Research Article

Least Squares Problems with Absolute Quadratic Constraints

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This paper analyzes linear least squares problems with absolute quadratic constraints. We develop
a generalized theory following Bookstein's conic-fitting and Fitzgibbon's direct ellipse-specific
fitting. Under simple preconditions, it can be shown that a minimum always exists and can
be determined by a generalized eigenvalue problem. This problem is numerically reduced to
an eigenvalue problem by multiplications of Givens' rotations. Finally, four applications of this
approach are presented.

1. Introduction

The least squares methods cover a wide range of applications in signal processing and
system identification [1–5]. Many technical applications need robust and fast algorithms for
fitting ellipses to given points in the plane. In the past, effective methods were Bookstein's
conic-fitting or Fitzgibbon's direct ellipse-specific fitting, where an algebaic distance with
a quadratic constraint is minimized [6, 7]. In this paper, we develop an extended theory of
minimization of least squares with a quadratic constraint based on the ideas of Bookstein and
Fitzgibbon. Thereby, we show the existence of a minimal solution and present the uniqueness
regarding to the smallest positive generalized eigenvalue. So, arbitrary conic fitting problems
with quadratic constraints are possible.

Let $A \in \mathbb{R}^{n \times m}$ be matrix with $n \geq m \geq 2$, $C \in \mathbb{R}^{m \times m}$ be symmetric matrix, and $d \in \mathbb{R}$ a
real value. We consider the problem of finding a vector $x \in \mathbb{R}^m$ which minimizes the function
$F : \mathbb{R}^m \to \mathbb{R}$ defined by

$$F(x) := \|Ax\|_2^2 \quad \text{subject to } x^t C x = d. \tag{1.1}$$

The side condition $x^t C x = d$ introduces an absolute quadratic constraint. The problem (1.1) is not a special case of Gander's optimization as presented in [8], because in our case C is a real symmetric matrix in contrast to the approach of Gander, where the side condition considers real symmetric matrices $C^t C$ which are positive definite. For our considerations, we require the following three assumptions

Assumption 1.1. By replacing C by $(-C)$, we consider $d \geq 0$. For $d = 0$, the trivial solution $x = 0 \in \mathbb{R}^m$ fulfills (1.1). Therefore, we demand $d > 0$.

Assumption 1.2. The set $N := \{x \in \mathbb{R}^m, \ x^t C x = d\}$ is not empty, that is, the matrix C has not been less than one positive eigenvalue. If we assume that C has only nonpositive eigenvalues, it would be negative semidefinite and

$$x^t C x \leq 0 \tag{1.2}$$

holds. With $d > 0$ it follows that the set N would be empty in this case.

Assumption 1.3. In the following, we set $S = A^t A$ and assume that S is regular. S is sometimes called scatter matrix.

In the following two sections, we introduce the theoretical basics of this optimization. The main result is the solution of a generalized eigenvalue problem. Afterwards, we reduce this system numerically to an eigenvalue problem. In Section 5, we present four typical applications for conic fitting problems with quadratic constraints. These approximations contain the ellipse fitting of Fitzgibbon, the hyperbola fitting of O'Leary, the conic fitting of Bookstein, and an optical application of shrinked aspheres [6, 7, 9, 10].

2. Existence of a Minimal Solution

Theorem 2.1. *If $S \in \mathbb{R}^{m \times m}$ is regular, then there exists a global minimum to the problem* (1.1).

Proof. The real regular matrix $S = A^t A$ is symmetric and positive definite. Therefore, a Cholesky decomposition $S = R^t R$ exists with a regular upper triangular matrix $R \in \mathbb{R}^{m \times m}$. In (1.1), we are looking for a solution $x \in \mathbb{R}^m$ minimizing

$$F(x) = \|Ax\|^2 = x^t S x = x^t R^t R x \quad \text{subject to } x^t C x = d. \tag{2.1}$$

With R regular we substitute x by $R^{-1} y$ for $y \in \mathbb{R}^m$. Thus, we obtain an equivalent problem to (2.1), where we want to find a vector $y \in \mathbb{R}^m$, minimizing

$$F(y) = y^t y = \|y\|^2 \quad \text{subject to } y^t \left(R^{-1}\right)^t C R^{-1} y = d. \tag{2.2}$$

