Research Article

Synchronization of Two Rank-One Chaotic Systems without and with Delay via Linear Delayed Feedback Control

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This paper illustrates the presence of chaos in rank-one chaotic systems with delay via a binary test (called 0-1 test) for chaos. Chaotic synchronization between two rank-one chaotic systems without and with delay is achieved by means of Lyapunov functional and linear delayed feedback control method. Numerical simulations are implemented to verify the effectiveness of the proposed chaos synchronization scheme.

1. Introduction

The introduction of the notion of chaotic synchronization by Pecora and Carroll [1] has stimulated a wide range of research activity in various fields. In 1999, Mainieri and Rehacek proposed a new method of synchronization, that is, projective synchronization [2]. Since then, many papers have been published on the projective synchronization of chaotic systems (see [3–14] and references cited therein). For some recent works on the synchronization of delayed chaotic systems, we refer the reader to [14–16].

Wang and Young first developed a general theory of rank-one attractors, that is, attractors with a single direction of instability and strong contraction in all other directions, then applied it to prove the existence of rank-one attractors in certain systems of differential equations [17–19]. In 2005, Wang and Oksasoglu presented a new mechanism of producing rank-one chaos in Chua's system [20]. In 2009, Chen and Han studied the existence of rank-one chaos in the neighborhood of a periodically kicked stable limit cycle close to a heteroclinic cycle of a planar equation [21]. However, to our knowledge, few papers have been published on the synchronization of rank-one chaotic systems.

In this paper, we study the synchronization between rank-one chaotic systems without and with delay using linear control method. The detailed arrangement is as follows: in Section 2, the rank-one chaotic systems proposed in [20, 21] are briefly described. The presence of chaos in rank-one chaotic systems with delay are tested using the 0-1 test for chaos recently proposed in [22]. In Section 3, the linear delayed feedback control method is adopted to realize the synchronization between two rank-one chaotic systems without and with delay. In Section 4, numerical simulations are shown to verify the validity and feasibility of the proposed method, and finally some conclusions are given in Section 5.

2. Systems Description

Let

$$P_{T,p}(t) = \frac{1}{p} \sum_{n=-\infty}^{+\infty} F_{n,T,p}(t), \tag{2.1}$$

where

$$F_{n,T,p}(t) = \begin{cases} 1, & nT \leq t < nT + p, \\ 0, & \text{elsewhere.} \end{cases} \tag{2.2}$$

$P_{T,p}(t)$ is a periodic pulse train with a pulse width of p and a period of T.

The rank-one chaotic system proposed in [20] can be described as follows:

$$\dot{x} = \alpha \left[y - \beta_1 x - \beta_3 x^3 \right] + \eta x P_{T,p}(t),$$

$$\dot{y} = x - y + z, \tag{2.3}$$

$$\dot{z} = -\beta y,$$

where η is used for controlling the magnitude of the forcing. When the system parameters are taken as the following values:

$$\alpha = 2, \quad \beta = 2.31135, \quad \beta_1 = 0.2495, \quad \beta_3 = -1, \quad \eta = 0.15, \quad p = 0.5, \tag{2.4}$$

there exist rank-one attractors in system (2.3) for large enough T, such as $T = 95$ [20].

The rank-one chaotic system proposed in [21] can be described as follows:

$$\dot{x} = 2Bxy + \varepsilon \left(x + ax^2 \right) - \varepsilon_1 x(t) P_{T,p}(t),$$

$$\dot{y} = 1 - 2Ax - By^2, \tag{2.5}$$

where ε_1 is used to control the magnitude of the forcing. When $A = 1$, $B = 1$, $\varepsilon = -1$, $a = -1.23$, $p = 0.5$, $\varepsilon_1 = 0.4$, there exist rank-one attractors in system (2.5) for $T = 10$ and $T = 8$ [21].

Without the particular statement, the above parameter values are adopted in this whole paper.

Incorporating delays into (2.3), (2.5) due to the effect of propagation delay in practical circuits, we consider the following systems:

$$\dot{x} = \alpha\left[y(t - \tau) - \beta_1 x(t) - \beta_3 x^3(t)\right] + \eta x(t) P_{T,p}(t),$$

$$\dot{y} = x(t - \tau) - y(t) + z(t - \tau), \tag{2.6}$$

$$\dot{z} = -\beta y(t - \tau);$$

$$\dot{x} = 2Bx(t)y(t) + \varepsilon\left[x(t) + ax^2(t)\right] - \varepsilon_1 x(t) P_{T,p}(t),$$

$$\dot{y} = 1 - 2Ax(t - \tau) - By^2(t). \tag{2.7}$$

The presence of chaos in systems (2.6), (2.7) will be confirmed via the 0-1 test for chaos. For convenience, we briefly describe the test as follows. For more details, we refer the reader to [22–24].

