Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 295209, 12 pages
doi:10.1155/2012/295209

Research Article

Multiple Positive Solutions for Singular Semipositone Periodic Boundary Value Problems with Derivative Dependence

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Received 29 March 2012; Accepted 19 May 2012

Academic Editor: Yansheng Liu

By constructing a special cone in $C^1[0,2\pi]$ and the fixed point theorem, this paper investigates second-order singular semipositone periodic boundary value problems with dependence on the first-order derivative and obtains the existence of multiple positive solutions. Further, an example is given to demonstrate the applications of our main results.

1. Introduction

In this paper, we are concerned with the existence of multiple positive solutions for the second-order singular semipositone periodic boundary value problems (PBVP, for short):

$$\begin{gathered} u''(t) + a(t)u(t) = f\left(t, u(t), u'(t)\right), \quad t \in (0, 2\pi), \\ u(0) = u(2\pi), \qquad u'(0) = u'(2\pi), \end{gathered} \tag{1.1}$$

where $a \in C[0, 2\pi]$, the nonlinear term $f(t, u, v)$ may be singular at $t = 0$, $t = 2\pi$, and $u = 0$, also may be negative for some value of t, u, and v.

In recent years, second-order singular periodic boundary value problems have been studied extensively because they can be used to model many systems in celestial mechanics such as the N-body problem (see [1–11] and references therein). By applying the

Krasnosel'skii's fixed point theorem, Jiang [5] proves the existence of one positive solution for the second-order PBVP

$$u''(t) + m^2 u = f(t,u), \quad t \in [0,2\pi],$$
$$u(0) = u(2\pi), \qquad u'(0) = u'(2\pi), \tag{1.2}$$

where $m \in (0,1/2)$ is a constant and $f \in C([0,2\pi] \times [0,+\infty),[0,+\infty))$. Zhang and Wang [6] used the same fixed point theorem to prove the existence of multiple positive solutions for PBVP (1.2) when $f(t,u)$ is nonnegative and singular at $u = 0$, not singular at $t = 0$, $t = 2\pi$. Lin et al. [7] only obtained the existence of one positive solution to PBVP (1.1) when $f(t,u,v) = f(t,u)$, f is semipositone and singular only at $u = 0$. All the above works were done under the assumption that the first-order derivative u' is not involved explicitly in the nonlinear term f.

Motivated by the works of [5–7], the present paper investigates the existence of multiple positive solutions to PBVP (1.1). PBVP (1.1) has two special features. The first one is that the nonlinearity f may depend on the first-order derivative of the unknown function u, and the second one is that the nonlinearity $f(t,u,v)$ is semipositone and singular at $t = 0$, $t = 2\pi$, and $u = 0$. We first construct a special cone different from that in [5–7] and then deduce the existence of multiple positive solutions by employing the fixed point theorem on a cone. Our results improve and generalize some related results obtained in [5–7].

A map $u \in C^1[0,2\pi] \cap C^2(0,2\pi)$ is said to be a positive solution to PBVP(1.1) if and only if u satisfies PBVP (1.1) and $u(t) > 0$ for $t \in [0,2\pi]$.

The contents of this paper are distributed as follows. In Section 2, we introduce some lemmas and construct a special cone, which will be used in Section 3. We state and prove the existence of at least two positive solutions to PBVP (1.1) in Section 3. Finally, an example is worked out to demonstrate our main results.

2. Some Preliminaries and Lemmas

Define the set functions

$$\Lambda = \left\{ a \in C[0,2\pi] : a \succ 0,\ t \in [0,2\pi],\ \left(\int_0^{2\pi} a^p \, dt \right)^{1/p} \le K(2q) \text{ for some } p \ge 1 \right\}, \tag{2.1}$$

where q is the conjugate exponent of p,

$$K(q) = \begin{cases} \dfrac{1}{q(2\pi)^{2/q}} \left(\dfrac{2}{2+q} \right)^{1-2/q} \left(\dfrac{\Gamma(1/q)}{\Gamma(1/2+1/q)} \right)^2, & 1 \le q < \infty, \\ \dfrac{2}{\pi}, & q = \infty, \end{cases} \tag{2.2}$$

where Γ is the Gamma function.

Given $a \in \Lambda$, let $G(t,s)$ be the Green function for the equation

$$u'' + a(t)u(t) = 0, \quad t \in (0, 2\pi),$$
$$u(0) = u(2\pi), \qquad u'(0) = u'(2\pi). \tag{2.3}$$

Now, the following Lemma follows immediately from the paper [7].