Now, we define $G : \mathbb{R}^m \to \mathbb{R}$ with $G(y) = y^t (R^{-1})^t C R^{-1} y - d$ and look for a solution y on the zero-set N_G of G with minimal distance to the point of origin. Let $y_0 \in N_G \neq \emptyset$ and $\overline{K_{r_0}^m}(0)$ be

the closed sphere of $\mathbb{R}^m$ in 0 with radius $r_0 = \|y_0\|$. Because of $y_0 \in N_G \cap \overline{K_{r_0}^m}(0)$ and G being continuous, the set

$$N_G \bigcap \overline{K_{r_0}^m}(0) = \left\{ y \in \overline{K_{r_0}^m}(0), \quad G(y) = 0 \right\} \tag{2.3}$$

is nonempty, closed and bounded. Therefore, for the continuous function $F(y) = \|y\|^2$ exists a minimal value y_M on $N_G \cap \overline{K_{r_0}^m}(0)$ with

$$\min_{y \in N_G \cap \overline{K_{r_0}^m}(0)} F(y) \leq r_0^2. \tag{2.4}$$

For all y from $N_G \setminus \overline{K_{r_0}^m}(0)$, it is $F(y) > r_0^2$. So, y_M is a minimal value of F in N_G. By the equivalence of (2.1), and (2.2) the assumption follows. □

3. Generalized Eigenvalue Problem

The minimization problem in (1.1) induces a generalized eigenvalue problem. The following theorem is already proven by Bookstein and Fitzgibbon for the special case of ellipse-fitting [6, 7].

Theorem 3.1. *If x_s is an extremum of $F(x)$ subject to $x^t C x = d$, then a positive $\lambda_0 \in \mathbb{R}$ exists with*

$$(S - \lambda_0 C)x_s = 0, \tag{3.1}$$

that is, x_s is an eigenvector to the generalized eigenvalue λ_0 and

$$F(x_s) = \lambda_0 \cdot d \tag{3.2}$$

holds.

Proof. Let $G : \mathbb{R}^m \to \mathbb{R}$ be a defined as $G(x) := d - x^t C x$. For $G(x) = 0$ and $d > 0$ follows $x \neq 0$. Further, G is continuously differentiable with $dG/dx = -2Cx \neq 0$ for all x of the zero-set of G. So, if x_s is a local extremum of $F(x)$ subject to $G(x) = 0$, then it is $\operatorname{rank}(dG/dx)(x_s) = 1$. Since F is also a continuously differentiable function in $\mathbb{R}^m$ with $m > 1$, it follows by using a Lagrange multiplier [11]: if x_s is a local extremum of $F(x)$ subject to $G(x) = 0$, then a $\lambda_0 \in \mathbb{R}$ exists, such that the Lagrange function $\phi : \mathbb{R}^{m+1} \to \mathbb{R}$ given as

$$\phi(x, \lambda) = F(x) + \lambda G(x) = x^t A^t A x + \lambda(d - x^t C x) \tag{3.3}$$

has a critical point in (x_s, λ_0). Therefore, x_s fulfills necessarily the equations:

$$\operatorname{grad}_x \phi(x, \lambda) = 2Sx - 2\lambda Cx = 0, \tag{3.4}$$

$$\frac{d\phi(x, \lambda)}{d\lambda} = x^t C x - d = 0. \tag{3.5}$$

The first equation describes a generalized eigenvalue problem with

$$(S - \lambda \cdot C)x = 0. \tag{3.6}$$

With $d > 0$, $x_s \neq 0$ and x_s fulfills (3.6), λ must be a generalized eigenvalue, and x_s is a corresponding eigenvector to λ of (3.6), so that $(S - \lambda C)$ is a singular matrix. If λ_0 is an eigenvalue and $x_0 \neq 0$ a corresponding eigenvector to λ_0 of (3.6), then every vector αx_0 is also a solution of (3.6) for λ_0. Now, we are looking for α, such that $x_s = \alpha x_0$ satisfies (3.5). For $\lambda_0 \neq 0$ and (3.4), it follows

$$d = \alpha^2 x_0^t C x_0 = \frac{\alpha^2 x_0^t S x_0}{\lambda_0}. \tag{3.7}$$

Because the left side and the numerator are positive, the denominator must also be chosen positive, that is, only positive eigenvalues solve (3.4) and (3.5). By the multiplication with λ_0,

$$d \cdot \lambda_0 = \alpha^2 x_0^t S x_0 = x_s^t S x_s = F(x_s) \tag{3.8}$$

follows and $x_s = \alpha \cdot x_0$ fulfills the constraint $G(x_s) = 0$. $\qquad\square$

Remark 3.2. Let x_0 be a generalized eigenvector to a positive eigenvalue λ_0 of problem (3.6). Then

$$x_{1,2} = \pm \sqrt{\frac{d\lambda_0}{x_0^t S x_0}} x_0 \tag{3.9}$$

are solutions of (3.8).

