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Review Article

Three-Step Fixed Point Iteration for Generalized Multivalued Mapping in Banach Spaces

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The convergence of three-step fixed point iterative processes for generalized multivalued nonexpansive mapping was considered in this paper. Under some different conditions, the sequences of three-step fixed point iterates strongly or weakly converge to a fixed point of the generalized multivalued nonexpansive mapping. Our results extend and improve some recent results.

1. Introduction

Let X be a Banach space and K a nonempty subset of X. The set K is called proximinal if for each $x \in X$, there exists an element $y \in K$ such that $\|x - y\| = d(x, K)$, where $d(x, K) = \inf\{\|x - z\| : z \in K\}$. Let $CB(K), C(K), P(K), F(T)$ denote the family of nonempty closed bounded subsets, nonempty compact subsets, nonempty proximinal bounded subsets of K, and the set of fixed points, respectively. A multivalued mapping $T : K \to CB(K)$ is said to be nonexpansive (quasi-nonexpansive) if

$$H(Tx, Ty) \le \|x - y\|, \quad x, y \in K, \\ (H(Tx, Tp) \le \|x - p\|, x \in K,\ p \in F(T)), \tag{1.1}$$

where $H(\cdot, \cdot)$ denotes the Hausdorff metric on $CB(X)$ defined by

$$H(A, B) := \max\left\{\sup_{x\in A} \inf_{y\in B} \|x - y\|,\ \sup_{y\in B} \inf_{x\in A} \|x - y\|\right\}, \quad A, B \in CB(X). \tag{1.2}$$

A point x is called a fixed point of T if $x \in Tx$. Since Banach's Contraction Mapping Principle was extended nicely to multivalued mappings by Nadler in 1969 (see [1]), many authors have studied the fixed point theory for multivalued mappings (e.g., see [2]). For single-valued nonexpansive mappings, Mann [3] and Ishikawa [4], respectively, introduced a new iteration procedure for approximating its fixed point in a Banach space as follows:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n y_n, \qquad y_n = (1-b_n)x_n + b_n T x_n, \tag{1.3}$$

where $\{\alpha_n\}$ and $\{b_n\}$ are sequences in $[0,1]$. Obviously, Mann iteration is a special case of Ishikawa iteration. Recently Song and Wang in [5, 6] introduce the following algorithms for multivalued nonexpansive mapping:

$$x_{n+1} = (1-\alpha_{\mathrm{n}})x_n + \alpha_n s_n, \tag{1.4}$$

where $s_n \in Tx_n, \gamma_n \in (0,+\infty)$ such that $\lim_{n\to\infty}\gamma_n = 0$ and $\|s_{n+1} - s_n\| \le H(Tx_{n+1}, Tx_n) + \gamma_n$,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n r_n, \qquad y_n = (1-b_n)x_n + b_n s_n, \tag{1.5}$$

where $\|s_n - r_n\| \le H(Tx_n, Ty_n) + \gamma_n$ and $\|s_{n+1} - r_n\| \le H(Tx_{n+1}, Ty_n) + \gamma_n$ for $s_n \in Tx_n$ and $r_n \in Ty_n$. They show some strong convergence results of the above iterates for multivalued nonexpansive mapping T under some appropriate conditions. However, the iteration scheme constructed by Song and Wang involves the following estimates,

$$\|s_n - r_n\| \le H(Tx_n, Ty_n) + \gamma_n, \qquad \|s_{n+1} - r_n\| \le H(Tx_{n+1}, Ty_n) + \gamma_n, \tag{1.6}$$

which are not easy to be computed and the scheme is more time consuming. It is observed that Song and Wang [6] did not use the above estimates in their proofs and the assumption on T, namely, $T(p) = \{p\}$ for any $p \in F(T)$ is quite strong. It is noted that the domain of T is compact, which is a strong condition. The aim of this paper is to construct an three iteration scheme for a generalized multivalued mappings, which removes the restriction of T, namely, $T(p) = \{p\}$ for any $p \in F(T)$ and also relax compactness of the domain of T. The generalized multivalued mappings was introduced in [7], if

$$\frac{1}{2}d(x, Tx) \le \|x - y\| \text{ implies } H(Tx, Ty) \le \|x - y\| \quad \forall x, y \in K, \tag{1.7}$$

where d is induced by the norm. Obviously, the condition is weaker than nonexpansiveness and stronger than quasinonexpansiveness, furthermore, there are some examples of a generalized nonexpansive multivalued mapping which is not a nonexpansive multivalued mapping (see [7, 8]).