Lemma 2.1. *$G(t,s)$ has the following properties:*

(G^1) *$G(t,s)$ is continuous in t and s for all $t, s \in [0, 2\pi]$;*

(G^2) *$G(t,s) > 0$ for all $(t,s) \in [0,2\pi] \times [0,2\pi]$, $G(0,s) = G(2\pi,s)$ and $\partial G/\partial t|_{(0,s)} = \partial G/\partial t|_{(2\pi,s)}$;*

(G^3) *denote $l_1 = \min_{0 \le t,s \le 2\pi} G(t,s)$ and $l_2 = \max_{0 \le t,s \le 2\pi} G(t,s)$, then $l_2 > l_1 > 0$;*

(G^4) *there exist functions $h, H \in C^2[0,2\pi]$ such that*

$$G(t,s) = \begin{cases} (\alpha+1)H(t)h(s) + (\beta-1)h(t)H(s) + cH(t)H(s) + dh(t)h(s), & 0 \le s \le t \le 2\pi, \\ \alpha H(t)h(s) + \beta h(t)H(s) + cH(t)H(s) + dh(t)h(s), & 0 \le t \le s \le 2\pi, \end{cases} \tag{2.4}$$

where α, β, c, d are constants, H, h are independent solutions of the linear differential equation $u'' + a(t)u(t) = 0$, and $H'(t)h(t) - h'(t)H(t) = 1$;

(G^5) *$G'_t(t,s)$ is bounded on $[0,2\pi] \times [0,2\pi]$.*

Denote $l_3 = \max_{0 \le t,s \le 2\pi} |G'_t(t,s)|$, then $l_3 > 0$.

Remark 2.2. Using paper [5], we can get $G(t,s)$ when $a(t) \equiv m^2$ and $m \in (0, 1/2)$, obtaining

$$G(t,s) = \begin{cases} \dfrac{\sin m(t-s) + \sin m(2\pi - t + s)}{2m(1-\cos 2m\pi)}, & 0 \le s \le t \le 2\pi, \\ \dfrac{\sin m(s-t) + \sin m(2\pi - s + t)}{2m(1-\cos 2m\pi)}, & 0 \le t \le s \le 2\pi, \end{cases}$$
$$G'_t(t,s) = \begin{cases} \dfrac{\cos m(t-s) - \cos m(2\pi - t + s)}{2(1-\cos 2m\pi)}, & 0 \le s \le t \le 2\pi, \\ \dfrac{-\cos m(s-t) + \cos m(2\pi - s + t)}{2(1-\cos 2m\pi)}, & 0 \le t < s \le 2\pi, \end{cases} \tag{2.5}$$
$$l_1 = \frac{\sin 2m\pi}{2m(1-\cos 2m\pi)}, \qquad l_2 = \frac{\sin m\pi}{m(1-\cos 2m\pi)}, \qquad l_3 = \frac{1}{2}.$$

Let $E = \{u \in C^1[0,2\pi] : u(0) = u(2\pi), u'(0) = u'(2\pi)\}$ with norm $\|u\| = \max\{\|u\|_0, \|u'\|_0\}$, where $\|u\|_0 = \max_{t \in [0,2\pi]} |u(t)|$. Then $(E, \|\cdot\|)$ is a Banach space. Let

$\sigma =: \min\{l_1/l_2, l_1/l_3\}$, $L =: l_3/l_1$, from Lemma 2.1, we know that σ, L are both constants and $0 < \sigma < 1$, $L > 0$.

Define

$$K = \{u \in E : \quad u(t) \geq \sigma\|u\|, \ |u'(t)| \leq L\|u\|, \ \forall t \in [0, 2\pi]\},$$
$$\Omega_r = \{u \in E : \quad \|u\| < r\}, \quad \forall r > 0. \tag{2.6}$$

It is easy to conclude that K is a cone of E and Ω_r is an open set of E.