Lemma 3.3. *If S is regular and C is symmetric, then all eigenvalues of* (3.1) *are real-valued and different to zero.*

Proof. With $\det(S) \neq 0$, $\lambda_0 \neq 0$ in (3.1). The Cholesky decomposition $S = R^t R$ with a regular upper triangular matrix R yields (3.1)

$$(R^t R - \lambda_0 C)x_s = R^t \left(I - \lambda_0 (R^t)^{-1} C R^{-1} \right) R x_s = 0. \tag{3.10}$$

With R invertible and the substitution $\mu_0 = \lambda_0^{-1}$, $y_s = R x_s$, we obtain an eigenvalue problem to the matrix $(R^t)^{-1} C R^{-1}$:

$$\left(\mu_0 I - (R^t)^{-1} C R^{-1} \right) y_s = 0. \tag{3.11}$$

Furthermore, we have

$$\left((R^t)^{-1} C R^{-1} \right)^t = \left(R^{-1} \right)^t C^t \left((R^t)^{-1} \right)^t = (R^t)^{-1} C^t R^{-1} = (R^t)^{-1} C R^{-1}. \tag{3.12}$$

Therefore, the matrix $(R^t)^{-1}CR^{-1}$ is symmetric and all eigenvalues μ_0 are real. With $\lambda_0 = \mu_0^{-1}$ for $\mu_0 \neq 0$ follows the proposition. □

Remark 3.4. Because of S regular and $\lambda \neq 0$ in

$$(S - \lambda_0 C)x_s = 0, \tag{3.13}$$

we can consider the equivalent problem with $\mu_0 = \lambda_0^{-1}$ instead of (3.11):

$$\left(\mu_0 I - S^{-1}C\right)x_s = 0. \tag{3.14}$$

This system is called *inverse* eigenvalue problem to (3.1). Here, the eigenspaces to the generalized eigenvalue λ_0 in (3.1) and to the eigenvalue $1/\lambda_0$ in (3.14) are identical. Therefore, the generalized eigenvectors in (3.1) are perpendicular.

Definition 3.5. The set of all eigenvalues of a matrix C is called spectrum $\sigma(C)$. We call the set of all eigenvalues to the generalized eigenvalue problem in (3.1) also spectrum and denote $\sigma(S,C)$. $\sigma^+(S,C)$ is defined as the set of all positive values in $\sigma(S,C)$.

Remark 3.6. In case of $rg(C) < m = rg(S)$, the inverse problem in (3.14) has eigenvalue 0 with multiplicity $rg(S) - rg(C)$. Otherwise, for $\mu \neq 0$ and $\mu \in \sigma(S^{-1}C)$ follows $1/\mu \in \sigma(S,C)$. Analogously, for $\mu \in \sigma((R^t)^{-1}CR^{-1})$ with $\mu \neq 0$, $1/\mu \in \sigma(S,C)$.

The following lemma is a modified result of Fitzgibbon [7].

Lemma 3.7. *The signs of the generalized eigenvalues of* (3.1) *are the same as those of C.*

Proof. With S being nonsingular, every generalized eigenvalue λ_0 of (3.1) is not zero. Therefore, it follows for the equivalent problem (3.11) that $\mu_0 = \lambda_0^{-1}$ is also a positive eigenvalue to $(R^{-1})^t CR^{-1}$, where R is an upper triangular matrix to the Cholesky decomposition of S. With Sylvester's Inertia Law, we know that the signs of eigenvalues of the matrices $(R^{-1})^t CR^{-1}$ are the same as those of C. □

For the following proofs, we need the lemma of Lagrange (see, e.g., [12]).