Consider a set of discrete data $\phi(n)(n = 1, 2, 3, \ldots)$, representing a one-dimensional observable dataset obtained from the underlying dynamics, such as a set of experimental data or a set of discrete data obtained from numerical solutions. For $c \in (0, \pi)$, define the translation variables as follows:

$$p_c(n) = \sum_{j=1}^{n}\phi(j)\cos jc, \quad q_c(n) = \sum_{j=1}^{n}\phi(j)\sin jc, \quad n = 1, 2, 3, \ldots, N. \tag{2.8}$$

To determine the growth of p_c and q_c, it is convenient to look at the mean square displacement (MSD), defined as

$$M_c(n) = \lim_{N \to \infty}\frac{1}{N}\sum_{j=1}^{N}\left[\left(p_c(j + n) - p_c(j)\right)^2 + \left(q_c(j + n) - q_c(j)\right)^2\right]. \tag{2.9}$$

The test for chaos is based on the asymptotic growth rate k_c of $M_c(n)$ as a function of n, which can be estimated as follows:

$$k_c = \lim_{n \to \infty}\frac{\log M_c(n)}{\log n}. \tag{2.10}$$

Numerically, k_c is determined by fitting a straight line to the graph of $\log M_c(n)$ versus $\log n$ through minimizing the absolute deviation.

For different c values, different k_c values can be calculated. The underlying dynamics should be examined by whether the median k of k_c values approach 0 or 1. Specifically, k is close to 0, signifying nonchaotic dynamics (i.e., periodic or quasi-periodic), or k is close to 1, signifying chaotic dynamics. Moreover, the dynamics of the translation components (p, q) are corresponding to the underlying dynamics. Bounded trajectories are corresponding

to periodic or quasi-periodic dynamics and Brownian-like (unbounded) trajectories are corresponding to chaotic dynamics.

For a sampled time series $x(t)$ for system (2.6), we have the following results:

(1) $k = 0.9980$ for $\tau = 0.001$, $k = 0.9974$ for $\tau = 0.003$, $k = 0.9978$ for $\tau = 0.01$, indicating that the dynamics of system (2.6) are chaotic. (a), (b), and (c) in Figure 1 depict Brownian-like trajectories, confirming that the dynamics of system (2.6) are chaotic for $\tau = 0.001$, $\tau = 0.003$, $\tau = 0.01$;

(2) $k = 0.0561$ for $\tau = 0.06$, indicating that the dynamics of system (2.6) are nonchaotic. (d) in Figure 1 depicts a bounded trajectory, confirming that the dynamics of system (2.6) is nonchaotic for $\tau = 0.06$.

For a sampled time series $x(t)$ for system (2.7), we have the following results: $k = 0.9977$ for $\tau = 0.03$, $T = 8$, and $k = 0.9978$ for $\tau = 0.03$, $T = 10$, indicating that the dynamics of system (2.7) are chaotic. (a) and (b) in Figure 2 depict Brownian-like trajectories, confirming that the dynamics of system (2.7) are chaotic for $T = 8$, $T = 10$, and $\tau = 0.03$.

3. Synchronization between Two Rank-one Chaotic Systems without and with Delay

The aim of this section is to achieve chaos synchronization for two rank-one chaotic systems without and with delay using linear delayed feedback control method.

We define the drive and response systems as follows:

$$\dot{X}_d = F(t, X_d(t)), \tag{3.1}$$

$$\dot{X}_r = G(t, X_r(t), X_r(t - \tau)) + U(t), \tag{3.2}$$

where $X_d(t) = [X_d^1(t), \ldots, X_d^m(t)]^T \in R^m$, $X_r(t) = [X_r^1(t), \ldots, X_r^n(t)]^T \in R^n$, $\tau \geq 0$, and $U(t) = [u_1(t), \ldots, u_n(t)]^T$ is a controller and that can be called the control function.

For brevity, let us define the synchronous error as follows:

$$[e_1(t), \ldots, e_n(t)]^T = X_r(t) - K_{n \times m} X_d(t), \tag{3.3}$$

where $K_{n \times m}$ is a constant scaling matrix.

Before presenting the main results, we introduce the following Barbalat's lemma and definition.

Lemma 3.1 (Barbalat's Lemma, [25, Lemma 1.2.2, page 4]). *Let f be a nonnegative function defined on $[0, +\infty)$ such that f is integrable on $[0, +\infty)$ and is uniformly continuous on $[0, +\infty)$. Then, $\lim_{t \to +\infty} f(t) = 0$.*