Let $T : K \to P(K)$ be a generalized nonexpansive multivalued mapping and $P_T(x) = \{y \in T(x) : \|x - y\| = d(x, T(x))\}$. The three-step mean multivalued iterative scheme is defined by $x_0 \in K$,

$$\begin{aligned} z_n &= (1 - a_n)x_n + a_n s_n, \\ y_n &= (1 - b_n - c_n)x_n + b_n t_n + c_n s_n, \\ x_{n+1} &= (1 - \alpha_n - \beta_n - \gamma_n)x_n + \alpha_n r_n + \beta_n t_n + \gamma_n s_n, \end{aligned} \tag{1.8}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{b_n + c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, and $\{\alpha_n + \beta_n + \gamma_n\}$ are appropriate sequence in $[0, 1]$, furthermore $s_n \in P_T(x_n), t_n \in P_T(z_n), r_n \in P_T(y_n)$. If $a_n = c_n = \beta_n = \gamma_n \equiv 0$ or $a_n = b_n = c_n = \beta_n = \gamma_n \equiv 0$, then iterative scheme (1.8) reduces to the Ishikawa and Mann multivalued iterative scheme. In fact let $\gamma_n \equiv 0$ or $c_n = \beta_n = \gamma_n \equiv 0$ or $b_n = c_n = \alpha_n = \gamma_n \equiv 0$, we also have the other three algorithms.

The mapping $T : K \to CB(K)$ is called hemicompact if, for any sequence x_n in K such that $d(x_n, T(x_n)) \to 0$ as $n \to \infty$, there exists a subsequence x_{n_k} of x_n such that $x_{n_k} \to p \in K$. We note that if K is compact, then every multivalued mapping $T : K \to CB(K)$ is hemicompact. The following definition was introduced in [9].

Definition 1.1. A multivalued mapping $T : K \to CB(K)$ is said to satisfy Condition (A) if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0, f(x) > 0$ for $x \in (0, \infty)$ such that

$$d(x, Tx) \geq f(d(x, F(T))) \quad \forall x \in K. \tag{1.9}$$

where $F(T) \neq \emptyset$ is the fixed point set of the multivalued mapping T. From now on, $F(T)$ stands for the fixed point set of the multivalued mapping T.

2. Preliminaries

A Banach space X is said to be satisfy Opial's condition [10] if, for any sequence $\{x_n\}$ in X, $x_n \rightharpoonup x (n \to \infty)$ implies the following inequality:

$$\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\|, \tag{2.1}$$

for all $y \in X$ with $y \neq x$. It is known that Hilbert spaces and $l_p (1 < p < \infty)$ have the Opial's condition.

Lemma 2.1 (see [7, 11]). *Let $\{x_n\}, \{y_n\}$, and $\{z_n\}$ be sequence in uniformly convex Banach space X. Suppose that $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ are sequence in $[0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$, $\limsup_n \|x_n\| \leq d$, $\limsup_n \|y_n\| \leq d$, $\limsup_n \|z_n\| \leq d$, and $\lim_n \|\alpha_n x_n + \beta_n y_n + \gamma_n z_n\| = d$. If $\liminf_n \alpha_n > 0$ and $\liminf_n \beta_n > 0$, then $\lim_n \|x_n - y_n\| = 0$.*

Lemma 2.2 (see [7, 11]). *Let X be a uniformly convex Banach space and $B_r := \{x \in X : \|x\| \leq r\}, r > 0$. Then there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\begin{aligned}
&\|\lambda x + \mu y + \xi z + \vartheta\omega\|^2 \\
&\quad\leq \lambda\|x\|^2 + \mu\|y\|^2 + \xi\|z\|^2 + \vartheta\|\omega\|^2 - \frac{1}{3}\vartheta\left(\lambda g(\|x-\omega\|) + \mu g(\|y-\omega\|) + \xi g(\|z-\omega\|)\right),
\end{aligned} \tag{2.2}$$

for all $x, y, z, \omega \in B_r$ and $\lambda, \mu, \xi, \vartheta \in [0,1]$ with $\lambda + \mu + \xi + \vartheta = 1$.