Lemma 2.3 (see [12]). *Let E be a Banach space and P a cone in E. Suppose Ω_1 and Ω_2 are bounded open sets of E such that $\theta \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$ and suppose that $A : P \cap (\overline{\Omega_2} \setminus \Omega_1) \rightarrow P$ is a completely continuous operator such that*

(1) $\inf_{u \in P \cap \partial\Omega_1} \|Au\| > 0$ *and* $u \neq \lambda Au$ *for* $u \in P \cap \partial\Omega_1$, $\lambda \geq 1$; $u \neq \lambda Au$ *for* $u \in P \cap \partial\Omega_2$, $0 < \lambda \leq 1$, *or*

(2) $\inf_{u \in P \cap \partial\Omega_2} \|Au\| > 0$ *and* $u \neq \lambda Au$ *for* $u \in P \cap \partial\Omega_2$, $\lambda \geq 1$; $u \neq \lambda Au$ *for* $u \in P \cap \partial\Omega_1$, $0 < \lambda \leq 1$.

Then A has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

For convenience, let us list some conditions for later use.

(H_0) $a(t) \in \Lambda$, $f : (0, 2\pi) \times (0, +\infty) \times R \rightarrow R$ *is continuous and there exists a constant $M > 0$ such that*

$$0 \leq f(t, u, v) + M \leq g(t)h(u, v), \quad \forall (t, u, v) \in (0, 2\pi) \times (0, +\infty) \times R, \tag{2.7}$$

where $g \in C((0, 2\pi), R^+)$, $h \in C((0, +\infty) \times R, R^+)$, *and* $0 < \int_0^{2\pi} g(t)dt < +\infty$;

(H_1) *there exist* $r_1 > \sigma^{-1} 2\pi M l_2$ *and* $a(t) \in L[0, 2\pi]$ *with* $\int_0^{2\pi} a(t)dt > (\geq) r_1 l_1^{-1}$ *such that*

$$M + f(t, u, v) \geq (>) a(t), \quad \forall t \in (0, 2\pi), \ u \in (0, r_1], \ v \in [-(Lr_1 + 2\pi M l_3), (Lr_1 + 2\pi M l_3)]; \tag{2.8}$$

(H_2) *there exists* $R_1 > r_1$ *such that*

$$\max\{l_2, l_3\} \int_0^{2\pi} g(t)dt < R_1 M_0^{-1}, \tag{2.9}$$

where $M_0 =: \max\{h(u, v) : u \in [\sigma R_1 - 2\pi M l_2, R_1], \ v \in [-(LR_1 + 2\pi M l_3), (LR_1 + 2\pi M l_3)]\}$;

(H_3) *there exists* $[\alpha^*, \beta^*] \subset (0, 2\pi)$ *such that*

$$\lim_{u \to +\infty} \frac{f(t, u, v)}{u} = +\infty \quad \text{uniformly with respect to } t \in [\alpha^*, \beta^*], \ v \in R. \tag{2.10}$$

3. Main Results

Theorem 3.1. *Assume that conditions* (H_0)–(H_3) *are satisfied, then PBVP* (1.1) *has at least two positive solutions* $u_1, u_2 \in C^1[0,2\pi] \cap C^2(0,2\pi)$ *such that* $r_1 < \|u_1 + M\omega\| < R_1 < \|u_2 + M\omega\|$, *where* $\omega(t) =: \int_0^{2\pi} G(t,s)ds$.

Proof. We consider the following PBVP:

$$\begin{gathered} u''(t) + a(t)u(t) = f\left(t, u(t) - M\omega(t), u'(t) - M\omega'(t)\right) + M, \quad t \in (0,2\pi), \\ u(0) = u(2\pi), \qquad u'(0) = u'(2\pi). \end{gathered} \tag{3.1}$$

It is easy to see that if $u \in C^1[0,2\pi] \cap C^2(0,2\pi)$ and $r_1 < \|u\| < R_1$ is a positive solution of PBVP (3.1) with $u(t) > M\omega(t)$ for $t \in [0,2\pi]$, then $x(t) = u(t) - M\omega(t)$ is a positive solution of PBVP (1.1) and $r_1 < \|x + M\omega\| < R_1$.

As a result, we will only concentrate our study on PBVP (3.1).

Define an operator $T : K \setminus \{\theta\} \to E$ by

$$(Tu)(t) =: \int_0^{2\pi} G(t,s)\left[f\left(s, u(s) - M\omega(s), u'(s) - M\omega'(s)\right) + M\right]ds, \quad \forall t \in [0,2\pi], \tag{3.2}$$

where $G(t,s)$ is the Green function to problem (2.3).

(1) We first show that $T : K \cap (\overline{\Omega_R} \setminus \Omega_{r_1}) \to K$ is completely continuous for any $R > r_1$.