Definition 3.2 (See [2]). The drive system (3.1) and the response system (3.2) are said to realize projective synchronization, if there exists a tracking control $U(t) = [u_1(t), \ldots, u_n(t)]^T$ such that the synchronous errors satisfy

$$\lim_{t \to +\infty} e_i(t) = 0 \quad (i = 1, 2, \ldots, n). \tag{3.4}$$
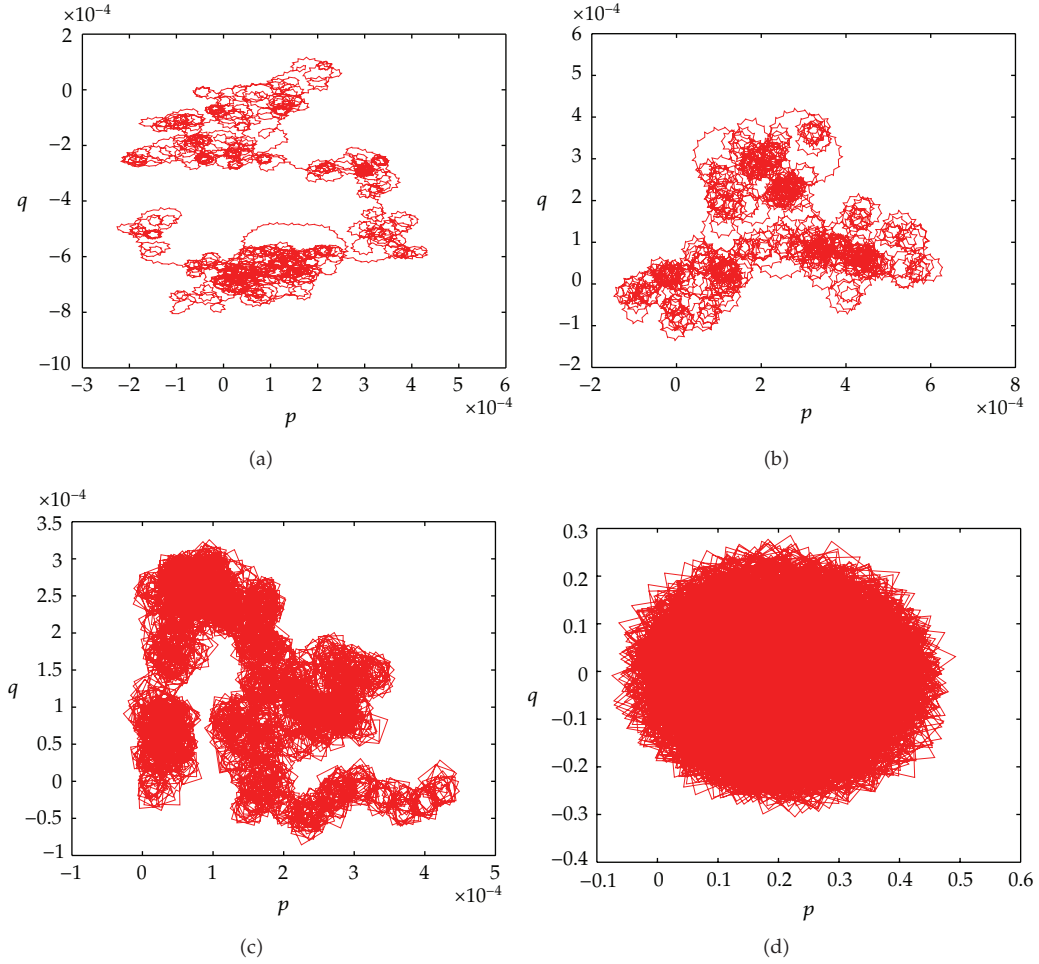
Figure 1: Dynamics of the translation components (p, q) of system (2.6): (a), (b), and (c) show Brownian-like trajectories for $\tau = 0.001$, $\tau = 0.003$, $\tau = 0.01$, respectively; (d) shows a bounded trajectory for $\tau = 0.06$.

Consider the drive system (3.5) and the response system (3.6):

$$\dot{x}_1 = \alpha\left[y_1(t) - \beta_1 x_1(t) - \beta_3 x_1^3(t)\right] + \eta x_1(t) P_{T,p}(t),$$

$$\dot{y}_1 = x_1(t) - y_1(t) + z_1(t), \tag{3.5}$$

$$\dot{z}_1 = -\beta y_1(t),$$

$$\dot{x}_2 = \alpha\left[y_2(t - \tau) - \beta_1 x_2(t) - \beta_3 x_2^3(t)\right] + \eta x_2(t) P_{T,p}(t) + u_1(t),$$

$$\dot{y}_2 = x_2(t - \tau) - y_2(t) + z_2(t - \tau) + u_2(t), \tag{3.6}$$

$$\dot{z}_2 = -\beta y_2(t - \tau) + u_3(t),$$

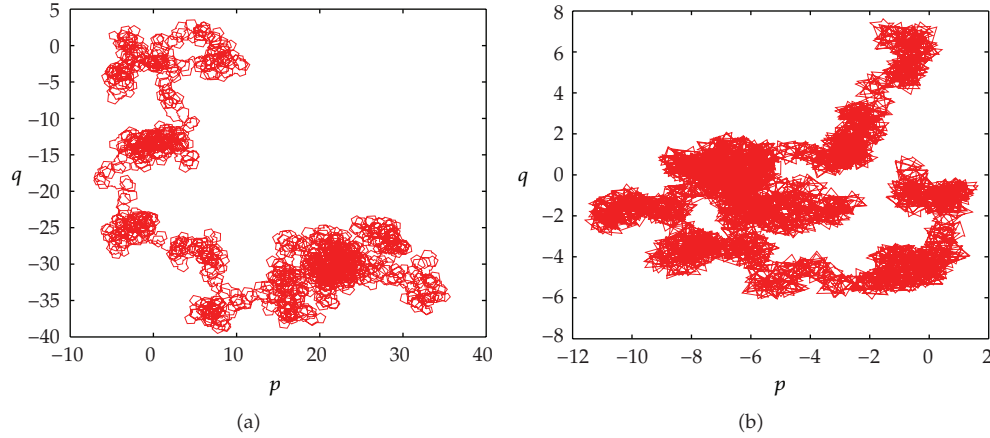where $u(t) = [u_1(t), u_2(t), u_3(t)]^T$ is the control function.

