or $\alpha^*(y) = \eta$. Hence,

$$\alpha^*(x * y) \supseteq \eta = \alpha^*(x) \cap \alpha^*(y). \tag{3.33}$$

Now, if $x, y \in e_E(\beta;\delta)$, then $x * y \in e_E(\beta;\delta)$. Thus,

$$\beta^*(x * y) = \beta(x * y) \subseteq \beta(x) \cup \beta(y) = \beta^*(x) \cup \beta^*(y). \tag{3.34}$$

If $x \notin e_E(\beta;\delta)$ or $y \notin e_E(\beta;\delta)$, then $\beta^*(x) = \rho$ or $\beta^*(y) = \rho$. Hence,

$$\beta^*(x * y) \subseteq \rho = \beta^*(x) \cup \beta^*(y). \tag{3.35}$$

Therefore, $\langle(\alpha^*,\beta^*);E\rangle$ is a double-framed soft algebra over U. □

The following example shows that the converse of Theorem 3.14 is not true in general.

Example 3.15. Suppose that there are ten houses in the initial universe set U given by

$$U = \{h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8, h_9, h_{10}\}. \tag{3.36}$$

Let a set of parameters $E = \{e_0, e_1, e_2, e_3\}$ be a set of status of houses in which

e_0 stands for the parameter "beautiful,"

e_1 stands for the parameter "cheap,"

e_2 stands for the parameter "in good location,"

e_3 stands for the parameter "in green surroundings,"

with the following binary operation:

$$\begin{array}{c|cccc} * & e_0 & e_1 & e_2 & e_3 \\ \hline e_0 & e_0 & e_1 & e_2 & e_3 \\ e_1 & e_1 & e_0 & e_3 & e_2 \\ e_2 & e_2 & e_3 & e_0 & e_1 \\ e_3 & e_3 & e_2 & e_1 & e_0 \end{array} \tag{3.37}$$

Then $(E, *, e_0)$ is a BCI-algebra. Consider a double-framed soft set $\langle(\alpha, \beta); E\rangle$ over U as follows:

$$\begin{gathered} \alpha : E \longrightarrow \mathcal{P}(U), \qquad x \longmapsto \begin{cases} U & \text{if } x = e_0, \\ \{h_2, h_4, h_6, h_8, h_{10}\} & \text{if } x = e_1, \\ \{h_3, h_6, h_9\} & \text{if } x = e_2, \\ \{h_8\} & \text{if } x = e_3, \end{cases} \\ \beta : E \longrightarrow \mathcal{P}(U), \qquad x \longmapsto \begin{cases} \{h_6\} & \text{if } x = e_0, \\ \{h_3, h_6, h_9\} & \text{if } x = e_1, \\ \{h_2, h_4, h_6, h_8, h_{10}\} & \text{if } x = e_2, \\ U & \text{if } x = e_3. \end{cases} \end{gathered} \tag{3.38}$$

Note that $i_E(\alpha; \gamma) = \{e_0, e_1\} = e_E(\beta; \delta)$ for $\gamma = \{h_2, h_4, h_6, h_8, h_{10}\}$ and $\delta = \{h_3, h_6, h_9\}$. Let $\langle(\alpha^*, \beta^*); E\rangle$ be a double-framed soft set over U defined by

$$\begin{gathered} \alpha^* : E \longrightarrow \mathcal{P}(U), \qquad x \longmapsto \begin{cases} \alpha(x) & \text{if } x \in\ i_E(\alpha; \gamma), \\ \emptyset & \text{otherwise}, \end{cases} \\ \beta^* : E \longrightarrow \mathcal{P}(U), \qquad x \longmapsto \begin{cases} \beta(x) & \text{if } x \in\ e_E(\beta; \delta), \\ U & \text{otherwise}, \end{cases} \end{gathered} \tag{3.39}$$

that is,

$$\begin{gathered} \alpha^* : E \longrightarrow \mathcal{P}(U), \qquad x \longmapsto \begin{cases} U & \text{if } x = e_0, \\ \{h_2, h_4, h_6, h_8, h_{10}\} & \text{if } x = e_1, \\ \emptyset & \text{if } x \in\ \{e_2, e_3\}, \end{cases} \\ \beta^* : E \longrightarrow \mathcal{P}(U), \qquad x \longmapsto \begin{cases} \{h_6\} & \text{if } x = e_0, \\ \{h_3, h_6, h_9\} & \text{if } x = e_1, \\ U & \text{if } x \in\ \{e_2, e_3\}. \end{cases} \end{gathered} \tag{3.40}$$

It is routine to verify that $\langle(\alpha^*, \beta^*); E\rangle$ is a double-framed soft algebra over U. But $\langle(\alpha, \beta); E\rangle$ is not a double-framed soft algebra over U since $\alpha(e_1) \cap \alpha(e_2) \nsubseteq \alpha(e_3) = \alpha(e_1 * e_2)$ and/or $\beta(e_1) \cup \beta(e_2) \nsupseteq \beta(e_3) = \beta(e_1 * e_2)$.