For any $u \in K \cap (\overline{\Omega_R} \setminus \Omega_{r_1})$, from ($H_1$), we have $u(t) - M\omega(t) \geq \sigma r_1 - 2\pi M l_2 > 0$. So, by Lemma 2.1 and (3.2),

$$(Tu)(0) = (Tu)(2\pi), \quad (Tu)'(0) = (Tu)'(2\pi), \tag{3.3}$$

$$\begin{aligned} (Tu)(t) &= \int_0^{2\pi} G(t,s)\left[f\left(s, u(s) - M\omega(s), u'(s) - M\omega'(s)\right) + M\right]ds \\ &\geq \frac{l_1}{l_2} l_2 \int_0^{2\pi} G(t,s)\left[f\left(s, u(s) - M\omega(s), u'(s) - M\omega'(s)\right) + M\right]ds \\ &\geq \frac{l_1}{l_2} \max_{\tau \in [0,2\pi]} \int_0^{2\pi} G(\tau,s)\left[f\left(s, u(s) - M\omega(s), u'(s) - M\omega'(s)\right) + M\right]ds \\ &= \frac{l_1}{l_2}\|Tu\|_0 \geq \sigma \|Tu\|_0, \quad \forall t \in [0,2\pi], \end{aligned} \tag{3.4}$$

$$\begin{aligned}
\left|(Tu)'(t)\right| &= \left|\int_0^{2\pi} G_t'(t,s)\left[f\left(s,u(s)-M\omega(s),u'(s)-M\omega'(s)\right)+M\right]ds\right| \\
&\le \int_0^{2\pi} \left|G_t'(t,s)\right|\left[f\left(s,u(s)-M\omega(s),u'(s)-M\omega'(s)\right)+M\right]ds \\
&\le \frac{l_3}{l_1}l_1\int_0^{2\pi}\left[f\left(s,u(s)-M\omega(s),u'(s)-M\omega'(s)\right)+M\right]ds \\
&\le \frac{l_3}{l_1}\int_0^{2\pi} G(\tau,s)\left[f\left(s,u(s)-M\omega(s),u'(s)-M\omega'(s)\right)+M\right]ds \\
&= \frac{l_3}{l_1}(Tu)(\tau), \quad \forall t,\tau\in[0,2\pi].
\end{aligned} \tag{3.5}$$

From (3.5), we have $(Tu)(t) \ge (l_1/l_3)\max_{\tau\in[0,2\pi]}|(Tu)'(\tau)| \ge \sigma\|(Tu)'\|_0$. Therefore, $(Tu)(t)\ge\sigma\|Tu\|$, $|(Tu)'(t)|\le L\|Tu\|$, *for all* $t\in[0,2\pi]$, *that is,* $T: K\cap(\overline{\Omega_R}\setminus\Omega_{r_1}) \to K$.

Assume that $u_n, u_* \in K\cap(\overline{\Omega_R}\setminus\Omega_{r_1})$ with $\|u_n-u_*\| \to 0$, $n\to+\infty$. Thus, from (H_1), we have

$$\begin{gathered}
\lim_{n\to+\infty} f\left(t,u_n(t)-M\omega(t),u_n'(t)-M\omega'(t)\right) \\
= f\left(t,u_*(t)-M\omega(t),u_*'(t)-M\omega'(t)\right), \quad t\in(0,2\pi), \\
\left|f\left(t,u_n(t)-M\omega(t),u_n'(t)-M\omega'(t)\right)\right| \le M+M_1 g(t), \quad t\in(0,2\pi), \\
\left[M+M_1g(t)\right]\in L[0,2\pi],
\end{gathered} \tag{3.6}$$

where $M_1 =: \max\{h(u,v): u\in[\sigma r_1-2\pi M l_2, R],\ v\in[-(LR+2\pi M l_3),(LR+2\pi M l_3)]\}$.

Lemma 2.1 and Lebesgue-dominated convergence theorem guarantee that

$$\begin{aligned}
\|Tu_n-Tu_*\| \le \max\{l_2,l_3\}\int_0^{2\pi}&\left|f\left(t,u_n(t)-M\omega(t),u_n'(t)-M\omega'(t)\right)\right. \\
&\left.-f\left(t,u_*(t)-M\omega(t),u_*'(t)-M\omega'(t)\right)\right|dt \longrightarrow 0, \quad n\longrightarrow+\infty.
\end{aligned} \tag{3.7}$$

So, $T: K\cap(\overline{\Omega_R}\setminus\Omega_{r_1}) \to K$ is continuous.