(a)

(b)

Figure 2: Dynamics of the translation components (p, q) of system (2.7) with $\tau = 0.03$: (a) and (b) show Brownian-like trajectories for $T = 8$, $T = 10$, respectively.

In this case, we define the synchronous errors as follows:

$$
\begin{aligned}
e_1(t) &= x_2(t) - k_1 x_1(t), \\
e_2(t) &= y_2(t) - k_2 y_1(t), \\
e_3(t) &= z_2(t) - k_3 z_1(t).
\end{aligned}
\tag{3.7}
$$

Now, we present the main result for the synchronization between system (3.5) and system (3.6).

Theorem 3.3. *If the linear controller $u(t) = [u_1(t), u_2(t), u_3(t)]^T$ is given as follows:*

$$
\begin{aligned}
u_1(t) &= \alpha\big[k_1 y_1(t) - k_2 y_1(t - \tau)\big] - m_1\big[x_2(t) - k_1 x_1(t)\big], \\
u_2(t) &= k_2 x_1(t) - k_1 x_1(t - \tau) + k_2 z_1(t) - k_3 z_1(t - \tau) - m_2\big[y_2(t) - k_2 y_1(t)\big], \\
u_3(t) &= \beta\big[k_2 y_1(t - \tau) - k_3 y_1(t)\big] - m_3\big[z_2(t) - k_3 z_1(t)\big],
\end{aligned}
\tag{3.8}
$$

where $k_1 = \pm 1$, $m_1 > 1 + \eta/p - \alpha\beta_1 - 0.1875\alpha\beta_3$, $m_2 > \alpha + \beta - 1$, $m_3 > 1$. Then, the drive system (3.5) and the response system (3.6) realize projective synchronization.

Proof. Because $|x_1(t)| < 0.25$ for sufficiently large $t \geq t_0^*$ [20], we make the following prior assumption:

$$
|x_2(t)| < 0.25, \quad t \geq t_0 \ (t_0 \geq t_0^*).
\tag{3.9}
$$

Consider the Lyapunov functional defined by

$$
V(t) = |e_1(t)| + |e_2(t)| + |e_3(t)| + \int_{t-\tau}^{t} \big[|e_1(s)| + (\alpha + \beta)|e_2(s)| + |e_3(s)|\big] ds, \quad t \geq t_0.
\tag{3.10}
$$

Calculating the upper right Dini derivative $D^+V(t)$ of $V(t)$ along the solutions of system (3.5) and system (3.6), we get

$$
\begin{aligned}
D^+V(t) \leq & \tilde{D}_1(t) + \tilde{D}_2(t) + \tilde{D}_3(t) + |e_1(t)| - |e_1(t - \tau)| \\
& + (\alpha + \beta)[|e_2(t)| - |e_2(t - \tau)|] + |e_3(t)| - |e_3(t - \tau)|,
\end{aligned}
\tag{3.11}
$$

where

$$
\tilde{D}_i(t) = D^+|e_i(t)| = \begin{cases} \dot{e}_i(t), & e_i(t) > 0, \\ -\dot{e}_i(t), & e_i(t) < 0, \quad (i = 1, 2, 3). \\ |\dot{e}_i(t)|, & e_i(t) = 0, \end{cases}
\tag{3.12}
$$

There are the following three cases to consider for $\tilde{D}_1(t)$.

(i) If $e_1(t) > 0$ (i.e., $x_2(t) > k_1 x_1(t)$), then

$$
\begin{aligned}
\tilde{D}_1(t) = & -(\alpha\beta_1 + m_1)e_1(t) + \alpha e_2(t - \tau) - \alpha\beta_3\left[x_2^3(t) - k_1 x_1^3(t)\right] + \eta e_1(t) P_{T,p}(t) \\
\leq & -\left(\alpha\beta_1 - \frac{\eta}{p} + m_1\right)|e_1(t)| + \alpha|e_2(t - \tau)| - \alpha\beta_3|e_1(t)|\left|x_2^2(t) + k_1 x_1(t)x_2(t) + k_1^2 x_1^2(t)\right| \\
\leq & -\left[\alpha\beta_1 - \frac{\eta}{p} + 0.1875\alpha\beta_3 + m_1\right]|e_1(t)| + \alpha|e_2(t - \tau)|.
\end{aligned}
\tag{3.13}
$$

