Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 246516, 20 pages
doi:10.1155/2012/246516

Research Article

Coupled Fixed Point Theorems of Single-Valued Mapping for *c*-Distance in Cone Metric Spaces

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Received 27 March 2012; Accepted 24 April 2012

Academic Editor: Francis T. K. Au

A new concept of the c-distance in cone metric space has been introduced recently in 2011. The aim of this paper is to extend and generalize some coupled fixed-point theorems on c-distance in cone metric space. Some examples are given.

1. Introduction

In 2007, Huang and Zhang [1] introduced the concept of cone metric space where each pair of points is assigned to a member of a real Banach space with a cone. Then, several authors have studied the existence and uniqueness of the fixed point and common fixed point for self-map f by considered different types of contractive conditions. Some of these works are noted in [2–12].

In [13], Bhaskar and Lakshmikantham introduced the concept of coupled fixed point for a given partially ordered set X. Lakshmikantham and Ćirić [14] proved some more coupled fixed-point theorems in partially ordered set.

In [15], Sabetghadam et al. considered the corresponding definition of coupled fixed point for the mapping in complete cone metric space and proved some coupled fixed point theorems. Then, several authors have studied the existence and uniqueness of the coupled fixed point and coupled common fixed point by considered different types of contractive conditions. Some of these works are noted in [16–23].

Recently, Cho et al. [23] introduced a new concept of the c-distance in cone metric spaces (also see [24]) and proved some fixed-point theorems in ordered cone metric spaces. This is more general than the classical Banach contraction mapping principle. Sintunavarat et al. [25] extended and developed the Banach contraction theorem on c-distance of Cho et al. [23]. Wang and Guo [24] proved some common fixed point theorems for this new distance.

Several authors have studied on the generalized distance in cone metric space. Some of this works are noted in [26–29].

In [28], Cho et al. proved some coupled fixed point theorems in ordered cone metric spaces by using the concept of c-distance.

Recall the following definition.

Definition 1.1 (see [15]). Let (X, d) be a cone metric space. An element $(x, y) \in X \times X$ is said to be a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

The following theorems are the main results given in [15].

Theorem 1.2 (see [15]). *Let (X, d) be a complete cone metric space. Suppose that the mapping $F : X \times X \to X$ satisfies the following contractive condition for all $x, y, u, v \in X$:*

$$d(F(x,y), F(u,v)) \preceq kd(x,u) + ld(y,v), \tag{1.1}$$

where k, l are nonnegative constants with $k + l < 1$. Then F has a unique coupled fixed point.

Theorem 1.3 (see [15]). *Let (X, d) be a complete cone metric space. Suppose the mapping $F : X \times X \to X$ satisfies the following contractive condition for all $x, y, u, v \in X$:*

$$d(F(x,y), F(u,v)) \preceq kd(F(x,y), x) + ld(F(u,v), u), \tag{1.2}$$

where k, l are nonnegative constants with $k + l < 1$. Then F has a unique coupled fixed point.

In this paper we proved some coupled fixed point results for c-distance in cone metric space. Our theorems extend and develop some theorems of Sabetghadam et al. [15] on c-distance of Cho et al. [23] in cone metric space.

2. Preliminaries

Let E be a real Banach space and θ denote to the zero element in E. A cone P is a subset of E such that

(1) P is nonempty set closed and $P \neq \{\theta\}$,

(2) if a, b are nonnegative real numbers and $x, y \in P$ then $ax + by \in P$, and

(3) $x \in P$ and $-x \in P$ implies $x = \theta$.

For any cone $P \subset E$, the partial ordering $\preceq$ with respect to P is defined by $x \preceq y$ if and only if $y - x \in P$. The notation of $\prec$ stands for $x \preceq y$ but $x \neq y$. Also, we used $x \ll y$ to indicate that $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P. A cone P is called normal if there exists a number K such that

$$\theta \preceq x \preceq y \Longrightarrow \|x\| \leq K\|y\| \tag{2.1}$$

for all $x, y \in E$. The least positive number K satisfying the above condition is called the normal constant of P.

Definition 2.1 (see [1]). Let X be a nonempty set and E be a real Banach space equipped with the partial ordering $\preceq$ with respect to the cone P. Suppose that the mapping $d : X \times X \to E$ satisfies the following condition:

(1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$,

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$, and

(3) $d(x, y) \preceq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 2.2 (see [1]). Let (X, d) be a cone metric space and $\{x_n\}$ be a sequence in X and $x \in X$.

One has the following:

(1) for all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that $d(x_n, x) \ll c$ for all $n > N$, then x_n is said to be convergent and x is the limit of $\{x_n\}$. We denote this by $x_n \to x$,

(2) for all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that $d(x_n, x_m) \ll c$ for all $n, m > N$ then $\{x_n\}$ is called a Cauchy sequence in X, and

(3) a cone metric space (X, d) is called complete if every Cauchy sequence in X is convergent.

Lemma 2.3 (see [8]). *(1) If E be a real Banach space with a cone P and $a \preceq \lambda a$ where $a \in P$ and $0 \leq \lambda < 1$, then $a = \theta$.*

(2) If $c \in \operatorname{int} P$, $\theta \preceq a_n$ and $a_n \to \theta$, then there exists a positive integer N such that $a_n \ll c$ for all $n \geq N$.

Next we give the notation of c-distance on a cone metric space which is a generalization of ω-distance of Kada et al. [30] with some properties.

Definition 2.4 (see [23]). Let (X, d) be a cone metric space. A function $q : X \times X \to E$ is called a c-distance on X if the following conditions hold:

(q1) $\theta \preceq q(x, y)$ for all $x, y \in X$,

(q2) $q(x, y) \preceq q(x, y) + q(y, z)$ for all $x, y, z \in X$,

(q3) for each $x \in X$ and $n \geq 1$, if $q(x, y_n) \preceq u$ for some $u = u_x \in P$, then $q(x, y) \preceq u$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$, and

(q4) for all $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \ll e$ such that $q(z, x) \ll e$ and $q(z, y) \ll e$ imply $d(x, y) \ll c$.

Example 2.5 (see [23]). Let $E = \mathbb{R}$ and $P = (x \in E : x \geq 0)$. Let $X = [0, \infty)$ and define a mapping $d : X \times X \to E$ by $d(x, y) = |x - y|$ for all $x, y \in X$. Then (X, d) is a cone metric space. Define a mapping $q : X \times X \to E$ by $q(x, y) = y$ for all $x, y \in X$. Then q is a c-distance on X.