Lemma 3.8 (Lemma of Lagrange). *For $M \subset \mathbb{R}^n$, $f : M \to \mathbb{R}$, $g = (g_1, \ldots, g_k) : M \to \mathbb{R}^k$, and $N_g = \{x \in M, g(x) = 0 \in \mathbb{R}^k\}$, let $\lambda \in \mathbb{R}^k$, so that $x_s \in N_g$ is a minimal value of the function $\Phi_\lambda : M \to \mathbb{R}$ with*

$$\Phi_\lambda(x) = f(x) + \sum_{i=1}^{k} \lambda_i g_i(x). \tag{3.15}$$

Then x_s is a minimal solution of f in N_g.

Definition 3.9. Let λ^* be the smallest positive value of $\sigma^+(S,C)$ and x_s^* a corresponding generalized eigenvector to λ^* to the constraint $x_s^{*t} C x_s^* = d$.

Lemma 3.10. *Let $S - \lambda C$ be a positive semidefinite matrix for $\lambda \in \sigma^+(S, C)$. Then a generalized eigenvector x_s corresponding to λ is a local minimum of* (1.1).

Proof. We consider $\Phi : \mathbb{R}^m \to \mathbb{R}$ with

$$\Phi(x) = x^t S x + \lambda(d - x^t C x). \tag{3.16}$$

With $\operatorname{grad}_x \Phi(x) = 2(Sx - \lambda Cx)$ holds $\operatorname{grad}_x \Phi(x_s) = 0$ and since the Hessian matrix $(S - \lambda C)$ of Φ is positive semidefinite, the vector x_s is a minimal solution of Φ. Then, x_s minimizes also $F(x)$ subject to $x^t C x = d$ by the Lemma 3.8. □

Remark 3.11. In fact, in Lemma 3.10, we require only a semidefinite matrix, because $x_s \neq 0$ fulfills $x_s^t(S - \lambda C)x_s = 0$ in (3.1).

Lemma 3.12. *The matrix $(S - \lambda^* C)$ is positive semidefinite.*

Proof. Let μ be an arbitrary eigenvalue of $((\lambda^*)^{-1}I - (R^t)^{-1}CR^{-1})$, where R is the upper triangular matrix of the Cholesky decomposition from S. With

$$\det\left((\lambda^*)^{-1}I - (R^t)^{-1}CR^{-1} - \mu I\right) = \det\left(\left((\lambda^*)^{-1} - \mu\right)I - (R^t)^{-1}CR^{-1}\right) = 0, \tag{3.17}$$

it follows that $((\lambda^*)^{-1} - \mu)$ is an eigenvalue of $(R^t)^{-1}CR^{-1}$. This value is corresponding in (3.11) with the inverse eigenvalue of problem (3.1). Furthermore, it yields

$$\frac{1}{\lambda^*} - \mu \leq \max\left\{\frac{1}{\lambda}, \ \lambda \in \sigma^+(S, C)\right\} = \frac{1}{\min\{\lambda, \lambda \in \sigma^+(S, C)\}} = \frac{1}{\lambda^*}. \tag{3.18}$$

So $\mu \geq 0$ follows, that is, $((\lambda^*)^{-1}I - (R^t)^{-1}CR^{-1})$ is positive semidefinite, and for $y \in \mathbb{R}^m$ we obtain

$$y^t\left((\lambda^*)^{-1}I - (R^t)^{-1}CR^{-1}\right) \ y \geq 0. \tag{3.19}$$

By setting $y = Rx$ and with regular R, we get

$$0 \leq x^t\left((\lambda^*)^{-1}R^t R - C\right)x = x^t\left((\lambda^*)^{-1}S - C\right)x. \tag{3.20}$$

With $\lambda^* > 0$, $x^t(S - \lambda^* C)x \geq 0$ for $x \in \mathbb{R}^m$, that is, $S - \lambda^* C$ is positive semidefinite. □

Theorem 3.13. *For the smallest value λ^* in $\sigma^+(S, C)$ exists a corresponding generalized eigenvector x_s^*, which minimizes $F(x)$ subject to $x^t C x = d$.*