3. Main Results

Lemma 3.1. *Let X be a real Banach space and K be a nonempty convex subset of $X, T : K \to P(K)$ be a generalized multivalued nonexpansive mapping with $F(T) \neq \emptyset$ such that P_T is nonexpansive. Let $\{x_n\}$ be a sequence in K defined by* (1.8)*, then one has the following conclusion:*

$$\lim_n \|x_n - p\| \text{ exists for any } p \in F(T). \tag{3.1}$$

Proof. Let $p \in F(T)$, then $p \in P_T(p) = \{p\}$. Since T is quasi-nonexpansive, thus we obtain

$$\begin{aligned}
\|z_n - p\| &\leq (1-a_n)\|x_n - p\| + a_n\|s_n - p\| \\
&\leq (1-a_n)\|x_n - p\| + a_n d(s_n, P_T(p)) \\
&\leq (1-a_n)\|x_n - p\| + a_n H(P_T(x_n), P_T(p)) \\
&\leq (1-a_n)\|x_n - p\| + a_n\|x_n - p\| \\
&\leq \|x_n - p\|,
\end{aligned} \tag{3.2}$$

similarly $\|y_n - p\| \leq \|x_n - p\|$, then we have

$$\begin{aligned}
\|x_{n+1} - p\| &\leq (1-\alpha_n-\beta_n-\gamma_n)\|x_n - p\| + \alpha_n\|r_n - p\| \\
&\quad + \beta_n\|t_n - p\| + \gamma_n\|s_n - p\| \\
&\leq (1-\alpha_n-\beta_n-\gamma_n)\|x_n - p\| + \alpha_n H(P_T(y_n), P_T(p)) \\
&\quad + \beta_n H(P_T(z_n), P_T(p)) + \gamma_n H(P_T(x_n), P_T(p)) \\
&\leq \|x_n - p\|.
\end{aligned} \tag{3.3}$$

Then $\{\|x_n - p\|\}$ is a decreasing sequence and hence $\lim_n\|x_n - p\|$ exists for any $p \in F(T)$. □

Lemma 3.2. *Let X be a uniformly convex Banach space and K be a nonempty convex subset of $X, T : K \to P(K)$ be a generalized multivalued nonexpansive mapping with $F(T) \neq \emptyset$ such that P_T is nonexpansive. Let $\{x_n\}$ be a sequence in K defined by* (1.8)*, if the coefficient satisfy one of the following control conditions:*

(i) $\liminf_n \alpha_n > 0$ *and one of the following holds:*

(a) $\limsup_n(\alpha_n + \beta_n + \gamma_n) < 1$ *and* $\limsup_n(b_n + c_n) < 1$,

(b) $0 < \liminf_n \beta_n \le \limsup_n(\alpha_n + \beta_n + \gamma_n) < 1$ *and* $\limsup_n c_n < 1$,

(c) $0 < \liminf_n b_n \le \limsup_n(b_n + c_n) < 1$ *and* $\limsup_n a_n < 1$,

(d) $0 < \liminf_n c_n \le \limsup_n(b_n + c_n) < 1$;

(ii) $0 < \liminf_n \beta_n \le \limsup_n(\alpha_n + \beta_n + \gamma_n) < 1$ *and* $\limsup_n a_n < 1$;

(iii) $0 < \liminf_n \gamma_n \le \limsup_n(\alpha_n + \beta_n + \gamma_n) < 1$;

(iv) $0 < \liminf_n(\alpha_n b_n + \beta_n)$ *and* $0 < \liminf_n a_n \le \limsup_n a_n < 1$;

then we have $\lim_n d(x_n, Tx_n) = 0$.

Proof. By Lemma 3.1, we know that $\lim_n\|x_n - p\|$ exists for any $p \in F(T)$, then it follows that $\{s_n - p\}$, $\{t_n - p\}$, and $\{r_n - p\}$ are all bounded. We may assume that these sequences belong to B_r where $r > 0$. Note that $p \in P_T(p) = \{p\}$ for any fixed point $p \in F(T)$ and T is quasi-nonexpansive. By Lemma 2.2, we get