Let $\langle(\alpha, \beta); A\rangle$ and $\langle(\alpha, \beta); B\rangle$ be double-framed soft sets over U. The $(\alpha_\wedge, \beta_\vee)$-*product* of $\langle(\alpha, \beta); A\rangle$ and $\langle(\alpha, \beta); B\rangle$ is defined to be a double-framed soft set $\langle(\alpha_{A\wedge B}, \beta_{A\vee B}); A \times B\rangle$ over U in which

$$\begin{aligned} \alpha_{A\wedge B} &: A \times B \longrightarrow \mathcal{P}(U), \qquad (x, y) \longmapsto \alpha(x) \cap \alpha(y), \\ \beta_{A\vee B} &: A \times B \longrightarrow \mathcal{P}(U), \qquad (x, y) \longmapsto \beta(x) \cup \beta(y). \end{aligned} \tag{3.41}$$

Theorem 3.16. *For any BCK/BCI-algebras E and F as sets of parameters, let $\langle(\alpha, \beta); E\rangle$ and $\langle(\alpha, \beta); F\rangle$ be double-framed soft algebras over U. Then the $(\alpha_\wedge, \beta_\vee)$-product of $\langle(\alpha, \beta); E\rangle$ and $\langle(\alpha, \beta); F\rangle$ is also a double-framed soft algebra over U.*

Proof. Note that $(E \times F, \circledast, (0, 0))$ is a BCK/BCI-algebra. For any $(x, y), (a, b) \in E \times F$, we have

$$\begin{aligned} \alpha_{E\wedge F}((x, y) \circledast (a, b)) &= \alpha_{E\wedge F}(x * a, y * b) \\ &= \alpha(x * a) \cap \alpha(y * b) \supseteq (\alpha(x) \cap \alpha(a)) \cap (\alpha(y) \cap \alpha(b)) \\ &= (\alpha(x) \cap \alpha(y)) \cap (\alpha(a) \cap \alpha(b)) \\ &= \alpha_{E\wedge F}(x, y) \cap \alpha_{E\wedge F}(a, b), \\ \beta_{E\vee F}((x, y) \circledast (a, b)) &= \beta_{E\vee F}(x * a, y * b) \\ &= \beta(x * a) \cup \beta(y * b) \subseteq (\beta(x) \cup \beta(a)) \cup (\beta(y) \cup \beta(b)) \\ &= (\beta(x) \cup \beta(y)) \cup (\beta(a) \cup \beta(b)) \\ &= \beta_{E\vee F}(x, y) \cup \beta_{E\vee F}(a, b). \end{aligned} \tag{3.42}$$

Hence, $\langle(\alpha_{E\wedge F}, \beta_{E\vee F}); E \times F\rangle$ is a double-framed soft algebra over U. □

Theorem 3.16 is illustrated from the following example.

Example 3.17. Consider the initial universe set U given by

$$U = \{h_1, h_2, h_3, h_4, h_5, h_6\}, \tag{3.43}$$

which consists of six women patients, and take five parameters as follows:

e_0 stands for the parameter "chest pain,"

e_1 stands for the parameter "headache,"

e_2 stands for the parameter "mental depression,"

e_3 stands for the parameter "migraine,"

e_4 stands for the parameter "neurosis."

Let $E = \{e_0, e_1, e_2\}$ and $F = \{e_0, e_3, e_4\}$ with the following Cayley tables:

$*$	e_0	e_1	e_2
e_0	e_0	e_0	e_0
e_1	e_1	e_0	e_1
e_2	e_2	e_2	e_0

$*$	e_0	e_3	e_4
e_0	e_0	e_0	e_0
e_3	e_3	e_0	e_0
e_4	e_4	e_4	e_0

(3.44)

Then

$$E \times F = \{(e_0, e_0), (e_0, e_3), (e_0, e_4), (e_1, e_0), (e_1, e_3), (e_1, e_4), (e_2, e_0), (e_2, e_3), (e_2, e_4)\} \tag{3.45}$$

is a BCK-algebra with the following Cayley table:

$*$	(e_0,e_0)	(e_0,e_3)	(e_0,e_4)	(e_1,e_0)	(e_1,e_3)	(e_1,e_4)	(e_2,e_0)	(e_2,e_3)	(e_2,e_4)
(e_0,e_0)	(e_0,e_0)	(e_0,e_0)	(e_0,e_0)	(e_0,e_0)	(e_0,e_0)	(e_0,e_0)	(e_0,e_0)	(e_0,e_0)	(e_0,e_0)
(e_0,e_3)	(e_0,e_3)	(e_0,e_0)	(e_0,e_0)	(e_0,e_3)	(e_0,e_0)	(e_0,e_0)	(e_0,e_3)	(e_0,e_0)	(e_0,e_0)
(e_0,e_4)	(e_0,e_4)	(e_0,e_4)	(e_0,e_0)	(e_0,e_4)	(e_0,e_4)	(e_0,e_0)	(e_0,e_4)	(e_0,e_4)	(e_0,e_0)
(e_1,e_0)	(e_1,e_0)	(e_1,e_0)	(e_1,e_0)	(e_0,e_0)	(e_0,e_0)	(e_0,e_0)	(e_1,e_0)	(e_1,e_0)	(e_1,e_0)
(e_1,e_3)	(e_1,e_3)	(e_1,e_0)	(e_1,e_0)	(e_0,e_3)	(e_0,e_0)	(e_0,e_0)	(e_1,e_3)	(e_1,e_0)	(e_1,e_0)
(e_1,e_4)	(e_1,e_4)	(e_1,e_4)	(e_1,e_0)	(e_0,e_4)	(e_0,e_4)	(e_0,e_0)	(e_1,e_4)	(e_1,e_4)	(e_1,e_0)
(e_2,e_0)	(e_2,e_0)	(e_2,e_0)	(e_2,e_0)	(e_2,e_0)	(e_2,e_0)	(e_2,e_0)	(e_0,e_0)	(e_0,e_0)	(e_0,e_0)
(e_2,e_3)	(e_2,e_3)	(e_2,e_0)	(e_2,e_0)	(e_2,e_3)	(e_2,e_0)	(e_2,e_0)	(e_0,e_3)	(e_0,e_0)	(e_0,e_0)
(e_2,e_4)	(e_2,e_4)	(e_2,e_4)	(e_2,e_0)	(e_2,e_4)	(e_2,e_4)	(e_2,e_0)	(e_0,e_4)	(e_0,e_4)	(e_0,e_0)

(3.46)

Let $\langle(\alpha, \beta); E\rangle$ and $\langle(\alpha, \beta); F\rangle$ be double-framed soft sets over U given by

$$\alpha : E \longrightarrow \mathcal{P}(U), \qquad x \longmapsto \begin{cases} \{h_1, h_2, h_3, h_4, h_5\} & \text{if } x \in \{e_0, e_1\}, \\ \{h_1, h_2\} & \text{if } x = e_2, \end{cases}$$

$$\beta : E \longrightarrow \mathcal{P}(U), \qquad x \longmapsto \begin{cases} \{h_3, h_6\} & \text{if } x \in \{e_0, e_1\}, \\ \{h_2, h_3, h_6\} & \text{if } x = e_2, \end{cases} \tag{3.47}$$

$$\alpha : F \longrightarrow \mathcal{P}(U), \qquad x \longmapsto \begin{cases} \{h_1, h_2, h_3, h_4, h_5\} & \text{if } x = e_0, \\ \{h_2, h_3, h_4\} & \text{if } x = e_3, \\ \{h_3, h_4\} & \text{if } x = e_4, \end{cases}$$

$$\beta : F \longrightarrow \mathcal{P}(U), \qquad x \longmapsto \begin{cases} \{h_3, h_6\} & \text{if } x \in \{e_0, e_1\}, \\ \{h_1, h_3, h_4, h_6\} & \text{if } x = e_2, \end{cases} \tag{3.48}$$

respectively. Then $\langle(\alpha,\beta);E\rangle$ and $\langle(\alpha,\beta);F\rangle$ are double-framed soft algebras over $\mathcal{U}$. Moreover, the $(\alpha_{\wedge},\beta_{\vee})$-product of $\langle(\alpha,\beta);E\rangle$ and $\langle(\alpha,\beta);F\rangle$ is a double-framed soft algebra over $\mathcal{U}$.

4. Conclusion

We have introduced the notion of double-framed soft algebras in a BCK/BCI-algebras and provided several examples. We have considered the characterization of double-framed soft algebra. We have shown the following.

(1) Every double-framed soft subset of a double-framed soft algebra is also a double-framed soft algebra.

(2) The double-framed soft int-uni set of two double-framed soft algebras is a double-framed soft algebra.

(3) The $(\alpha_{\wedge}, \beta_{\vee})$-product of double-framed soft algebras $\langle(\alpha,\beta);E\rangle$ and $\langle(\alpha,\beta);F\rangle$ is also a double-framed soft algebra.

Given a double-framed soft algebra, we have made a new double-framed soft algebra. We have provided an example which shows that the new constructed double-framed soft set is a double-framed soft algebra, but the given double-framed soft set is not a double-framed soft algebra.

Future research will focus on applying the idea/result in this paper to the ideal theory of BCK/BCI-algebras and related algebraic structures.

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