Proof. The matrix $(S - \lambda^* C)$ is positive semidefinite by Lemma 3.12, and with Lemma 3.10 it follows that x_s^* is a local minimum of problem (1.1). Furthermore, we know by Theorem 3.1 that if x_s is a local extremum of $F(x)$ subject to $x^t C x = d$, then a positive value λ_s exists with

$$(S - \lambda_s C)x_s = 0 \tag{3.21}$$

and it is $F(x_s) = \lambda_s d$. Because of the existence of a minimum x_E in Theorem 2.1 a value $\lambda_E \in \sigma^+(S, C)$ exists in problem (3.21) regarding to x_E. Otherwise, for an arbitrary local minimum x_s,

$$F(x_s^*) = \lambda^* d = \min \ \{\lambda d, \ \lambda \in \sigma^+(S, C)\} \le \lambda_s d = F(x_s). \tag{3.22}$$

So, $\lambda^* = \lambda_E$ follows and x_s^* is a minimum of $F(x)$ subject to $x^t C x = d$. □

Example 3.14. We minimize $F : \mathbb{R}^2 \to \mathbb{R}$ with $F(\xi_1, \xi_2) = \xi_1^2 + \xi_2^2$ subject to $\xi_1^2 - \xi_2^2 = 1$. So, we have $d = 1$, S the identity matrix $I_2 \in \mathbb{R}^{2 \times 2}$, and $C \in \mathbb{R}^{2 \times 2}$ a diagonal matrix with values -1 and 1. Then, we get the following generalized eigenvalue problem:

$$\det(I_2 - \lambda C) = (1 - \lambda)(-1 - \lambda) = 0 \tag{3.23}$$

with eigenvalues 1 and -1. Because of Theorem 3.1, we consider a generalized eigenvector $(\alpha, 0)^t$ with $\alpha \in \mathbb{R} \setminus \{0\}$ only for $\lambda = 1$. Then, $(1, 0)^t$ and $(-1, 0)^t$ are solutions subject to $\xi_1^2 - \xi_2^2 = 1$. This result is conform to the geometric interpretation, since we are looking for $x = (\xi_1, \xi_2)^t$ on the hyperbola $\xi_1^2 - \xi_2^2 = 1$ with minimal distance to the origin.

4. Reduction to an Eigenvalue Problem of Dimension $rg(C)$

In numerical applications, a generalized eigenvalue problem is mostly reduced to an eigenvalue problem, for example, by multiplication with S^{-1}. Thus, we obtain the inverse problem (3.14) from (3.1) (see, e.g., [13]). But, S may be ill-conditioned, so that a solution of (3.14) may be numerical instable. Therefore, we present another reduction of (3.1).

Many times, C is a sparse matrix with $r := \text{rank}(C) \le \text{rank}(S)$. This symmetric matrix C is diagonalizable in $C = P^t D P$ with P orthogonal and D diagonal. Further, we assume that the first r diagonal entrees in D are different to 0. For the characteristic polynomial in (3.1), it follows

$$p(\lambda) = \det(S - \lambda C) = \det(P S P^t - \lambda D) = 0. \tag{4.1}$$

The order of p is r. We decompose these matrices in

$$PSP^t = \left(\begin{array}{c|c} S_1 & S_2 \\ \hline S_2^t & S_3 \end{array}\right), \qquad D = \left(\begin{array}{c|c} D_1 & 0 \\ \hline 0 & 0 \end{array}\right) \tag{4.2}$$

with $S_1, D_1 \in \mathbb{R}^{r \times r}$, $S_2 \in \mathbb{R}^{r \times (m-r)}$, and $S_3 \in \mathbb{R}^{(m-r) \times (m-r)}$. Now, we eliminate S_2 in PSP^t by multiplications with Givens rotations $G_k \in \mathbb{R}^{m \times m}$, $k = 1, \dots, l$, so that it follows:

$$G_l \cdots G_2 G_1 P S P^t = \left(\begin{array}{c|c} \Sigma_1 & 0 \\ \hline \Sigma_2 & \Sigma_3 \end{array}\right), \qquad G_l \cdots G_2 G_1 D = \left(\begin{array}{c|c} \Delta_1 & 0 \\ \hline \Delta_2 & 0 \end{array}\right) \tag{4.3}$$

with $\Sigma_1, \Delta_1 \in \mathbb{R}^{r \times r}$, $\Sigma_2, \Delta_2 \in \mathbb{R}^{(m-r) \times r}$, and $\Sigma_3 \in \mathbb{R}^{(m-r) \times (m-r)}$. In (4.1), we achieve with orthogonal G_k, $k = 1, \ldots, l$

$$p(\lambda) = \det(PSP^t - \lambda D) = \det(G_l \cdots G_2 G_1 (PSP^t - \lambda D))$$

$$= \det \begin{pmatrix} \Sigma_1 - \lambda \Delta_1 & 0 \\ \Sigma_2 - \lambda \Delta_2 & \Sigma_3 \end{pmatrix} = \det(\Sigma_3) \det(\Sigma_1 - \lambda \Delta_1) = 0. \tag{4.4}$$