Lemma 2.6 (see [23]). *Let (X, d) be a cone metric space and q is a c-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $x, y, z \in X$. Suppose that u_n is a sequences in P converging to θ. Then the following hold.*

(1) *If $q(x_n, y) \preceq u_n$ and $q(x_n, z) \preceq u_n$, then $y = z$.*

(2) *If $q(x_n, y_n) \preceq u_n$ and $q(x_n, z) \preceq u_n$, then $\{y_n\}$ converges to z.*

(3) *If $q(x_n, x_m) \preceq u_n$ for $m > n$, then $\{x_n\}$ is a Cauchy sequence in X.*

(4) *If $q(y, x_n) \preceq u_n$, then $\{x_n\}$ is a Cauchy sequence in X.*

Remark 2.7 (see [23]). (1) $q(x, y) = q(y, x)$ does not necessarily for all $x, y \in X$.
(2) $q(x, y) = \theta$ is not necessarily equivalent to $x = y$ for all $x, y \in X$.

3. Main Results

In this section we prove some coupled fixed point theorems using c-distance in cone metric space. In whole paper cone metric space is over nonnormal cone with nonempty interior.

Theorem 3.1. *Let (X, d) be a complete cone metric space, and q is a c-distance on X. Let $F : X \times X \to X$ be a mapping and suppose that there exists mappings $k, l : X \times X \to [0, 1)$ such that the following hold:*

(a) *$k(F(x, y), F(u, v)) \leq k(x, y)$ and $l(F(x, y), F(u, v)) \leq l(x, y)$ for all $x, y, u, v \in X$,*

(b) *$k(x, y) = k(y, x)$ and $l(x, y) = l(y, x)$ for all $x, y \in X$,*

(c) *$(k + l)(x, y) < 1$ for all $x, y \in X$,*

(d) *$q(F(x, y), F(u, v)) \preceq k(x, y)q(x, u) + l(x, y)q(y, v)$ for all $x, y, u, v \in X$.*

Then F has a coupled fixed point $(x^*, y^*) \in X \times X$. Further, if $x_1 = F(x_1, y_1)$ and $y_1 = F(y_1, x_1)$, then $q(x_1, x_1) = \theta$ and $q(y_1, y_1) = \theta$. Moreover, the coupled fixed point is unique and is of the form (x^*, x^*) for some $x^* \in X$.

Proof. Choose $x_0, y_0 \in X$. Set $x_1 = F(x_0, y_0), y_1 = F(y_0, x_0), x_2 = F(x_1, y_1), y_2 = F(y_1, x_1), \ldots, x_{n+1} = F(x_n, y_n), y_{n+1} = F(y_n, x_n)$. Then we have the following:

$$\begin{aligned}
q(x_n, x_{n+1}) &= q\big(F\big(x_{n-1}, y_{n-1}\big), \mathrm{F}(x_n, y_n)\big) \\
&\preceq k\big(x_{n-1}, y_{n-1}\big)q(x_{n-1}, x_n) + l\big(x_{n-1}, y_{n-1}\big)q\big(y_{n-1}, y_n\big) \\
&= k\big(F\big(x_{n-2}, y_{n-2}\big), F\big(y_{n-2}, x_{n-2}\big)\big)q(x_{n-1}, x_n) \\
&\quad + l\big(F\big(x_{n-2}, y_{n-2}\big), F\big(y_{n-2}, x_{n-2}\big)\big)q\big(y_{n-1}, y_n\big) \\
&\preceq k\big(x_{n-2}, y_{n-2}\big)q(x_{n-1}, x_n) + l\big(x_{n-2}, y_{n-2}\big)q\big(y_{n-1}, y_n\big) \\
&\vdots \\
&\preceq k(x_0, y_0)q(x_{n-1}, x_n) + l(x_0, y_0)q\big(y_{n-1}, y_n\big).
\end{aligned} \tag{3.1}$$

And similarly

$$\begin{aligned}
q(y_n, y_{n+1}) &= q(F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \\
&\leq k(y_{n-1}, x_{n-1})q(y_{n-1}, y_n) + l(y_{n-1}, x_{n-1})q(x_{n-1}, x_n) \\
&= k(F(y_{n-2}, x_{n-2}), F(x_{n-2}, y_{n-2}))q(y_{n-1}, y_n) \\
&\quad + l(F(y_{n-2}, x_{n-2}), F(x_{n-2}, y_{n-2}))q(x_{n-1}, x_n) \\
&\leq k(y_{n-2}, x_{n-2})q(y_{n-1}, y_n) + l(y_{n-2}, x_{n-2})q(x_{n-1}, x_n) \\
&\vdots \\
&\leq k(y_0, x_0)q(y_{n-1}, y_n) + l(y_0, x_0)q(x_{n-1}, x_n) \\
&= k(x_0, y_0)q(y_{n-1}, y_n) + l(x_0, y_0)q(x_{n-1}, x_n).
\end{aligned} \tag{3.2}$$

Put, $q_n = q(x_n, x_{n+1}) + q(y_n, y_{n+1})$. Then we have

$$\begin{aligned}
q_n &= q(x_n, x_{n+1}) + q(y_n, y_{n+1}) \\
&\leq k(x_0, y_0)q(x_{n-1}, x_n) + l(x_0, y_0)q(y_{n-1}, y_n) + k(x_0, y_0)q(y_{n-1}, y_n) \\
&\quad + l(x_0, y_0)q(x_{n-1}, x_n) \\
&= (k(x_0, y_0) + l(x_0, y_0))(q(x_{n-1}, x_n) + q(y_{n-1}, y_n)) \\
&= (k(x_0, y_0) + l(x_0, y_0))_{q_{n-1}} \\
&= hq_{n-1} \\
&\vdots \\
&\leq h^n q_0,
\end{aligned} \tag{3.3}$$

where $h = k(x_0, y_0) + l(x_0, y_0) < 1$.

Let $m > n \geq 1$. It follows that

$$\begin{aligned}
q(x_n, x_m) &\leq q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + \cdots + q(x_{m-1}, x_m), \\
q(y_n, y_m) &\leq q(y_n, y_{n+1}) + q(y_{n+1}, x_{y+2}) + \cdots + q(y_{y-1}, y_m).
\end{aligned} \tag{3.4}$$

Then we have

$$\begin{aligned}
q(x_n, x_m) + q(y_n, y_m) &\leq q_n + q_{n+1} + \cdots + q_{m-1} \\
&\leq h^n q_0 + h^{n+1} q_0 + \cdots + h^{m-1} q_0 \\
&= \left(h^n + h^{n+1} + \cdots + h^{m-1}\right) q_0 \\
&\leq \frac{h^n}{1-h} q_0.
\end{aligned} \tag{3.5}$$