For any bounded $D\subset K\cap(\overline{\Omega_R}\setminus\Omega_{r_1})$, From Lemma 2.1 and ($H_1$), for any $u\in D$, we have

$$\begin{aligned}
\|Tu\| &\le \max\{l_2,l_3\}\int_0^{2\pi}\left[f\left(s,u(s)-M\omega(s),u'(s)-M\omega'(s)\right)+M\right]ds \\
&\le \max\{l_2,l_3\}\int_0^{2\pi} g(s)h\left(u(s)-M\omega(s),u'(s)-M\omega'(s)\right)ds \\
&\le \max\{l_2,l_3\}M_1\int_0^{2\pi} g(s)ds,
\end{aligned} \tag{3.8}$$

which means the functions belonging to $\{(TD)(t)\}$ and the functions belonging to $\{(TD)'(t)\}$ are uniformly bounded on $[0,2\pi]$. Notice that

$$\left|(Tu)'(t)\right| \le l_3 M_1 \int_0^{2\pi} g(s)ds, \quad t \in [0,2\pi],\, u \in D, \tag{3.9}$$

which implies that the functions belonging to $\{(TD)(t)\}$ are equicontinuous on $[0,2\pi]$. From Lemma 2.1, we have

$$G'_t(t,s) = \begin{cases} (\alpha+1)H'(t)h(s) + (\beta-1)h'(t)H(s) + cH'(t)H(s) + dh'(t)h(s), & 0 \le s \le t \le 2\pi, \\ \alpha H'(t)h(s) + \beta h'(t)H(s) + cH'(t)H(s) + dh'(t)h(s), & 0 \le t < s \le 2\pi, \end{cases} \tag{3.10}$$

where α,β,c,d are constants, $h,H \in C^2[0,2\pi]$ are independent solutions of the linear differential equation $u'' + a(t)u(t) = 0$, and $H'(t)h(t) - h'(t)H(t) = 1$.

It is easy to see that $G'_t(t,s)$ is continuous in t and s for $0 \le s \le t \le 2\pi$ and $0 \le t < s \le 2\pi$. So, for any $t_1,\ t_2 \in [0,2\pi], t_1 < t_2$, we have

$$\begin{gathered} \int_0^{t_1} \left|G'_t(t_1,s) - G'_t(t_2,s)\right| g(s)ds \longrightarrow 0, \quad \text{as } t_1 \longrightarrow t_2^-, \text{ or } t_2 \longrightarrow t_1^+, \\ \int_{t_1}^{t_2} \left|G'_t(t_1,s) - G'_t(t_2,s)\right| g(s)ds \le 2l_3 \int_{t_1}^{t_2} g(s)ds \longrightarrow 0, \quad \text{as } t_1 \longrightarrow t_2^-, \text{ or } t_2 \longrightarrow t_1^+, \\ \int_{t_2}^{2\pi} \left|G'_t(t_1,s) - G'_t(t_2,s)\right| g(s)\ ds \longrightarrow 0, \quad \text{as } t_1 \longrightarrow t_2^-\text{, or } t_2 \longrightarrow t_1^+. \end{gathered} \tag{3.11}$$

Therefore,

$$\begin{aligned} &\left|(Tu)'(t_1) - (Tu)'(t_2)\right| \\ &\quad = \left|\int_0^{2\pi} \left[G'_t(t_1,s) - G'_t(t_2,s)\right]\left[f\left(s,u(s) - M\omega(s),u'(s) - M\omega'(s)\right) + M\right]ds\right| \\ &\quad \le M_1 \int_0^{2\pi} \left|G'_t(t_1,s) - G'_t(t_2,s)\right| g(s)ds \\ &\quad = M_1 \left\{\int_0^{t_1} \left|G'_t(t_1,s) - G'_t(t_2,s)\right| g(s)ds + \int_{t_1}^{t_2} \left|G'_t(t_1,s) - G'_t(t_2,s)\right| g(s)ds \right. \\ &\qquad\quad \left. + \int_{t_2}^{2\pi} \left|G'_t(t_1,s) - G'_t(t_2,s)\right| g(s)ds \right\} \longrightarrow 0, \quad \text{as } t_1 \longrightarrow t_2^- \text{ or } t_2 \longrightarrow t_1^+. \end{aligned} \tag{3.12}$$

Thus, the functions belonging to $\{TD'(t)\}$ are equicontinuous on $[0,2\pi]$. By Arzela-Ascoli theorem, TD is relatively compact in $C^1[0,2\pi]$.