(ii) If $e_1(t) < 0$ (i.e., $x_2(t) < k_1 x_1(t)$), then

$$
\begin{aligned}
\tilde{D}_1(t) = & (\alpha\beta_1 + m_1)e_1(t) - \alpha e_2(t - \tau) + \alpha\beta_3\left[x_2^3(t) - k_1^3 x_1^3(t)\right] - \eta e_1(t) P_{T,p}(t) \\
\leq & -\left[\alpha\beta_1 - \frac{\eta}{p} + 0.1875\alpha\beta_3 + m_1\right]|e_1(t)| + \alpha|e_2(t - \tau)|.
\end{aligned}
\tag{3.14}
$$

(iii) If $e_1(t) = 0$ (i.e., $x_2(t) = k_1 x_1(t)$), then

$$
\tilde{D}_1(t) = \alpha|e_2(t - \tau)| = -\left[\alpha\beta_1 - \frac{\eta}{p} + 0.1875\alpha\beta_3 + m_1\right]|e_1(t)| + \alpha|e_2(t - \tau)|.
\tag{3.15}
$$

From (i)–(iii), we have

$$
\tilde{D}_1(t) \leq -\left[\alpha\beta_1 - \frac{\eta}{p} + 0.1875\alpha\beta_3 + m_1\right]|e_1(t)| + \alpha|e_2(t - \tau)|, \quad t \geq t_0.
\tag{3.16}
$$

It is easy to see that

$$
\tilde{D}_2(t) = \begin{cases} e_1(t-\tau) - (1+m_2)e_2(t) + e_3(t-\tau), & e_2(t) > 0, \\ -e_1(t-\tau) + (1+m_2)e_2(t) - e_3(t-\tau), & e_2(t) < 0, \\ |e_1(t-\tau) + e_3(t-\tau)|, & e_2(t) = 0, \end{cases}
$$

$$
\tilde{D}_3(t) = \begin{cases} -\beta e_2(t-\tau) - m_3 e_3(t), & e_3(t) > 0, \\ \beta e_2(t-\tau) + m_3 e_3(t), & e_3(t) < 0, \\ \beta |e_2(t-\tau)|, & e_3(t) = 0. \end{cases}
\tag{3.17}
$$

Considering $\tilde{D}_2(t)$, $\tilde{D}_3(t)$ in the same way, we can obtain

$$
\tilde{D}_2(t) \le -(1+m_2)|e_2(t)| + |e_1(t-\tau)| + |e_3(t-\tau)|, \quad t \ge t_0,
$$

$$
\tilde{D}_3(t) \le -m_3|e_3(t)| + \beta|e_2(t-\tau)|, \quad t \ge t_0.
\tag{3.18}
$$

Hence, we have

$$
\begin{aligned}
D^+V(t) &\le \tilde{D}_1(t) + \tilde{D}_2(t) + \tilde{D}_3(t) + |e_1(t)| - |e_1(t-\tau)| \\
&\quad + (\alpha + \beta)\big[|e_2(t)| - |e_2(t-\tau)|\big] + |e_3(t)| - |e_3(t-\tau)| \\
&\le -\left[\alpha\beta_1 - \frac{\eta}{p} + 0.1875\alpha\beta_3 + m_1 - 1\right]|e_1(t)| - (1+m_2-\alpha-\beta)|e_2(t)| \\
&\quad - (m_3-1)|e_3(t)|, \quad t \ge t_0.
\end{aligned}
\tag{3.19}
$$

So there exists $c > 0$ such that

$$
D^+V(t) \le -c(|e_1(t)| + |e_2(t)| + |e_3(t)|), \quad t \ge t_0.
\tag{3.20}
$$

Integrating on both sides of the above inequality leads to

$$
V(t) + c\int_{t_0}^t [|e_1(s)| + |e_2(s)| + |e_3(s)|]ds \le V(t_0) < +\infty, \quad t \ge t_0,
\tag{3.21}
$$

which implies

$$
|e_1(t)| + |e_2(t)| + |e_3(t)| \in L^1[0,+\infty).
\tag{3.22}
$$

From the boundedness of $x_i(t)$, $y_i(t)$, and $z_i(t)$ $(i = 1, 2)$, it follows that $e_1(t), e_2(t), e_3(t)$, and their derivatives remain bounded on $[0, +\infty)$. Hence, $|e_1(t)| + |e_2(t)| + |e_3(t)|$ is uniformly continuous. By Lemma 3.1, it follows that

$$\lim_{t \to +\infty} (|e_1(t)| + |e_2(t)| + |e_3(t)|) = 0. \tag{3.23}$$

Hence,

$$\lim_{t \to +\infty} |e_1(t)| = 0, \qquad \lim_{t \to +\infty} |e_2(t)| = 0, \qquad \lim_{t \to +\infty} |e_3(t)| = 0. \tag{3.24}$$

The proof is complete. □

Remark 3.4. When $k_1 = k_2 = k_3 = 1$, $\tau = 0$, the linear controller $u(t) = [u_1(t), u_2(t), u_3(t)]^T$ is given as follows:

$$u_1(t) = -m_1[x_2(t) - x_1(t)], \qquad u_2(t) = -m_2[y_2(t) - y_1(t)], \qquad u_3(t) = -m_3[z_2(t) - z_1(t)]. \tag{3.25}$$

In this case, the drive system (3.5) and the response system (3.6) realize complete synchronization.