Because of $p(0) = \det(PSP^t) = \det(S) \neq 0$ and $p(0) = \det(\Sigma_3) \det(\Sigma_1)$, the submatrices Σ_1, Σ_3 are regular and the generalized eigenvalues of $\det(\Sigma_1 - \lambda \Delta_1)$ are different to zero. So, with $y \in \mathbb{R}^r$

$$(\Sigma_1 - \lambda \Delta_1) y = 0 \tag{4.5}$$

can be transformed to an equivalent eigenvalue problem with

$$\left(\Delta_1^{-1} \Sigma_1 - \lambda I \right) y = 0. \tag{4.6}$$

This system can be solved by finding the matrix X with $\Delta_1 X = \Sigma_1$ using the Gaussian elimination and determining the eigenvalues of X computing the QR algorithm [13]. Because all steps are equivalent, we have $\sigma(\Delta_1^{-1} \Sigma_1) = \sigma(S, C)$, that is, the eigenvalues of (3.1) and (4.6) are the same.

With Theorem 3.13, we are looking for the smallest value $\lambda^* \in \sigma^+(S, C)$ and a corresponding generalized eigenvector x_s^* to minimize the problem (3.1). So,

$$0 = (S - \lambda^* C) x_s^* = (SP^t - \lambda^* CP^t) P x_s^* \tag{4.7}$$

yields. By substitution of y_s^* for Px_s^*, we obtain

$$0 = G_l \cdots G_2 G_1 (PSP^t - \lambda^* D) y_s^* = \begin{pmatrix} \Sigma_1 - \lambda^* \Delta_1 & 0 \\ \Sigma_2 - \lambda^* \Delta_2 & \Sigma_3 \end{pmatrix} y_s^*. \tag{4.8}$$

We decompose y_s^* into the subvectors $y_{s,r}^* \in \mathbb{R}^r$ and $y_{s,m-r}^* \in \mathbb{R}^{m-r}$ with $(y_s^*)^t = (y_{s,r}^* | y_{s,m-r}^*)^t$. Then, $y_{s,r}^*$ is a generalized eigenvector for λ^* of the problems (4.5) and (4.6).

Let $y_{s,r}^*$ be an eigenvector to the smallest positive eigenvalue λ^* of (4.6). Since Σ_3 is regular, it follows in (4.8)

$$y_{s,m-r}^* = \Sigma_3^{-1} (\Sigma_2 - \lambda^* \Delta_2) y_{s,r}^* \tag{4.9}$$

and a generalized eigenvector x_s^* for λ^* in (3.1) is given as:

$$x_s^* = P^t \begin{pmatrix} y_{s,r}^* \\ \Sigma_3^{-1} (\Sigma_2 - \lambda^* \Delta_2) y_{s,r}^* \end{pmatrix}. \tag{4.10}$$

5. Applications in Conic Fitting

5.1. Fitzgibbon's Ellipse Fitting

First, we would like to find an ellipse for a given set of points in $\mathbb{R}^2$. Generally, a conic in $\mathbb{R}^2$ is implicitly defined as the zero set of $f : \mathbb{R}^6 \times \mathbb{R}^2 \to \mathbb{R}$ to a constant parameter $a = (\alpha_1, \ldots, \alpha_6)^t \in \mathbb{R}^6$:

$$f(a, \xi, \eta) = \alpha_1 \xi^2 + \alpha_2 \xi \eta + \alpha_3 \eta^2 + \alpha_4 \xi + \alpha_5 \eta + \alpha_6. \tag{5.1}$$

The equation $f(a, \xi, \eta) = 0$ can also be written with $x = (\xi, \eta)^t$ as

$$0 = x^t A x + b^t x + c \quad \text{with } A = \begin{pmatrix} \alpha_1 & \alpha_2/2 \\ \alpha_2/2 & \alpha_3 \end{pmatrix}, \quad b = \begin{pmatrix} \alpha_4 \\ \alpha_5 \end{pmatrix}, \quad c = \alpha_6. \tag{5.2}$$