From (3.5) we have

$$q(x_n, x_m) \preceq \frac{h^n}{1-h} q_0, \tag{3.6}$$

and also

$$q(y_n, y_m) \preceq \frac{h^n}{1-h} q_0. \tag{3.7}$$

Thus, Lemma 2.6(3) shows that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. Since X is complete, there exists x^* and $y^* \in X$ such that $x_n \to x^*$ and $y_n \to y^*$ as $n \to \infty$. By (q3) we have the following:

$$q(x_n, x^*) \preceq \frac{h^n}{1-h} q_0, \tag{3.8}$$

and also

$$q(y_n, y^*) \preceq \frac{h^n}{1-h} q_0. \tag{3.9}$$

On the other hand,

$$\begin{aligned}
q(x_n, F(x^*, y^*)) &= q(F(x_{n-1}, y_{n-1}), F(x^*, y^*)) \\
&\preceq k(x_{n-1}, y_{n-1}) q(x_{n-1}, x^*) + l(x_{n-1}, y_{n-1}) q(y_{n-1}, y^*) \\
&= k(F(x_{n-2}, y_{n-2}), F(y_{n-2}, x_{n-2})) q(x_{n-1}, x^*) \\
&\quad + l(F(x_{n-2}, y_{n-2}), F(y_{n-2}, x_{n-2})) q(y_{n-1}, y^*) \\
&\preceq k(x_{n-2}, y_{n-2}) q(x_{n-1}, x^*) + l(x_{n-2}, y_{n-2}) q(y_{n-1}, y^*) \\
&\vdots \\
&\preceq k(x_0, y_0) q(x_{n-1}, x^*) + l(x_0, y_0) q(y_{n-1}, y^*) \\
&\preceq k(x_0, y_0) \frac{h^{n-1}}{1-h} q_0 + l(x_0, y_0) \frac{h^{n-1}}{1-h} q_0 \\
&= (k(x_0, y_0) + l(x_0, y_0)) \frac{h^{n-1}}{1-h} q_0 \\
&= h \frac{h^{n-1}}{1-h} q_0 \\
&= \frac{h^n}{1-h} q_0.
\end{aligned} \tag{3.10}$$

By Lemma 2.6 (1), (3.8), and (3.10), we have $x^* = F(x^*, y^*)$. By similar way we have $y^* = F(y^*, x^*)$. Therefore (x^*, y^*) is a coupled fixed point of F.

Suppose that $x_1 = F(x_1, y_1)$ and $y_1 = F(y_1, x_1)$, then we have

$$\begin{aligned} q(x_1, x_1) &= q(F(x_1, y_1), F(x_1, y_1)) \\ &\preceq k(x_1, y_1)q(x_1, x_1) + l(x_1, y_1)q(y_1, y_1), \end{aligned} \tag{3.11}$$

and also

$$\begin{aligned} q(y_1, y_1) &= q(F(y_1, x_1), F(y_1, x_1)) \\ &\preceq k(y_1, x_1)q(y_1, y_1) + l(y_1, x_1)q(x_1, x_1) \\ &= k(x_1, y_1)q(y_1, y_1) + l(x_1, y_1)q(x_1, x_1) \end{aligned} \tag{3.12}$$

which implies that

$$\begin{aligned} q(x_1, x_1) + q(y_1, y_1) &\preceq k(x_1, y_1)q(x_1, x_1) + k(x_1, y_1)q(y_1, y_1) + l(x_1, y_1)q(y_1, y_1) \\ &\quad + l(x_1, y_1)q(x_1, x_1) \\ &= (k(x_1, y_1) + l(x_1, y_1))(q(x_1, x_1) + q(y_1, y_1)) \\ &= (k + l)(x_1, y_1)(q(x_1, x_1) + q(y_1, y_1)). \end{aligned} \tag{3.13}$$

Since $(k + l)(x_1, y_1) < 1$, Lemma 2.3 (1) shows that $q(x_1, x_1) + q(y_1, y_1) = \theta$. But $q(x_1, x_1) \succeq \theta$ and $q(y_1, y_1) \succeq \theta$, hence $q(x_1, x_1) = \theta$ and $q(y_1, y_1) = \theta$.

Finally, suppose that there is another coupled fixed point (x', y') then we have

$$\begin{aligned} q(x^*, x') &= q(F(x^*, y^*), F(x', y')) \\ &\preceq k(x^*, y^*)q(x^*, x') + l(x^*, y^*)q(y^*, y') \end{aligned} \tag{3.14}$$

and also

$$\begin{aligned} q(y^*, y') &= q(F(y^*, x^*), F(y', x')) \\ &\preceq k(y^*, x^*)q(y^*, y') + l(y^*, x^*)q(x^*, x') \\ &= k(x^*, y^*)q(y^*, y') + l(x^*, y^*)q(x^*, x'), \end{aligned} \tag{3.15}$$

which implies that

$$\begin{aligned} q(x^*, x') + q(y^*, y') &\preceq k(x^*, y^*)q(x^*, x') + k(x^*, y^*)q(y^*, y') + l(x^*, y^*)q(y^*, y') \\ &\quad + l(x^*, y^*)q(x^*, x') \\ &= (k(x^*, y^*) + l(x^*, y^*))(q(x^*, x') + q(y^*, y')) \\ &= (k + l)(x^*, y^*)(q(x^*, x') + q(y^*, y')). \end{aligned} \tag{3.16}$$

Since $(k+l)(x^*, y^*) < 1$, Lemma 2.3 (1) shows that $q(x^*, x') + q(y^*, y') = \theta$. But $q(x^*, x') \succeq \theta$ and $q(y^*, y') \succeq \theta$. Hence $q(x^*, x') = \theta$ and $q(y^*, y') = \theta$. Also we have $q(x^*, x^*) = \theta$ and $q(y^*, y^*) = \theta$. Hence Lemma 2.6 part 1 shows that $x^* = x'$ and $y^* = y'$, which implies that $(x^*, y^*) = (x', y')$. Similarly, we prove that $x^* = y'$ and $y^* = x'$. Hence, $x^* = y^*$. Therefore, the coupled fixed point is unique and is of the form (x^*, x^*) for some $x^* \in X$. □

From above Theorem we have the following corollaries.

Corollary 3.2. *Let (X, d) be a complete cone metric space, and q is a c-distance on X. Suppose that the mapping $F : X \times X \to X$ satisfies the following contractive condition:*

$$q(F(x, y), F(u, v)) \preceq kq(x, u) + lq(y, v), \tag{3.17}$$

for all $x, y, u, v \in X$, where k, l are nonnegative constants with $k + l < 1$. Then F has a coupled fixed point $(x^*, y^*) \in X \times X$. Further, if $x_1 = F(x_1, y_1)$ and $y_1 = F(y_1, x_1)$ then $q(x_1, x_1) = \theta$ and $q(y_1, y_1) = \theta$. Moreover, the coupled fixed point is unique and is of the form (x^*, x^*) for some $x^* \in X$.