Remark 3.5. When $k_1 = k_2 = k_3 = -1$, $\tau = 0$, the linear controller $u(t) = [u_1(t), u_2(t), u_3(t)]^T$ is given as follows:

$$u_1(t) = -m_1[x_2(t) + x_1(t)], \qquad u_2(t) = -m_2[y_2(t) + y_1(t)], \qquad u_3(t) = -m_3[z_2(t) + z_1(t)]. \tag{3.26}$$

In this case, the drive system (3.5) and the response system (3.6) realize antisynchronization.

Consider the drive system (3.27) and the response system (3.28):

$$\dot{x}_1 = 2Bx_1(t)y_1(t) + \varepsilon\left[x_1(t) + ax_1^2(t)\right] - \varepsilon_1 x_1(t)P_{T,p}(t), \tag{3.27}$$
$$\dot{y}_1 = 1 - 2Ax_1(t) - By_1^2(t),$$

$$\dot{x}_2 = 2Bx_2(t)y_2(t) + \varepsilon\left[x_2(t) + ax_2^2(t)\right] - \varepsilon_1 x_2(t)P_{T,p}(t) + u_1(t), \tag{3.28}$$
$$\dot{y}_2 = 1 - 2Ax_2(t - \tau) - By_2^2(t) + u_2(t),$$

where $u = [u_1(t), u_2(t)]^T$ is the control function.

Our goal is to determine the control function $u = [u_1(t), u_2(t)]^T$. For brevity, let us define

$$e_1(t) = x_2(t) - x_1(t), \qquad e_2(t) = y_2(t) - y_1(t). \tag{3.29}$$

Now, we present the main result for the synchronization between system (3.27) and system (3.28).

Theorem 3.6. *If the linear controller $u = [u_1(t), u_2(t)]^T$ is given as follows:*

$$u_1(t) = -m_1[x_2(t) - x_1(t)],$$
$$u_2(t) = 2A[x_1(t - \tau) - x_1(t)] - m_2[y_2(t) - y_1(t)], \tag{3.30}$$

where $m_1 > 2A + \varepsilon a$, $m_2 > 4B$. Then, e_1, e_2 converge to zero as time t tends to infinity, that is, the drive system (3.27) *and the response system* (3.28) *asymptotically synchronize.*

Proof. Because $0 < x_1(t) < 1$, $|y_1(t)| < 1$ for sufficiently large $t \geq t_0^*$ [21], we make the following prior assumption:

$$0 < x_2(t) < 1, \qquad |y_2(t)| < 1, \qquad t \geq t_0 \ (t_0 \geq t_0^*). \tag{3.31}$$

Consider the Lyapunov functional defined by

$$V(t) = |\ln x_2(t) - \ln x_1(t)| + |y_2(t) - y_1(t)| + 2A \int_{t-\tau}^{t} |x_2(s) - x_1(s)| ds, \quad t \geq t_0. \tag{3.32}$$

Calculating the upper right Dini derivative $D^+V(t)$ of $V(t)$ along the solutions of system (3.27) and system (3.28), we get

$$D^+V(t) \leq \tilde{D}_1(t) + \tilde{D}_2(t) + 2A[|x_2(t) - x_1(t)| - |x_2(t - \tau) - x_1(t - \tau)|], \tag{3.33}$$

where

$$\tilde{D}_1(t) = \begin{cases} \dfrac{\dot{x}_2}{x_2} - \dfrac{\dot{x}_1}{x_1} = 2B[y_2(t) - y_1(t)] + \left[\varepsilon a - \dfrac{m_1}{x_2(t)}\right][x_2(t) - x_1(t)], & x_2(t) > x_1(t), \\[3mm] \dfrac{\dot{x}_1}{x_1} - \dfrac{\dot{x}_2}{x_2} = -2B[y_2(t) - y_1(t)] - \left[\varepsilon a - \dfrac{m_1}{x_2(t)}\right][x_2(t) - x_1(t)], & x_2(t) < x_1(t), \\[3mm] \left|\dfrac{\dot{x}_2}{x_2} - \dfrac{\dot{x}_1}{x_1}\right| = 2B|y_2(t) - y_1(t)|, & x_2(t) = x_1(t), \end{cases}$$

$$\tilde{D}_2(t) = \begin{cases} \dot{y}_2 - \dot{y}_1 = -2A[x_2(t - \tau) - x_1(t - \tau)] - [B(y_2(t) + y_1(t)) + m_2]e_2(t), & y_2(t) > y_1(t), \\[2mm] \dot{y}_1 - \dot{y}_2 = 2A[x_2(t - \tau) - x_1(t - \tau)] + [B(y_2(t) + y_1(t)) + m_2]e_2(t), & y_2(t) < y_1(t), \\[2mm] |\dot{y}_2 - \dot{y}_1| = 2A|x_2(t - \tau) - x_1(t - \tau)|, & y_2(t) = y_1(t). \end{cases} \tag{3.34}$$

There are the following three cases to consider for $\tilde{D}_1(t)$.