Hence, $T: \quad K \cap (\overline{\Omega_R} \setminus \Omega_{r_1}) \rightarrow K$ is completely continuous for any $R > r_1$.

(2) We now show that

$$\inf_{u\in K\cap\partial\Omega_{r_1}}\|Tu\|>0,\qquad u\neq\lambda Tu,\quad \forall u\in K\cap\partial\Omega_{r_1},\ \lambda\geq 1. \tag{3.13}$$

For any $u\in K\cap\partial\Omega_{r_1}$, we have

$$\begin{gathered}0<\sigma r_1-2\pi Ml_2\leq u(t)-M\omega(t)\leq r_1,\\ \left|u'(t)-M\omega'(t)\right|\leq\left|u'(t)\right|+M\left|\omega'(t)\right|\leq Lr_1+2\pi Ml_3,\quad \forall t\in[0,2\pi].\end{gathered} \tag{3.14}$$

From (H_1) and (3.2),

$$\begin{aligned}(Tu)(t)&=\int_0^{2\pi}G(t,s)\left[f\left(s,u(s)-M\omega(s),u'(s)-M\omega'(s)\right)+M\right]ds\\ &\geq l_1\int_0^{2\pi}a(s)ds>l_1r_1l_1^{-1}=r_1>0.\end{aligned} \tag{3.15}$$

Suppose that there exist $\lambda_0\geq 1$ and $u_0\in K\cap\partial\Omega_{r_1}$ such that $u_0=\lambda_0Tu_0$, that is, for $t\in[0,2\pi]$,

$$\begin{aligned}u_0(t)\geq(Tu_0)(t)&=\int_0^{2\pi}G(t,s)\left[f\left(s,u_0(s)-M\omega(s),u_0'(s)-M\omega'(s)\right)+M\right]ds\\ &\geq l_1\int_0^{2\pi}a(s)ds>l_1r_1l_1^{-1}=r_1.\end{aligned} \tag{3.16}$$

This is in contradiction with $u_0\in K\cap\partial\Omega_{r_1}$ and (3.13) holds.

(3) Next, we show that

$$u\neq\lambda Tu\quad \forall u\in K\cap\partial\Omega_{R_1},\ 0<\lambda\leq 1. \tag{3.17}$$

Suppose this is false, then there exist $\lambda_0\in(0,1]$ and $u_0\in K\cap\partial\Omega_{R_1}$ with $u_0=\lambda_0Tu_0$, that is, for $t\in[0,2\pi]$, we have

$$\begin{aligned}u_0(t)\leq(Tu_0)(t)&=\int_0^{2\pi}G(t,s)\left[f\left(s,u_0(s)-M\omega(s),u_0'(s)-M\omega'(s)\right)+M\right]ds,\\ \left|u_0'(t)\right|=\lambda_0\left|(Tu_0)'(t)\right|&\leq\int_0^{2\pi}\left|G_t'(t,s)\right|\left[f\left(s,u(s)-M\omega(s),u'(s)-M\omega'(s)\right)+M\right]ds.\end{aligned} \tag{3.18}$$

From (H_2), we have

$$\begin{gathered}0<\sigma R_1-2\pi Ml_2\leq u_0(t)-M\omega(t)\leq R_1,\\ \left|u_0'(t)-M\omega'(t)\right|\leq\left|u_0'(t)\right|+M\left|\omega'(t)\right|\leq LR_1+2\pi Ml_3,\quad \forall t\in[0,2\pi].\end{gathered} \tag{3.19}$$

Therefore, by (3.18), (3.19), and (H_2), it follows that

$$\begin{aligned} u_0(t) &\le l_2 \int_0^{2\pi} g(s) h\big(u_0(s) - M\omega(s), u_0'(s) - M\omega'(s)\big) ds \\ &\le l_2 M_0 \int_0^{2\pi} g(s) ds < R_1, \quad \forall t \in [0, 2\pi], \\ \left|u_0'(t)\right| &\le l_3 \int_0^{2\pi} g(s) h\big(u_0(s) - M\omega(s), u_0'(s) - M\omega'(s)\big) ds \\ &\le l_3 M_0 \int_0^{2\pi} g(s) ds < R_1, \quad \forall t \in [0, 2\pi]. \end{aligned} \tag{3.20}$$

Thus, $\|u\| < R_1$. This is in contradiction with $u_0 \in K \cap \partial\Omega_{R_1}$ and (3.17) holds.