The eigenvalues λ_1, λ_2 of A characterize a conic uniquely [14]. Thus, we need $4\lambda_1 \lambda_2 = 4\alpha_1 \alpha_3 - \alpha_2^2 > 0$ for ellipses in $f(a, x) = 0$. Furthermore, every scaled vector μa with $\mu \in \mathbb{R} \setminus \{0\}$ describes the same zero-set of f. So, we can impose the constraint for ellipses with $4\alpha_1 \alpha_3 - \alpha_2^2 = 1$. For n $(n \geq 6)$ given points $(\xi_i, \eta_i)^t \in \mathbb{R}^2$, we want to find a parameter $a \in \mathbb{R}^6$, which minimizes $F : \mathbb{R}^6 \to \mathbb{R}$ with

$$F(a) = \sum_{i=1}^{n} f(a, \xi_i, \eta_i)^2 \quad \text{subject to } 4\alpha_1 \alpha_3 - \alpha_2^2 = 1. \tag{5.3}$$

This ellipse fitting problem is established and solved by Fitzgibbon [7]. With the following matrices $D \in \mathbb{R}^{n \times 6}$, $C \in \mathbb{R}^{6 \times 6}$

$$D = \begin{pmatrix} \xi_1^2 & \xi_1 \eta_1 & \eta_1^2 & \xi_1 & \eta_1 & 1 \\ \xi_2^2 & \xi_2 \eta_2 & \eta_2^2 & \xi_1 & \eta_2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \xi_n^2 & \xi_n \eta_n & \eta_n^2 & \xi_n & \eta_n & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{5.4}$$

and $F(a) = \sum_{i=1}^{n} f(a, \xi_i, \eta_i)^2 = \|Da\|_2^2$, we achieve the equivalent problem:

$$\min_{a \in \mathbb{R}^6} \|Da\|_2^2 \quad \text{subject to } a^t C a = 1. \tag{5.5}$$

For $S = D^t D$, we have a special case of (1.1). Assuming S is a regular matrix and since the eigenvalues of C are -2, -1, 0, and 2, by lemma 3.3 we know that the generalized eigenvalue problem

$$(S - \lambda C)a = 0 \tag{5.6}$$

has exactly one positive solution $\lambda^* \in \mathbb{R}$. Because of Theorem 3.13 a corresponding generalized eigenvector a^* to λ^* minimizes the problem (5.5) and a^* consists of the coefficients of an implicit given ellipse.

A numerically stable noniterative algorithm to solve this optimization problem is presented by Halir and Flusser [15]. In comparison with Section 4, their method uses a special block decomposition of the matrices D and C.

5.2. Hyperbola Fitting

Instead of ellipses, O'Leary and Zsombor-Murray want to find a hyperbola to a set of scattered data $x_i \in \mathbb{R}^2$ [9]. A hyperbola is a conic, which can uniquely be characterized by $4\lambda_1\lambda_2 = 4\alpha_1\alpha_3 - \alpha_2^2 < 0$ [14]. So, we consider the constraint $\alpha_2^2 - 4\alpha_1\alpha_3 = 1$ and obtain the optimization problem:

$$\min_{a \in \mathbb{R}^6} \|Da\|_2^2 \quad \text{subject to } a^t(-C)a = 1 \tag{5.7}$$

with D and C being chosen in 5.1. The matrix $(-C)$ has two positive eigenvalues. In this case, a solution is given by a generalized eigenvector to the smallest value in $\sigma^+(S, -C)$. But O'Leary and Zsombor-Murray determine the best hyperbolic fit by evaluation of $\kappa_i = \alpha_{2,i}^2 - 4\alpha_{1,i}\alpha_{3,i}$, where the eigenvector $a_i = (a_{i,1}, \ldots, a_{i,6})^t$ is associated to a positive value of $\sigma^+(S, -C)$.