Corollary 3.3. *Let (X, d) be a complete cone metric space and q is a c-distance on X. Suppose that the mapping $F : X \times X \to X$ satisfies the following contractive condition:*

$$q(F(x, y), F(u, v)) \preceq k\left[q(x, u) + q(y, v)\right], \tag{3.18}$$

for all $x, y, u, v \in X$, where $k \in [0, 1/2)$. Then F has a coupled fixed point $(x^*, y^*) \in X \times X$. Further, if $x_1 = F(x_1, y_1)$ and $y_1 = F(y_1, x_1)$, then $q(x_1, x_1) = \theta$ and $q(y_1, y_1) = \theta$. Moreover, the coupled fixed point is unique and is of the form (x^*, x^*) for some $x^* \in X$.

Theorem 3.4. *Let (X, d) be a complete cone metric space, and q is a c-distance on X. Suppose that the mapping $F : X \times X \to X$ is continuous, and suppose that there exists mappings $k, l, r : X \times X \to [0, 1)$ such that the following hold:*

(a) $k(F(x, y), F(u, v)) \le k(x, y)$, $l(F(x, y), F(u, v)) \le l(x, y) r(F(x, y), F(u, v)) \le r(x, y)$ *for all $x, y, u, v \in X$,*

(b) $(k + l + r)(x, y) < 1$ *for all $x, y \in X$, and*

(c) $q(F(x, y), F(u, v)) \preceq k(x, y)q(x, u) + l(x, y)q(x, F(x, y)) + r(x, y)q(u, F(u, v))$ *for all $x, y, u, v \in X$.*

Then F has a coupled fixed point $(x^*, y^*) \in X \times X$. Further, if $x_1 = F(x_1, y_1)$ and $y_1 = F(y_1, x_1)$, then $q(x_1, x_1) = \theta$ and $q(y_1, y_1) = \theta$. Moreover, the coupled fixed point is unique and is of the form (x^*, x^*) for some $x^* \in X$.

Proof. Choose $x_0, y_0 \in X$. Set $x_1 = F(x_0, y_0), y_1 = F(y_0, x_0), x_2 = F(x_1, y_1), y_2 = F(y_1, x_1), \ldots, x_{n+1} = F(x_n, y_n), y_{n+1} = F(y_n, x_n)$. Then we have the following:

$$\begin{aligned}
q(x_n, x_{n+1}) &= q\big(F(x_{n-1}, y_{n-1}), F(x_n, y_n)\big) \\
&\leq k(x_{n-1}, y_{n-1}) q(x_{n-1}, x_n) \\
&\quad + l(x_{n-1}, y_{n-1}) q\big(x_{n-1}, F(x_{n-1}, y_{n-1})\big) \\
&\quad + r(x_{n-1}, y_{n-1}) q\big(x_n, F(x_n, y_n)\big) \\
&= k\big(F(x_{n-2}, y_{n-2}), F(y_{n-2}, x_{n-2})\big) q(x_{n-1}, x_n) \\
&\quad + l\big(F(x_{n-2}, y_{n-2}), F(y_{n-2}, x_{n-2})\big) q(x_{n-1}, x_n) \\
&\quad + r\big(F(x_{n-2}, y_{n-2}), F(y_{n-2}, x_{n-2})\big) q(x_n, x_{n+1}) \\
&\leq k(x_{n-2}, y_{n-2}) q(x_{n-1}, x_n) + l(x_{n-2}, y_{n-2}) q(x_{n-1}, x_n) \\
&\quad + r(x_{n-2}, y_{n-2}) q(x_n, x_{n+1}) \\
&\;\;\vdots \\
&\leq k(x_0, y_0) q(x_{n-1}, x_n) + l(x_0, y_0) q(x_{n-1}, x_n) \\
&\quad + r(x_0, y_0) q(x_n, x_{n+1}).
\end{aligned} \tag{3.19}$$

Then, we have

$$\begin{aligned}
q(x_n, x_{n+1}) &\leq \frac{k(x_0, y_0) + l(x_0, y_0)}{1 - r(x_0, y_0)} q(x_{n-1}, x_n) \\
&= h q(x_{n-1}, x_n) \\
&\leq h^2 q(x_{n-2}, x_{n-1}) \\
&\;\;\vdots \\
&\leq h^n q(x_0, x_1),
\end{aligned} \tag{3.20}$$

where $h = (k(x_0, y_0) + l(x_0, y_0))/(1 - r(x_0, y_0)) < 1$.

Similarly we have

$$\begin{aligned}
q(y_n, y_{n+1}) &\leq \frac{k(y_0, x_0) + l(y_0, x_0)}{1 - l(y_0, x_0) - r(y_0, x_0)} q(y_{n-1}, y_n) \\
&= d q(y_{n-1}, y_n)
\end{aligned}$$

$$
\begin{aligned}
&\preceq d^2 q(y_{n-2}, y_{n-1}) \\
&\vdots \\
&\preceq d^n q(y_0, y_1),
\end{aligned}
\tag{3.21}
$$

where $d = (k(y_0, x_0) + l(y_0, x_0))/(1 - l(y_0, x_0) - r(y_0, x_0)) < 1$.

Let $m > n \geq 1$. Then it follows that

$$
\begin{aligned}
q(x_n, x_m) &\preceq q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + \cdots + q(x_{m-1}, x_m) \\
&\preceq \left(h^n + h^{n+1} + \cdots + h^{m-1}\right) q(x_0, x_1) \\
&\preceq \frac{h^n}{1-h} q(x_0, x_1)
\end{aligned}
\tag{3.22}
$$

and also

$$
\begin{aligned}
q(y_n, y_m) &\preceq q(y_n, y_{n+1}) + q(y_{n+1}, y_{n+2}) + \cdots + q(y_{m-1}, y_m) \\
&\preceq \left(d^n + d^{n+1} + \cdots + d^{m-1}\right) q(y_0, y_1) \\
&\preceq \frac{d^n}{1-d} q(y_0, y_1).
\end{aligned}
\tag{3.23}
$$

Thus, Lemma 2.6 (3) shows that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. Since X is complete, there exists x^* and $y^* \in X$ such that $x_n \to x^*$ and $y_n \to y^*$ as $n \to \infty$. Since F is continuous, then $x^* = \lim x_{n+1} = \lim F(x_n, y_n) = F(\lim x_n, \lim y_n) = F(x^*, y^*)$. Similarly, $y^* = F(y^*, x^*)$. Therefore, (x^*, y^*) is a coupled fixed point of F.