$$\begin{aligned}
\|z_n - p\|^2 &\le (1 - a_n)\|x_n - p\|^2 + a_n\|s_n - p\|^2 \\
&\le (1 - a_n)\|x_n - p\|^2 + a_n H(P_T(x_n), P_T(p))^2 \\
&\le \|x_n - p\|^2, \\
\|y_n - p\|^2 &\le (1 - b_n - c_n)\|x_n - p\|^2 + b_n\|t_n - p\|^2 + c_n\|s_n - p\|^2 \\
&\quad - \frac{1}{3}(1 - b_n - c_n)\left(b_n g(\|t_n - x_n\|) + c_n g(\|s_n - x_n\|)\right) \\
&\le (1 - b_n - c_n)\|x_n - p\|^2 + b_n H(P_T(z_n), P_T(p))^2 + c_n H(P_T(x_n), P_T(p))^2 \\
&\quad - \frac{1}{3}(1 - b_n - c_n) b_n g(\|t_n - x_n\|) \\
&\le \|x_n - p\|^2 - \frac{1}{3}(1 - b_n - c_n) b_n g(\|t_n - x_n\|),
\end{aligned} \tag{3.4}$$

and therefore we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le (1 - \alpha_n - \beta_n - \gamma_n)\|x_n - p\|^2 + \alpha_n\|r_n - p\|^2 + \beta_n\|t_n - p\|^2 + \gamma_n\|s_n - p\|^2 \\
&\quad - \frac{1}{3}(1 - \alpha_n - \beta_n - \gamma_n)\left[\alpha_n g(\|x_n - r_n\|) + \beta_n g(\|x_n - t_n\|) + \gamma_n g(\|x_n - s_n\|)\right] \\
&\le (1 - \alpha_n - \beta_n - \gamma_n)\|x_n - p\|^2 + \alpha_n H(P_T(y_n), P_T(p))^2 + \beta_n H(P_T(z_n), P_T(p))^2 \\
&\quad + \gamma_n H(P_T(x_n), P_T(p))^2 \\
&\quad - \frac{1}{3}(1 - \alpha_n - \beta_n - \gamma_n)\left[\alpha_n g(\|x_n - r_n\|) + \beta_n g(\|x_n - t_n\|) + \gamma_n g(\|x_n - s_n\|)\right]
\end{aligned}$$

$$\leq \|x_n - p\|^2 - \frac{\alpha_n}{3}(1-b_n-c_n)b_n g(\|t_n - x_n\|) - \frac{1}{3}(1-\alpha_n-\beta_n-\gamma_n) \times \left[\alpha_n g(\|x_n - r_n\|) + \beta_n g(\|x_n - t_n\|) + \gamma_n g(\|x_n - s_n\|)\right]. \tag{3.5}$$

Then

$$(1-\alpha_n-\beta_n-\gamma_n)\alpha_n g(\|x_n - r_n\|) \leq 3\left(\|x_n - p\|^2 - \|x_{n+1} - p\|^2\right), \tag{3.6}$$

$$(1-\alpha_n-\beta_n-\gamma_n)\beta_n g(\|x_n - t_n\|) \leq 3\left(\|x_n - p\|^2 - \|x_{n+1} - p\|^2\right), \tag{3.7}$$

$$(1-\alpha_n-\beta_n-\gamma_n)\gamma_n g(\|x_n - s_n\|) \leq 3\left(\|x_n - p\|^2 - \|x_{n+1} - p\|^2\right), \tag{3.8}$$

$$\alpha_n(1-b_n-c_n)b_n g(\|t_n - x_n\|) \leq 3\left(\|x_n - p\|^2 - \|x_{n+1} - p\|^2\right). \tag{3.9}$$

Since $\lim_n\|x_n - p\|$ exists for any $p \in F(T)$, it follows from (3.6) that $\lim_n(1-\alpha_n-\beta_n-\gamma_n)\alpha_n g(\|x_n - r_n\|) = 0$. From g is continuous strictly increasing with $g(0) = 0$ and $0 < \liminf_n \alpha_n \leq \limsup_n(\alpha_n + \beta_n + \gamma_n) < 1$, then

$$\lim_n \|x_n - r_n\| = 0. \tag{3.10}$$

Using a similarly method together with inequalities (3.7) and $0 < \liminf_n \beta_n \leq \limsup_n(\alpha_n + \beta_n + \gamma_n) < 1$, then

$$\lim_n \|x_n - t_n\| = 0. \tag{3.11}$$

Similarly, from (3.8) and $0 < \liminf_n \gamma_n \leq \limsup_n(\alpha_n + \beta_n + \gamma_n) < 1$, we have $\lim_n\|x_n - s_n\| = 0$, since $s_n \in Tx_n$, then $0 \leq \lim_n d(x_n, Tx_n) \leq \lim_n\|x_n - s_n\| = 0$, thus we get (iii). In the sequence we prove (i) (a). From iterative scheme (1.8), we have

$$\begin{aligned}\|s_n - x_n\| &\leq \|s_n - r_n\| + \|r_n - x_n\| \leq H(P_T(x_n), P_T(y_n)) + \|r_n - x_n\| \\ &\leq \|x_n - y_n\| + \|r_n - x_n\| \\ &\leq b_n\|x_n - t_n\| + c_n\|x_n - s_n\| + \|r_n - x_n\|.\end{aligned} \tag{3.12}$$