(4) Choose $N^* = (1 + 2\pi M l_2)[\sigma l_1(\beta^* - \alpha^*)]^{-1} + 1$. From ($H_3$), there exists $R_2 > \max\{R_1, 1\}$ such that

$$f(t, u, v) \ge N^* u, \quad \forall u \ge R_2,\ v \in R,\ t \in \left[\alpha^*, \beta^*\right]. \tag{3.21}$$

Now, we show that

$$\inf_{u \in K \cap \partial\Omega_R} \|Tu\| > 0, \qquad u \ne \lambda T u, \quad \forall u \in K \cap \partial\Omega_R,\ \lambda \ge 1, \tag{3.22}$$

where $R = (R_2 + 2\pi M l_2)\sigma^{-1}$.

For any $u \in K \cap \partial\Omega_R$, we have

$$u(t) - M\omega(t) \ge \sigma R - 2\pi M l_2 = R_2, \quad \forall t \in [0, 2\pi]. \tag{3.23}$$

This and (3.21) together with (3.2) imply

$$\begin{aligned} (Tu)(t) &= \int_0^{2\pi} G(t,s)\left[f\big(s, u(s) - M\omega(s), u'(s) - M\omega'(s)\big) + M\right] ds \\ &\ge l_1 \int_{\alpha^*}^{\beta^*} \left[f\big(s, u(s) - M\omega(s), u'(s) - M\omega'(s)\big)\right] ds \\ &\ge l_1 N^* R_2\left(\beta^* - \alpha^*\right) > 0. \end{aligned} \tag{3.24}$$

Suppose that there exist $\lambda_0 \geq 1$ and $u_0 \in K \cap \partial\Omega_R$ such that $u_0 = \lambda_0 T u_0$, then, for $t \in [\alpha^*, \beta^*]$, we have

$$\begin{aligned} u_0(t) \geq (Tu_0)(t) &= \int_0^{2\pi} G(t,s)\left[f\left(s, u_0(s) - M\omega(s), u_0'(s) - M\omega'(s)\right) + M\right] ds \\ &\geq l_1 \int_{\alpha^*}^{\beta^*} \left[f\left(s, u(s) - M\omega(s), u'(s) - M\omega'(s)\right)\right] ds \\ &\geq l_1 N^* R_2\left(\beta^* - \alpha^*\right) > \left(R_2 + 2\pi M l_2\right)\sigma^{-1} = R. \end{aligned} \tag{3.25}$$

This is in contradiction with $u_0 \in K \cap \partial\Omega_R$ and (3.22) holds.

Now, (3.13), (3.17), (3.22), and Lemma 2.3 guarantee that T has two fixed points $u_1 \in K \cap (\Omega_{R_1} \setminus \overline{\Omega_{r_1}})$, $u_2 \in K \cap (\Omega_R \setminus \overline{\Omega_{R_1}})$ with $r_1 < \|u_1\|_1 < R_1 < \|u_2\|_1 < R$. Clear, PBVP (3.1) has at least two positive solutions $u_1, u_2 \in C^1[0, 2\pi] \cap C^2(0, 2\pi)$. □

Remark 3.2. From the proof of Theorem 3.1, when $f(t, u, v)$ is nonnegative (i.e., $M = 0$ in (H_0)), Theorem 3.1 still holds.

Corollary 3.3. *Assume that (H_0)–(H_2) hold, then PBVP* (1.1) *has at least one positive solution $u(t)$ such that $r_1 < \|u + M\omega\| < R_1$, where $\omega(t) =: \int_0^{2\pi} G(t,s)ds$.*

Corollary 3.4. *Assume that (H_0) and (H_3) hold, and*

(H_4) there exist $R_1 > \sigma^{-1} 2\pi M l_2$ such that

$$\max\{l_2, l_3\} \int_0^{2\pi} g(t)dt < R_1 M_0^{-1}, \tag{3.26}$$

where $M_0 =: \max\{h(u,v) : u \in [\sigma R_1 - 2\pi M l_2, R_1]$ and $v \in [-(LR_1 + 2\pi M l_3), (LR_1 + 2\pi M l_3)]\}$. Then PBVP (1.1) *has at least one positive solution $u(t)$ such that $\|u + M\omega\| > R_1$, where $\omega(t) =: \int_0^{2\pi} G(t,s)ds$.*