Suppose that $x_1 = F(x_1, y_1)$ and $y_1 = F(y_1, x_1)$, then we have

$$
\begin{aligned}
q(x_1, x_1) &= q(F(x_1, y_1), F(x_1, y_1)) \\
&\preceq k(x_1, x_1) q(x_1, x_1) + l(x_1, x_1) q(x_1, F(x_1, y_1)) \\
&\quad + r(x_1, x_1) q(x_1, F(x_1, y_1)) \\
&= k(x_1, x_1) q(x_1, x_1) + l(x_1, x_1) q(x_1, x_1) + r(x_1, x_1) q(x_1, x_1) \\
&= (k + l + r)(x_1, x_1) q(x_1, x_1).
\end{aligned}
\tag{3.24}
$$

Since $(k+l+r)(x_1, x_1) < 1$, Lemma 2.3 (1) shows that $q(x_1, x_1) = \theta$. By similar way, $q(y_1, y_1) = \theta$.

Finally, suppose that there is another coupled fixed point (x', y'), then we have

$$\begin{aligned}
q(x^*, x') &= q(F(x^*, y^*), F(x', y')) \\
&\preceq k(x^*, y^*)q(x^*, x') + l(x^*, y^*)q(x^*, F(x^*, y^*)) + r(x^*, y^*)q(x', F(x', y')) \\
&= k(x^*, y^*)q(x^*, x') + l(x^*, y^*)q(x^*, x^*) + r(x^*, y^*)q(x', x') \\
&= k(x^*, y^*)q(x^*, x') \\
&\preceq k(x^*, y^*)q(x^*, x') + l(x^*, y^*)q(x^*, x') + r(x^*, y^*)q(x^*, x') \\
&= (k + l + r)(x^*, y^*)q(x^*, x').
\end{aligned} \tag{3.25}$$

Since $(k+l+r)(x^*, y^*) < 1$, Lemma 2.3 (1) shows that $q(x^*, x') = \theta$. Also we have $q(x^*, x^*) = \theta$. Hence Lemma 2.6 (1) show that $x^* = x'$. By similar way we have $y^* = y'$ which implies that $(x^*, y^*) = (x', y')$. Similarly, we prove that $x^* = y'$ and $y^* = x'$. Hence, $x^* = y^*$. Therefore, the coupled fixed point is unique and is of the form (x^*, x^*) for some $x^* \in X$. □

From the above theorem, we have the following corollaries.

Corollary 3.5. *Let (X, d) be a complete cone metric space, and q is a c-distance on X. Suppose that the mapping $F : X \times X \to X$ is continuous, and suppose that there exist mappings $l, r : X \times X \to [0, 1)$ such that the following hold:*

(a) *$l(F(x, y), F(u, v)) \le l(x, y)$ and $r(F(x, y), F(u, v)) \le r(x, y)$ for all $x, y, u, v \in X$,*

(b) *$(l + r)(x, y) < 1$ for all $x, y \in X$, and*

(c) *$q(F(x, y), F(u, v)) \preceq l(x, y)q(x, F(x, y)) + r(x, y)q(u, F(u, v))$ for all $x, y, u, v \in X$.*

Then F has a coupled fixed point $(x^*, y^*) \in X \times X$. Further, if $x_1 = F(x_1, y_1)$ and $y_1 = F(y_1, x_1)$, then $q(x_1, x_1) = \theta$ and $q(y_1, y_1) = \theta$. Moreover, the coupled fixed point is unique and is of the form (x^*, x^*) for some $x^* \in X$.

Corollary 3.6. *Let (X, d) be a complete cone metric space, and q is a c-distance on X. Suppose that the mapping $F : X \times X \to X$ is continuous and satisfies the following contractive condition:*

$$q(F(x, y), F(u, v)) \preceq kq(x, u) + lq(x, F(x, y)) + rq(u, F(u, v)), \tag{3.26}$$

for all $x, y, u, v \in X$, where k, l, r are nonnegative constants with $k + l + r < 1$. Then F has a coupled fixed point $(x^*, y^*) \in X \times X$. Further, if $x_1 = F(x_1, y_1)$ and $y_1 = F(y_1, x_1)$, then $q(x_1, x_1) = \theta$ and $q(y_1, y_1) = \theta$. Moreover, the coupled fixed point is unique and is of the form (x^*, x^*) for some $x^* \in X$.

Corollary 3.7. *Let (X, d) be a complete cone metric space, and q is a c-distance on X. Suppose that the mapping $F : X \times X \to X$ is continuous and satisfies the following contractive condition for all:*

$$q(F(x, y), F(u, v)) \preceq lq(x, F(x, y)) + rq(u, F(u, v)), \tag{3.27}$$

for all $x, y, u, v \in X$, where l, r are nonnegative constants with $l + r < 1$. Then F has a coupled fixed point $(x^*, y^*) \in X \times X$. Further, if $x_1 = F(x_1, y_1)$ and $y_1 = F(y_1, x_1)$ then $q(x_1, x_1) = \theta$ and $q(y_1, y_1) = \theta$. Moreover, the coupled fixed point is unique and is of the form (x^*, x^*) for some $x^* \in X$.

Corollary 3.8. *Let (X, d) be a complete cone metric space, and q is a c-distance on X. Suppose that the mapping $F : X \times X \to X$ is continuous and satisfies the following contractive condition for all:*

$$q(F(x,y), F(u,v)) \preceq l\left[q(x, F(x,y)) + q(u, F(u,v))\right], \tag{3.28}$$

for all $x, y, u, v \in X$, where $l \in [0, 1/2)$ is a constant. Then F has a coupled fixed point $(x^*, y^*) \in X \times X$. Further, if $x_1 = F(x_1, y_1)$ and $y_1 = F(y_1, x_1)$, then $q(x_1, x_1) = \theta$ and $q(y_1, y_1) = \theta$. Moreover, the coupled fixed point is unique and is of the form (x^*, x^*) for some $x^* \in X$.

Finally, we provide another result without condition (b) in Theorem 3.1, and we do not require that F is continuous.

Theorem 3.9. *Let (X, d) be a complete cone metric space, and q is a c-distance on X. Let $F : X \times X \to X$ be a mapping, and suppose that there exists mappings $k, l, r : X \times X \to [0, 1)$ such that the following hold:*

(a) $k(F(x,y), F(u,v)) \leq k(x,y)$, $l(F(x,y), F(u,v)) \leq l(x,y)$ *and* $r(F(x,y), F(u,v)) \leq r(x,y)$ *for all* $x, y, u, v \in X$,

(b) $(k + 2l + r)(x,y) < 1$ *for all* $x, y \in X$, *and*

(c) $(1 - r(x,y))q(F(x,y), F(u,v)) \preceq k(x,y)q(x, F(x,y)) + l(x,y)q(x, F(u,v))$ *for all* $x, y, u, v \in X$.

Then F has a coupled fixed point $(x^*, y^*) \in X \times X$. Further, if $x_1 = F(x_1, y_1)$ and $y_1 = F(y_1, x_1)$, then $q(x_1, x_1) = \theta$ and $q(y_1, y_1) = \theta$. Moreover, the coupled fixed point is unique and is of the form (x^*, x^*) for some $x^* \in X$.