(i) If $x_2(t) > x_1(t)$, then

$$\tilde{D}_1(t) = 2B[y_2(t) - y_1(t)] + \left[\varepsilon a - \frac{m_1}{x_2(t)}\right][x_2(t) - x_1(t)]$$

$$\le 2B|y_2(t) - y_1(t)| + (\varepsilon a - m_1)|x_2(t) - x_1(t)|. \tag{3.35}$$

(ii) If $x_2(t) < x_1(t)$, then

$$\tilde{D}_1(t) = -2B[y_2(t) - y_1(t)] - \left[\varepsilon a - \frac{m_1}{x_2(t)}\right][x_2(t) - x_1(t)]$$

$$\le 2B|y_2(t) - y_1(t)| + (\varepsilon a - m_1)|x_2(t) - x_1(t)|. \tag{3.36}$$

(iii) If $x_2(t) = x_1(t)$, then

$$\tilde{D}_1(t) = 2B|y_2(t) - y_1(t)|$$

$$= 2B|y_2(t) - y_1(t)| + (\varepsilon a - m_1)|x_2(t) - x_1(t)|. \tag{3.37}$$

From (i)–(iii), we have

$$\tilde{D}_1(t) \le (\varepsilon a - m_1)|x_2(t) - x_1(t)| + 2B|y_2(t) - y_1(t)|, \quad t \ge t_0. \tag{3.38}$$

Considering $\tilde{D}_2(t)$ in the same way, we can obtain

$$\tilde{D}_2(t) \le 2A|x_2(t - \tau) - x_1(t - \tau)| + (2B - m_2)|y_2(t) - y_1(t)|, \quad t \ge t_0. \tag{3.39}$$

Hence, we have

$$D^+V(t) \le \tilde{D}_1(t) + \tilde{D}_2(t) + 2A[|x_2(t) - x_1(t)| - |x_2(t - \tau) - x_1(t - \tau)|]$$

$$\le (2A + \varepsilon a - m_1)|x_2(t) - x_1(t)| + (4B - m_2)|y_2(t) - y_1(t)|, \quad t \ge t_0. \tag{3.40}$$

So there exists $\alpha > 0$ such that

$$D^+V(t) \le -\alpha(|x_2(t) - x_1(t)| + |y_2(t) - y_1(t)|), \quad t \ge t_0. \tag{3.41}$$

Integrating on both sides of the above inequality leads to

$$V(t) + \alpha \int_{t_0}^{t} (|x_2(s) - x_1(s)| + |y_2(s) - y_1(s)|)ds \le V(t_0) < +\infty, \quad t \ge t_0, \tag{3.42}$$
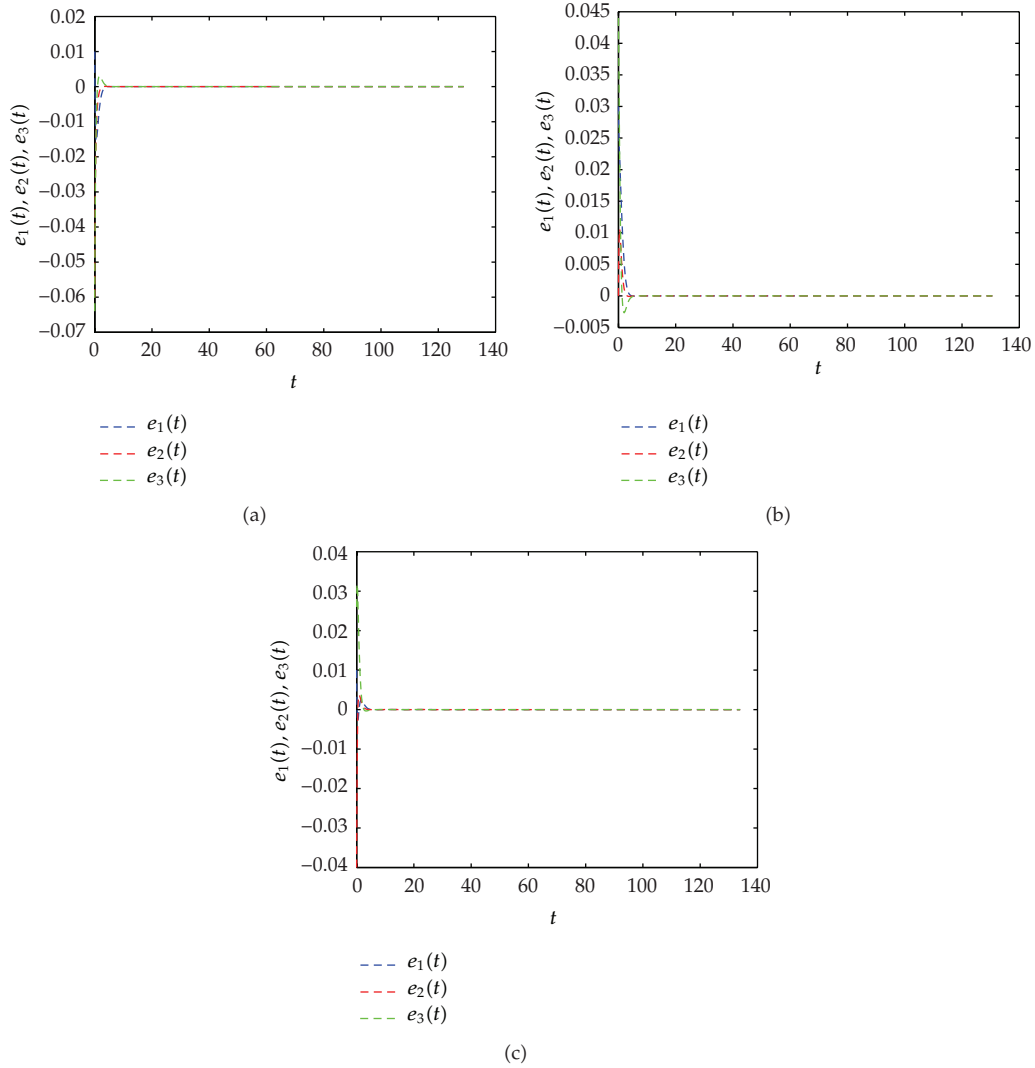
(a)