Example 3.5. Consider the following second-order singular semipositone PBVP:

$$\begin{gathered} u'' + \frac{1}{16}u = \frac{u^{9/4} + (u')^2 + 1}{8\pi u\sqrt{t(2\pi - t)}} - \frac{\sqrt{3}}{30\pi}\cos\frac{t}{12}, \quad t \in (0, 2\pi), \\ u(0) = u(2\pi), \qquad u'(0) = u'(2\pi). \end{gathered} \tag{3.27}$$

4. Conclusion

PBVP (3.27) has at least two positive solutions $u_1, u_2 \in C^1[0, 2\pi] \cap C^2(0, 2\pi)$ and $u_1(t), u_2(t) > 0$ for $t \in [0, 2\pi]$.

To see this, we will apply Theorem 3.1 with $m = 1/4$, $f(t, u, v) = ((u^{9/4} + v^2 + 1)/8\pi u\sqrt{t(2\pi - t)}) - (\sqrt{3}/30\pi)\cos(t/12)$, $g(t) = 1/\sqrt{t(2\pi - t)}$, $h(u, v) = (u^{9/4} + v^2 + 1)/8\pi u$, $M = 1/20\pi$.

From Remark 2.2, it is easy to see that $l_1 = 2$, $l_2 = 2\sqrt{2}$, $l_3 = 1/2$, $\sigma = \sqrt{2}/2$, and $L = 1/4$.

By simple computation, we easily get $0 \leq f(t,u,v) + M \leq g(t)h(u,v)$ and $\int_0^{2\pi} g(t)dt = \pi$. So (H_0) holds.

Taking $r_1 = 1/2$, $a(t) = 1/4\pi\sqrt{t(2\pi - t)}$, then $\sigma^{-1}2\pi M l_2 = \sqrt{2} \cdot 2\pi \cdot (1/20\pi) \cdot 2\sqrt{2} = 2/5 < r_1$, $\int_0^{2\pi} a(t)dt = 1/4 = r_1 l_1^{-1}$ and for any $t \in (0, 2\pi)$, $u \in (0, 1/2]$, $v \in [-7/40,\ 7/40]$,

$$\begin{aligned}
\frac{u^{9/4} + v^2 + 1}{8\pi u\sqrt{t(2\pi - t)}} - \frac{\sqrt{3}}{30\pi}\cos\frac{t}{12} + \frac{1}{20\pi} &\geq \frac{1/2^{9/4} + 1}{4\pi\sqrt{t(2\pi - t)}} - \frac{\sqrt{3}}{30\pi} + \frac{1}{20\pi} \\
&\geq \frac{1/2^{9/4}}{4(\pi)^2} - \frac{\sqrt{3}}{30\pi} + \frac{1}{20\pi} + \frac{1}{4\pi\sqrt{t(2\pi - t)}} \\
&> \frac{1}{4\pi\sqrt{t(2\pi - t)}}.
\end{aligned} \tag{4.1}$$

Thus, (H_1) holds.

Taking $R_1 = 4$, then for $u \in [(9/5)\sqrt{2}, 4]$, $|v| \leq 21/20$, we have

$$M_0 \leq \frac{5}{72\sqrt{2}\pi}\left(2^{9/2} + \left(\frac{21}{20}\right)^2 + 1\right) < \frac{5}{72\sqrt{2}\pi}(23 + 2 + 1) = \frac{65}{36\sqrt{2}\pi}. \tag{4.2}$$

So, $M_0 \max\{l_2, l_3\} \int_0^{2\pi} g(t)dt = 2\sqrt{2}\pi M_0 < 65/18 < 4 = R_1$. That is, (H_2) holds.

Let $[\alpha^*, \beta^*] = [\pi/2, \pi]$, then it is easy to check that (H_3) holds.

Thus all the conditions of Theorem 3.1 are satisfied, so PBVP (3.27) has at least two positive solutions $u_1,\ u_2 \in C^1[0, 2\pi] \cap C^2(0, 2\pi)$ and $u_1(t), u_2(t) > 0$ for $t \in [0, 2\pi]$.

Acknowledgments

Research supported by the Project of Shandong Province Higher Educational Science and Technology Program (J09LA08) and Reward Fund for Excellent Young and Middle-Aged Scientists of Shandong Province (BS2011SF022), China.

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