Proof. Choose $x_0, y_0 \in X$. Set $x_1 = F(x_0, y_0), y_1 = F(y_0, x_0), x_2 = F(x_1, y_1), y_2 = F(y_1, x_1), \ldots, x_{n+1} = F(x_n, y_n), y_{n+1} = F(y_n, x_n)$. Observe that

$$(1 - r(x,y))q(F(x,y), F(u,v)) \preceq k(x,y)q(x, F(x,y)) + l(x,y)q(x, F(u,v)), \tag{3.29}$$

equivalently

$$q(F(x,y), F(u,v)) \preceq k(x,y)q(x, F(x,y)) + l(x,y)q(x, F(u,v)) + r(x,y)q(F(x,y), F(u,v)). \tag{3.30}$$

Then we have the following:

$$
\begin{aligned}
q(x_n, x_{n+1}) &= q(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\
&\le k(x_{n-1}, y_{n-1}) q(x_{n-1}, F(x_{n-1}, y_{n-1})) \\
&\quad + l(x_{n-1}, y_{n-1}) q(x_{n-1}, F(x_n, y_n)) \\
&\quad + r(x_{n-1}, y_{n-1}) q(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\
&= k(F(x_{n-2}, y_{n-2}), F(y_{n-2}, x_{n-2})) q(x_{n-1}, x_n) \\
&\quad + l(F(x_{n-2}, y_{n-2}), F(y_{n-2}, x_{n-2})) q(x_{n-1}, x_{n+1}) \\
&\quad + r(F(x_{n-2}, y_{n-2}), F(y_{n-2}, x_{n-2})) q(x_n, x_{n+1}) \\
&\le k(x_{n-2}, y_{n-2}) q(x_{n-1}, x_n) + l(x_{n-2}, y_{n-2}) q(x_{n-1}, x_{n+1}) \\
&\quad + r(x_{n-2}, y_{n-2}) q(x_n, x_{n+1}) \\
&\;\vdots \\
&\le k(x_0, y_0) q(x_{n-1}, x_n) + l(x_0, y_0) q(x_{n-1}, x_{n+1}) \\
&\quad + r(x_0, y_0) q(x_n, x_{n+1}) \\
&\le k(x_0, y_0) q(x_{n-1}, x_n) + l(x_0, y_0) q(x_{n-1}, x_n) \\
&\quad + l(x_0, y_0) q(x_n, x_{n+1}) + r(x_0, y_0) q(x_n, x_{n+1}).
\end{aligned}
\tag{3.31}
$$

Then, we have

$$
\begin{aligned}
q(x_n, x_{n+1}) &\le \frac{k(x_0, y_0) + l(x_0, y_0)}{1 - l(x_0, y_0) - r(x_0, y_0)} q(x_{n-1}, x_n) \\
&= h q(x_{n-1}, x_n) \\
&\le h^2 q(x_{n-2}, x_{n-1}) \\
&\;\vdots \\
&\le h^n q(x_0, x_1),
\end{aligned}
\tag{3.32}
$$

where $h = (k(x_0, y_o) + l(x_0, y_o))/(1 - l(x_0, y_o) - r(x_0, y_o)) < 1$.

Similarly we have

$$\begin{aligned}
q(y_n, y_{n+1}) &\preceq \frac{k(y_0, x_0) + l(y_0, x_0)}{1 - l(y_0, x_0) - r(y_0, x_0)} q(y_{n-1}, y_n) \\
&= d q(y_{n-1}, y_n) \\
&\preceq d^2 q(y_{n-2}, y_{n-1}) \\
&\vdots \\
&\preceq d^n q(y_0, y_1),
\end{aligned} \tag{3.33}$$

where $d = (k(y_0, x_0) + l(y_0, x_0))/(1 - l(y_0, x_0) - r(y_0, x_0)) < 1$.

Let $m > n \geq 1$. Then it follows that

$$\begin{aligned}
q(x_n, x_m) &\preceq q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + \cdots + q(x_{m-1}, x_m) \\
&\preceq \left(h^n + h^{n+1} + \cdots + h^{m-1}\right) q(x_0, x_1) \\
&\preceq \frac{h^n}{1-h} q(x_0, x_1)
\end{aligned} \tag{3.34}$$

and also

$$\begin{aligned}
q(y_n, y_m) &\preceq q(y_n, y_{n+1}) + q(y_{n+1}, y_{n+2}) + \cdots + q(y_{m-1}, y_m) \\
&\preceq \left(d^n + d^{n+1} + \cdots + d^{m-1}\right) q(y_0, y_1) \\
&\preceq \frac{d^n}{1-d} q(y_0, y_1).
\end{aligned} \tag{3.35}$$

Thus, Lemma 2.6 (3) shows that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. Since X is complete, there exists x^* and $y^* \in$ X such that $x_n \to x^*$ and $y_n \to y^*$ as $n \to \infty$. By (q3) we have

$$q(x_n, x^*) \preceq \frac{h^n}{1-h} q(x_0, x_1), \tag{3.36}$$

$$q(y_n, y^*) \preceq \frac{d^n}{1-d} q(y_0, y_1). \tag{3.37}$$

On the other hand, we have

$$
\begin{aligned}
q(x_n, F(x^*, y^*)) &= q(F(x_{n-1}, y_{n-1}), F(x^*, y^*)) \\
&\preceq k(x_{n-1}, y_{n-1}) q(x_{n-1}, F(x_{n-1}, y_{n-1})) + l(x_{n-1}, y_{n-1}) q(x_{n-1}, F(x^*, y^*)) \\
&\quad + r(x_{n-1}, y_{n-1}) q(F(x_{n-1}, y_{n-1}), F(x^*, y^*)) \\
&= k(F(x_{n-2}, y_{n-2}), F(y_{n-2}, x_{n-2})) q(x_{n-1}, x_n) \\
&\quad + l(F(x_{n-2}, y_{n-2}), F(y_{n-2}, x_{n-2})) q(x_{n-1}, F(x^*, y^*)) \\
&\quad + r(F(x_{n-2}, y_{n-2}), F(y_{n-2}, x_{n-2})) q(x_n, F(x^*, y^*)) \\
&\preceq k(x_{n-2}, y_{n-2}) q(x_{n-1}, x_n) + l(x_{n-2}, y_{n-2}) q(x_{n-1}, F(x^*, y^*)) \\
&\quad + r(x_{n-2}, y_{n-2}) q(x_n, F(x^*, y^*)) \\
&\ \vdots \\
&\preceq k(x_0, y_0) q(x_{n-1}, x_n) + l(x_0, y_0) q(x_{n-1}, F(x^*, y^*)) \\
&\quad + r(x_0, y_0) q(x_n, F(x^*, y^*)) \\
&\preceq k(x_0, y_0) q(x_{n-1}, x_n) + l(x_0, y_0) q(x_{n-1}, x_n) \\
&\quad + l(x_0, y_0) q(x_n, F(x^*, y^*)) + r(x_0, y_0) q(x_n, F(x^*, y^*)).
\end{aligned}
\tag{3.38}
$$