To show that $\lim_n\|x_n - s_n\| = 0$, it suffices to show that there exist a subsequence $\{n_j\}$ of $\{n\}$ such that $\lim_{n_j}\|x_{n_j} - s_{n_j}\| = 0$. If $\liminf_j b_{n_j} > 0$, it follows from (3.9) that

$$\alpha_{n_j}\left(1-b_{n_j}-c_{n_j}\right)b_{n_j} g\left(\left\|t_{n_j} - x_{n_j}\right\|\right) \leq 3\left(\left\|x_{n_j} - p\right\|^2 - \left\|x_{n_j+1} - p\right\|^2\right). \tag{3.13}$$

Since $\lim_n\|x_n - p\|$ exists for any $p \in F(T)$, we have

$$\lim_{n_j} \alpha_{n_j}\left(1-b_{n_j}-c_{n_j}\right)b_{n_j} g\left(\left\|t_{n_j} - x_{n_j}\right\|\right) = 0. \tag{3.14}$$

From g is continuous strictly increasing with $g(0) = 0, \liminf_j \alpha_{n_j} > 0$ and $0 < \liminf_{n_j} b_{n_j} \leq \limsup_{n_j}(b_{n_j} + c_{n_j}) < 1$, we have

$$\lim_{n_j} \left\| t_{n_j} - x_{n_j} \right\| = 0. \tag{3.15}$$

This together with (3.10), (3.12), (3.15) gives

$$\lim_{j} \left(1 - c_{n_j}\right) \left\| s_{n_j} - x_{n_j} \right\| = 0. \tag{3.16}$$

Since $\liminf_{n_j}(1 - c_{n_j}) = 1 - \limsup_{n_j} c_{n_j} > 0$, we have $\lim_j \|s_{n_j} - x_{n_j}\| = 0$. On the other hand, if $\liminf_j b_{n_j} = 0$, then we may extract a subsequence $\{b_{n_k}\}$ of $\{b_{n_j}\}$ so that $\lim_k b_{n_k} = 0$. This together with (i) (a) and (3.10), (3.12) gives

$$\lim_{k}(1 - c_{n_k})\|s_{n_k} - x_{n_k}\| = 0, \text{ and so } \lim_{k}\|s_{n_k} - x_{n_k}\| = 0. \tag{3.17}$$

By Double Extract Subsequence Principle, we obtain the result.

If $0 < \liminf_n \beta_n \leq \limsup_n(\alpha_n + \beta_n + \gamma_n) < 1$ and $\limsup_n a_n < 1$, we will prove (ii),

$$\begin{aligned} \|s_n - x_n\| &\leq \|s_n - t_n\| + \|t_n - x_n\| \leq H(P_T(x_n), P_T(z_n)) + \|t_n - x_n\| \\ &\leq \|x_n - z_n\| + \|t_n - x_n\| \\ &\leq a_n\|x_n - s_n\| + \|t_n - x_n\|. \end{aligned} \tag{3.18}$$

Since $\limsup_n a_n < 1$, then

$$\liminf_{n}(1 - a_n) = 1 - \limsup_{n} a_n > 0. \tag{3.19}$$

This together with (3.11), (3.18), we obtain the result.

We will prove (i) (b), let $p \in F(T)$. By Lemma 3.1, we let $\lim_n \|x_n - p\| = d$ for some $d \geq 0$. From iterative scheme (1.8), we know

$$d = \lim_{n}\left\|x_{n+1} - p\right\| = \lim_{n}\left\|(1 - \alpha_n - \beta_n - \gamma_n)(x_n - p) + \alpha_n(r_n - p) + \beta_n(t_n - p) + \gamma_n(s_n - p)\right\|. \tag{3.20}$$