5.3. Bookstein's Conic Fitting

In Bookstein's method, the conic constraint is restricted to

$$2\left(\text{trace}^2 A - 2\det A\right) = 2\left(\lambda_1^2 + \lambda_2^2\right) = 2\alpha_1^2 + \alpha_2^2 + 2\alpha_3^2 = 1, \tag{5.8}$$

where λ_1, λ_2 are the eigenvalues of A in f [6]. There, it is $\lambda_{1,2} \in [-\sqrt{2}/2, \sqrt{2}/2]$ and at least one of them different to 0. But the constraint (5.8) is not a restriction to a class of conics. Here, we determine an arbitrary conic, which minimizes

$$F(a) = \sum_{i=1}^{n} f\left(a, \xi_i, \eta_i\right)^2 \quad \text{subject to } 2\alpha_1^2 + \alpha_2^2 + 2\alpha_3^2 = 1. \tag{5.9}$$

The resulting data matrix $D \in \mathbb{R}^{n \times 6}$ is the same as for Fitzgibbon's problem. The constraint matrix $C \in \mathbb{R}^{6 \times 6}$ has a diagonal shape with the entrees $(2, 1, 2, 0, 0, 0)$, that is, all eigenvalues of C are nonnegative. In the case of a regular matrix S, the problem (5.9) is solved for a generalized eigenvector to the smallest value in $\sigma^+(S, C)$.

5.4. Approximation of Shrinked Aspheres

After the molding process in optical applications, the shrinkage of rotation-symmetric aspheres is implicitly defined for $x = (\xi, \zeta)^t$ in

$$\alpha_1 \zeta^2 + \alpha_2 \zeta + \alpha_3 + \alpha_4 \xi^2 = 0 \quad \text{subject to } \alpha_2^2 - 4\alpha_1 \alpha_3 = 4r^2, \tag{5.10}$$

where $r \in \mathbb{R} \setminus \{0\}$ and $a = (a_1, \ldots, a_4)^t$ are aspheric-specific constants [10]. For $i = 1, \ldots, n$ with $n \geq 4$, the scattered data $x_i = (\xi_i, \zeta_i)^t \in \mathbb{R}^2$ of a shrinked asphere are given in this approximation problem. Here, we are looking for the conic parameter $a = (\alpha_1, \ldots, \alpha_4)^t$ for a fixed value r_{ref}, which minimizes

$$F(a) = \sum_{i=1}^{n} \left(\alpha_1 \zeta_i^2 + \alpha_2 \zeta_i + \alpha_3 + \alpha_4 \xi_i^2 \right)^2 \quad \text{subject to } \alpha_2^2 - 4\alpha_1 \alpha_3 = 4r_{\text{ref}}^2. \tag{5.11}$$

Analogously to Fitzgibbon, we have the matrices $D \in \mathbb{R}^{n \times 4}$ and $C \in \mathbb{R}^{4 \times 4}$ with

$$D = \begin{pmatrix} \zeta_1^2 & \zeta_1 & 1 & \xi_1^2 \\ \zeta_2^2 & \zeta_2 & 1 & \xi_2^2 \\ \vdots & \vdots & \vdots & \vdots \\ \zeta_n^2 & \zeta_n & 1 & \xi_n^2 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.12}$$

and with $F(a) = \|Da\|_2^2$ we get the following optimization problem:

$$\min_{a \in \mathbb{R}^4} \|Da\|_2^2 \quad \text{subject to } a^t C a = 4r_{\text{ref}}^2. \tag{5.13}$$

This is also an application of (1.1). The matrix C has the eigenvalues $-2, 0, 1$, and 2. So, the generalized eigenvalue problem in (3.1) with regular $S = D^t D \in \mathbb{R}^{4 \times 4}$ has two positive values in $\sigma^+(S, C)$. With Theorem 3.13, a generalized eigenvector $a^* \in \mathbb{R}^4$ to the smaller of both values solves (5.13).

The coefficients α_i in the problems (5.5) and (5.13) correspond not to the same monomials $\xi^k \zeta^l$. Hence, we have different matrices D and C.

6. Conclusion

In this paper, we present a minimization problem of least squares subject to absolute quadratic constraints. We develop a closed theory with the main result that a minimum is a solution of a generalized eigenvalue problem corresponding to the smallest positive eigenvalue. Further, we show a reduction to an eigensystem for numerical calculations. Finally, we study four applications about conic approximations. We analyze Fitzgibbon's method for direct ellipse-specific fitting, O'Leary's direct hyperbola approximation, Bookstein's conic fitting, and an optical application of shrinked aspheres. All these systems are attribute to the general optimization problem.

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