Then, we have

$$
\begin{aligned}
q(x_n, F(x^*, y^*)) &\preceq \frac{k(x_0, y_0) + l(x_0, y_0)}{1 - l(x_0, y_0) - r(x_0, y_0)} q(x_{n-1}, x_n) \\
&= h q(x_{n-1}, x_n) \\
&\preceq h h^{n-1} q(x_0, x_1) \\
&= h^n q(x_0, x_1) \\
&\preceq \frac{h^n}{1 - h} q(x_0, x_1).
\end{aligned}
\tag{3.39}
$$

By Lemma 2.6 (1), (3.36), and (3.39), we have $x^* = F(x^*, y^*)$.

By similar way we have

$$
q(y_n, F(y^*, x^*)) \preceq \frac{d^n}{1 - d} q(y_0, y_1).
\tag{3.40}
$$

By Lemma 2.6 (1), (3.37), and (3.40), we have $y^* = F(y^*, x^*)$. Therefore, (x^*, y^*) is a coupled fixed point of F.

Suppose that $x_1 = F(x_1, y_1)$ and $y_1 = F(y_1, x_1)$, then we have

$$\begin{aligned}
q(x_1, x_1) &= q(F(x_1, y_1), F(x_1, y_1)) \\
&\preceq k(x_1, y_1) q(x_1, F(x_1, y_1)) + l(x_1, y_1) q(x_1, F(x_1, y_1)) \\
&\quad + r(x_1, y_1) q(F(x_1, y_1), F(x_1, y_1)) \\
&= k(x_1, y_1) q(x_1, x_1) + l(x_1, y_1) q(x_1, x_1) + r(x_1, y_1) q(x_1, x_1) \\
&\preceq k(x_1, y_1) q(x_1, x_1) + l(x_1, y_1) q(x_1, x_1) + l(x_1, \mathrm{y}_1) q(x_1, x_1) \\
&\quad + r(x_1, y_1) q(x_1, x_1) \\
&= (k + 2l + r)(x_1, y_1) q(x_1, x_1).
\end{aligned} \tag{3.41}$$

Since $(k+2l+r)(x_1, y_1) < 1$, Lemma 2.3 (1) shows that $q(x_1, x_1) = \theta$. By similar way, $q(y_1, y_1) = \theta$.

Finally, suppose that there is another coupled fixed point (x', y'), then we have

$$\begin{aligned}
q(x^*, x') &= q(F(x^*, y^*), F(x', y')) \\
&\preceq k(x^*, y^*) q(x^*, F(x^*, y^*)) + l(x^*, y^*) q(x^*, F(x', y')) \\
&\quad + r(x^*, y^*) q(F(x^*, y^*), F(x', y')) \\
&= k(x^*, y^*) q(x^*, x^*) + l(x^*, y^*) q(x^*, x') + r(x^*, y^*) q(x^*, x') \\
&= l(x^*, y^*) q(x^*, x') + r(x^*, y^*) q(x^*, x') \\
&\preceq k(x^*, y^*) q(x^*, x') + l(x^*, y^*) q(x^*, x') + l(x^*, y^*) q(x^*, x') \\
&\quad + r(x^*, y^*) q(x^*, x') \\
&= (k + 2l + r)(x^*, y^*) q(x^*, x').
\end{aligned} \tag{3.42}$$

Since $(k+2l+r)(x^*, y^*) < 1$, Lemma 2.3 (1) shows that $q(x^*, x') = \theta$. Also we have $q(x^*, x^*) = \theta$. Hence Lemma 2.6 (1) shows that $x^* = x'$. By similar way we have $y^* = y'$ which implies that $(x^*, y^*) = (x', y')$. Similarly, we prove that $x^* = y'$ and $y^* = x'$. Hence, $x^* = y^*$. Therefore the coupled fixed point is unique and is of the form (x^*, x^*) for some $x^* \in X$. □

From the above theorem, we have the following corollaries.

Corollary 3.10. *Let (X, d) be a complete cone metric space, and q is a c -distance on X. Let $F : X \times X \to X$ be a mapping, and suppose that there exists mappings $k, l : X \times X \to [0, 1)$ such that the following hold:*

(a) *$k(F(x, y), F(u, v)) \leq k(x, y)$ and $l(F(x, y), F(u, v)) \leq l(x, y)$ for all $x, y, u, v \in X$,*

(b) *$(k + 2l)(x, y) < 1$ for all $x, y \in X$, and*

(c) *$q(F(x, y), F(u, v)) \preceq k(x, y) q(x, F(x, y)) + l(x, y) q(x, F(u, v))$ for all $x, y, u, v \in X$.*

Then F has a coupled fixed point $(x^*, y^*) \in X \times X$. Further, if $x_1 = F(x_1, y_1)$ and $y_1 = F(y_1, x_1)$, then $q(x_1, x_1) = \theta$ and $q(y_1, y_1) = \theta$. Moreover, the coupled fixed point is unique and is of the form (x^*, x^*) for some $x^* \in X$.

Corollary 3.11. *Let (X, d) be a complete cone metric space, and q is a c-distance on X. Suppose that the mapping $F : X \times X \to X$ satisfies the following contractive condition:*

$$(1 - r)q(F(x, y), F(u, v)) \preceq kq(x, F(x, y)) + lq(x, F(u, v)), \tag{3.43}$$

for all $x, y, u, v \in X$, where k, l, r are nonnegative constants with $k + 2l + r < 1$. Then F has a coupled fixed point $(x^*, y^*) \in X \times X$. Further, if $x_1 = F(x_1, y_1)$ and $y_1 = F(y_1, x_1)$, then $q(x_1, x_1) = \theta$ and $q(y_1, y_1) = \theta$. Moreover, the coupled fixed point is unique and is of the form (x^*, x^*) for some $x^* \in X$.

Corollary 3.12. *Let (X, d) be a complete cone metric space, and q is a c-distance on X. Suppose that the mapping $F : X \times X \to X$ satisfies the following contractive condition:*

$$q(F(x, y), F(u, v)) \preceq kq(x, F(x, y)) + lq(x, F(u, v)), \tag{3.44}$$

for all $x, y, u, v \in X$, where k, l are nonnegative constants with $k + 2l < 1$. Then F has a coupled fixed point $(x^*, y^*) \in X \times X$. Further, if $x_1 = F(x_1, y_1)$ and $y_1 = F(y_1, x_1)$, then $q(x_1, x_1) = \theta$ and $q(y_1, y_1) = \theta$. Moreover, the coupled fixed point is unique and is of the form (x^*, x^*) for some $x^* \in X$.