From Lemma 3.1, we have known that $\|z_n - p\| \leq \|x_n - p\|$ and $\|y_n - p\| \leq \|x_n - p\|$, then

$$\begin{aligned} \limsup_{n}\left\|r_n - p\right\| &\leq \limsup_{n} H(P_T(y_n), P_T(p)) \leq \limsup_{n}\left\|y_n - p\right\| \leq d, \\ \limsup_{n}\left\|t_n - p\right\| &\leq \limsup_{n} H(P_T(z_n), P_T(p)) \leq \limsup_{n}\left\|z_n - p\right\| \leq d, \\ \limsup_{n}\left\|s_n - p\right\| &\leq \limsup_{n} H(P_T(x_n), P_T(p)) \leq \limsup_{n}\left\|x_n - p\right\| \leq d. \end{aligned} \tag{3.21}$$

From (3.20) and Lemma 2.1, we have

$$\lim_{n}\|x_n - t_n\| = \lim_{n}\|r_n - x_n\| = 0. \tag{3.22}$$

Notice that

$$\begin{aligned}\|x_n - s_n\| &\le \|x_n - r_n\| + \|r_n - s_n\| \le \|x_n - r_n\| + H(P_T(y_n), P_T(x_n)) \\ &\le \|x_n - y_n\| + \|x_n - r_n\| \\ &\le b_n\|x_n - t_n\| + c_n\|x_n - s_n\| + \|x_n - r_n\|.\end{aligned} \tag{3.23}$$

Since $\limsup_n c_n < 1$, we have $\lim_n\|s_n - x_n\| = 0$, therefore $0 \le \lim_n d(x_n, Tx_n) \le \lim_n\|x_n - s_n\| = 0$.

We will prove (i) (c). From iterative scheme (1.8) and Lemma 3.1, we have

$$\begin{aligned}\|x_{n+1} - p\| &\le (1 - \alpha_n - \beta_n - \gamma_n)\|x_n - p\| + \alpha_n\|y_n - p\| + \beta_n\|z_n - p\| + \gamma_n\|x_n - p\| \\ &\le (1 - \alpha_n)\|x_n - p\| + \alpha_n\|y_n - p\|.\end{aligned} \tag{3.24}$$

which implies

$$\|x_{n+1} - p\| - \|x_n - p\| + \alpha_n\|x_n - p\| \le \alpha_n\|y_n - p\|. \tag{3.25}$$

Notice that $\liminf_n \alpha_n > 0$ and $\lim_n\|x_n - p\|$ exists. Hence from (3.25) we have

$$d = \lim_n\|x_n - p\| \le \liminf_n\|y_n - p\| \le \limsup\|y_n - p\| \le d. \tag{3.26}$$

Therefore, from iterative scheme (1.8) we have

$$d = \lim_n\|y_n - p\| = \lim_n\|(1 - b_n - c_n)(x_n - p) + b_n(t_n - p) + c_n(s_n - p)\|. \tag{3.27}$$

From Lemma 2.1, we have

$$\lim_n\|x_n - t_n\| = 0. \tag{3.28}$$

Notice that

$$\begin{aligned}\|s_n - x_n\| &\le \|s_n - t_n\| + \|t_n - x_n\| \le H(P_T(x_n), P_T(z_n)) + \|t_n - x_n\| \\ &\le \|x_n - z_n\| + \|t_n - x_n\| \\ &\le a_n\|x_n - s_n\| + \|t_n - x_n\|.\end{aligned} \tag{3.29}$$

Since $\limsup_n a_n < 1$, then $0 \le \lim_n d(x_n, Tx_n) \le \lim_n\|x_n - s_n\| = 0$.

By (3.27) and Lemma 2.1, we can similarly prove (i) (d).

Finally, we will prove (iv). From iterative scheme (1.8) and Lemma 3.1, we have

$$\begin{aligned}\|x_{n+1}-p\| &\le (1-\alpha_n-\beta_n-\gamma_n)\|x_n-p\|+\alpha_n\|r_n-p\|+\beta_n\|t_n-p\|+\gamma_n\|s_n-p\|\\ &\le (1-\alpha_n-\beta_n-\gamma_n)\|x_n-p\|+\alpha_n\|y_n-p\|+\beta_n\|z_n-p\|+\gamma_n\|x_n-p\|\\ &\le (1-\alpha_n-\beta_n)\|x_n-p\|+\alpha_n[(1-b_n)\|x_n-p\|+b_n\|z_n-p\|]+\beta_n\|z_n-p\|,\end{aligned} \tag{3.30}$$

which implies

$$\|x_{n+1}-p\|-\|x_n-p\|+(\alpha_n b_n+\beta_n)\|x_n-p\| \le (\alpha_n b_n+\beta_n)\|z_n-p\|. \tag{3.31}$$