Corollary 3.13. *Let (X, d) be a complete cone metric space, and q is a c-distance on X. Suppose that the mapping $F : X \times X \to X$ satisfies the following contractive condition:*

$$q(F(x, y), F(u, v)) \preceq k[q(x, F(x, y)) + q(x, F(u, v))], \tag{3.45}$$

for all $x, y, u, v \in X$, where $k \in [0, 1/3)$ is a constant. Then F has a coupled fixed point $(x^*, y^*) \in X \times X$. Further, if $x_1 = F(x_1, y_1)$ and $y_1 = F(y_1, x_1)$, then $q(x_1, x_1) = \theta$ and $q(y_1, y_1) = \theta$. Moreover, the coupled fixed point is unique and is of the form (x^*, x^*) for some $x^* \in X$.

Example 3.14. Consider Example 2.5. Define the mapping $F : X \times X \to X$ by $F(x, y) = (x + y)/4$ for all $(x, y) \in X \times X$. Then we have, $q(F(x, y), F(u, v)) = F(u, v) = (u + v)/4 = u/4 + v/4 \preceq (3/7)u + (3/7)v = kq(x, u) + lq(y, v)$ with $k = l = 3/7$, and $k + l = 6/7 < 1$. Therefore, the conditions of Theorem 3.1 are satisfied, and then F has a unique coupled fixed point $(x, y) = (0, 0)$ and $F(0, 0) = 0$ with $q(0, 0) = 0$.

Example 3.15. Consider Example 2.5. Define the mapping $F : X \times X \to X$ by $F(7/8, 7/8) = 1/4$ and $F(x, y) = (x + y)/4$ for all $(x, y) \in X \times X$ with $(x, y) \neq (7/8, 7/8)$. Since $d(F(1, 1), F(7/8, 7/8)) = d(1, 7/8) + d(1, 7/8)$, there is not $k, l \in [0, 1)$ such that $d(F(x, y), F(u, v)) \preceq kd(x, u) + ld(y, v)$ for all x, y, u, v in X. Since Theorem 1.2 of Sabetghadam et al. [15, Theorem 2.2] cannot applied to this example on cone metric space. To check this example on c-distance, we have

(1) If $(x, y) = (u, v) = (7/8, 7/8)$, then we have $q(F(7/8, 7/8), F(7/8, 7/8)) = F(7/8, 7/8) = 1/4 \preceq k(7/8) + l(7/8) = kq(7/8, 7/8) + lq(7/8, 7/8)$ with $k = l = 3/7$, and $k + l = 6/7 < 1$.

(2) If $(x, y) \neq (u, v) \neq (7/8, 7/8)$, then we have $q(F(x, y), F(u, v)) = \mathrm{F}(u, v) = (u+v)/4 = u/4+v/4 \preceq (3/7)u+(3/7)v = kq(x, u)+lq(y, v)$ with $k = l = 3/7$, and $k+l = 6/7 < 1$.

(3) If $(x, y) = (7/8, 7/8), (u, v) \neq (7/8, 7/8)$, then we have $q(F(7/8, 7/8), F(u, v)) = F(u, v) = (u + v)/4 = u/4 + v/4 \preceq (3/7)u + (3/7)v = kq(7/8, \mathrm{u}) + lq(7/8, v)$ with $k = l = 3/7$, and $k + l = 6/7 < 1$.

(4) If $(x, y) \neq (7/8, 7/8), (u, v) = (7/8, 7/8)$, then we have $q(F(x, y), F(7/8, 7/8)) = F(7/8, 7/8) = 1/4 \preceq k(7/8) + l(7/8) = kq(x, 7/8) + lq(y, 7/8)$ with $k = l = 3/7$, and $k + l = 6/7 < 1$.

Hence $q(F(x, y), F(u, v)) \preceq kq(x, u) + lq(y, v)$ for all x, y, u, v with $k = l = 3/7$, and $k + l = 6/7 < 1$. Therefore, the conditions of Theorem 3.1 are satisfied, and then F has a unique coupled fixed point $(x, y) = (0, 0)$ and $F(0, 0) = 0$ with $q(0, 0) = 0$.

Remark 3.16. In Example 3.14, it is easy to see that $d(F(x, y), F(u, v)) \preceq kd(x, u) + ld(y, v)$ for all x, y, u, v in X with $k = l = 1/4$. Therefore, the condition of Theorem 1.2 of Sabetghadam et al. [15, Theorem 2.2] is satisfied, and then F has a unique coupled fixed point $(x, y) = (0, 0)$, $F(0, 0) = 0$ with $q(0, 0) = 0$.

Example 3.17. Let $E = \mathbb{R}^2$ and $P = \{(x, y) \in E : x, y \geq 0\}$. Let $X = [0, 1)$ and define a mapping $d : X \times X \to E$ by $d(x, y) = (|x - y|, |x - y|)$ for all $x, y \in X$. Then (X, d) is a complete cone metric space, see [15]. Define a mapping $q : X \times X \to E$ by $q(x, y) = (y, y)$ for all $x, y \in X$. Then q is a c-distance on X. In fact (q1)–(q3) are immediate. Let $c \in E$ with $\theta \ll c$ and put $e = c/2$. If $q(z, x) \ll e$ and $q(z, y) \ll e$, then we have

$$\begin{aligned} d(x, y) &= (|x - y|, |x - y|) \\ &\preceq (x + y, x + y) \\ &= (x, x) + (y, y) \\ &= q(z, x) + q(z, y) \\ &\ll e + e \\ &= c. \end{aligned} \tag{3.46}$$

This shows that (q4) holds. Therefore, q is a c-distance on X. Define the mapping $F : X \times X \to X$ by $F(x, y) = (x + y)/4$ for all $(x, y) \in X \times X$. Then we have, $q(F(x, y), F(u, v)) = (F(u, v), F(u, v)) = ((u+v)/4, (u+v)/4) = (u/4, u/4)+(v/4, v/4) = (1/4)(u, u)+(1/4)(v, v) \preceq (3/7)(u, u) + (3/7)(v, v) = kq(x, u) + lq(y, v)$ with $k = l = 3/7$, and $k + l = 6/7 < 1$. Therefore, the conditions of Theorem 3.1 are satisfied, and then F has a unique coupled fixed point $(x, y) = (0, 0)$ and $F(0, 0) = 0$ with $q(0, 0) = 0$.

Acknowledgments

The authors would like to acknowledge the financial support received from Universiti Kebangsaan Malaysia under the research Grant no. OUP-UKM-FST-2012. The authors thank the referee for his/her careful reading of the manuscript and useful suggestions.

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