Notice that

$$0<\liminf_n(\alpha_n b_n+\beta_n), \qquad \lim_n\|x_n-p\| \text{ exists.} \tag{3.32}$$

Hence we have

$$d=\lim_n\|x_n-p\| \le \liminf_n\|z_n-p\| \le \limsup\|z_n-p\| \le d. \tag{3.33}$$

Thus, we have

$$d=\lim_n\|z_n-p\|=\lim_n(1-a_n)\|x_n-p\|+a_n\|s_n-p\|. \tag{3.34}$$

By Lemma 2.1 and $0<\liminf_n a_n \le \limsup_n a_n<1$, we have $0\le \lim_n d(x_n,Tx_n)\le \lim_n\|x_n-s_n\|=0$. □

Theorem 3.3. *Let X be a uniformly convex Banach space and K be a nonempty convex subset of X, $T:K\to P(K)$ be a generalized multivalued nonexpansive mapping with $F(T)\neq\emptyset$ such that P_T is nonexpansive. Let $\{x_n\}$ be a sequence in K defined by* (1.8)*, the coefficient satisfy the control conditions in Lemma 3.2 and T satisfies Condition (A) with respect to the sequence $\{x_n\}$, then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Lemma 3.2, we have $\lim_n d(x_n,Tx_n)=0$. Since T satisfies Condition (A) with respect to $\{x_n\}$. Then

$$f(d(x_n,F(T))) \le d(x_n,Tx_n) \longrightarrow 0. \tag{3.35}$$

Thus, we get $\lim_n d(x_n,F(T))=0$. The remainder of the proof is the same as in [6, Theorem 2.4], we omit it. □

Theorem 3.4. *Let X be a uniformly convex Banach space and K be a nonempty convex subset of X, $T:K\to P(K)$ be a generalized multivalued nonexpansive mapping with $F(T)\neq\emptyset$ such that P_T is nonexpansive. Let $\{x_n\}$ be a sequence in K defined by* (1.8)*, the coefficient satisfy the control conditions in Lemma 3.2 and T is hemicompact, then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Lemma 3.2, we have $\lim_n d(x_n, Tx_n) = 0$. Since T is hemicompact, then there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty}\|x_{n_k} - q\| = 0$ for some $q \in K$. Thus,

$$\begin{aligned} d(q, Tq) &\leq \|q - x_{n_k}\| + d(x_{n_k}, Tx_{n_k}) + H(Tx_{n_k}, Tq) \\ &\leq 2\|q - x_{n_k}\| + d(x_{n_k}, Tx_{n_k}) \longrightarrow 0. \end{aligned} \tag{3.36}$$

Hence, q is a fixed point of T. Now on take on q in place of p, we get that $\lim_{n\to\infty}\|x_n - q\|$ exists. It follows that $x_n \to q$ as $n \to \infty$. This completes the proof. □

Theorem 3.5. *Let X, T and $\{x_n\}$ be the same as in Lemma 3.2. If K be a nonempty weakly compact convex subset of a Banach space X and X satisfies Opial's condition, then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. The proof of the Theorem is the same as in [6, Theorem 2.5], we omit it. □

Remark 3.6. From the definition of iterative scheme (1.8), Theorems 3.3, 3.4, and 3.5 extend some results in [6, 12], and also give some new results are different from the [5]. In fact, we can present an example of a multivalued map T for which P_T is nonexpansive. A multivalued map $T : D \to CB(X)$ is $*$-nonexpansive [13] if for all $x, y \in D$ and $u_x \in T(x)$ with $d(x, u_x) = \inf\{d(x, z) : z \in T(x)\}$, there exists $u_y \in T(y)$ with $d(y, u_y) = \inf\{d(y, w) : \omega \in T(y)\}$ such that

$$d(u_x, u_y) \leq d(x, y). \tag{3.37}$$

It is clear that if T is $*$-nonexpansive, then P_T is nonexpansive. It is known that $*$-nonexpansiveness is different from nonexpansiveness for multivalued maps. Let $D = [0, \infty)$ and T be defined by $Tx = [x, 2x]$ for $x \in D$ [14]. Then $P_T(x) = x$ for $x \in D$ and thus it is nonexpansive. Note that T is $*$-nonexpansive but not nonexpansive (see [14]).

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