(b)

(c)

Figure 3: Synchronization errors $e_1(t)$, $e_2(t)$, and $e_3(t)$ for the drive system (3.5) and response system (3.6). (a) $\tau = 0.001$, $m_1 = 1.1761$, $m_2 = 3.31136$, $m_3 = 1.001$, $k_1 = 1$, $k_2 = -3$, $k_3 = -3$. (b) $\tau = 0.003$, $m_1 = 1.1761$, $m_2 = 3.31136$, $m_3 = 1.001$, $k_1 = -1$, $k_2 = 3$, $k_3 = 3$. (c) $\tau = 0.01$, $m_1 = 1.1761$, $m_2 = 3.31136$, $m_3 = 1.001$, $k_1 = 1$, $k_2 = -1$, $k_3 = 2$.

which implies

$$|x_2(t) - x_1(t)| + |y_2(t) - y_1(t)| \in L^1[0, +\infty). \tag{3.43}$$

From the boundedness of $x_i(t)$ and $y_i(t)(i = 1, 2)$, it follows that $x_2(t) - x_1(t)$, $y_2(t) - y_1(t)$ and their derivatives remain bounded on $[0, +\infty)$. Hence, $|x_2(t) - x_1(t)| + |y_2(t) - y_1(t)|$ is uniformly continuous. By Lemma 3.1, it follows that

$$\lim_{t \to +\infty} \left(|x_2(t) - x_1(t)| + |y_2(t) - y_1(t)| \right) = 0. \tag{3.44}$$
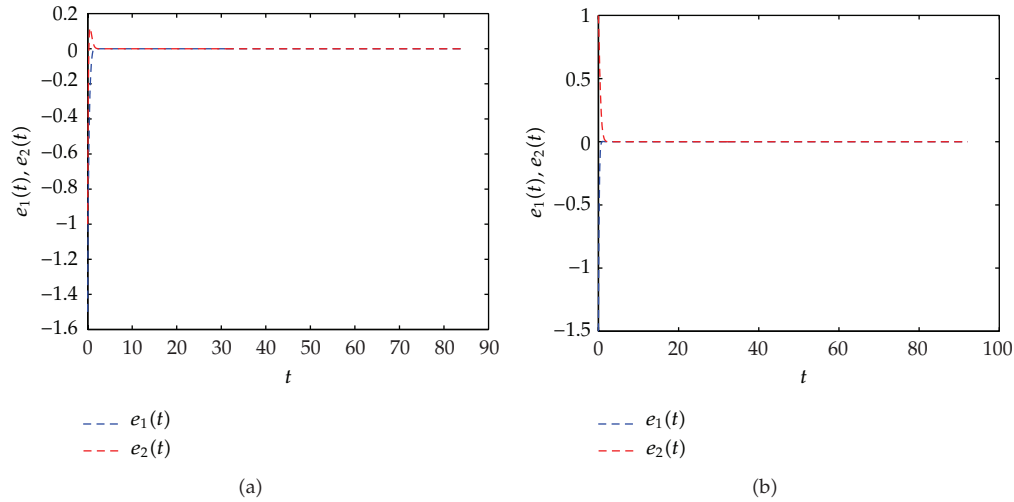
(a)

(b)

Figure 4: Synchronization errors $e_1(t)$, $e_2(t)$ for the drive system (3.27) and response system (3.28). (a) $\tau = 0.03$, $T = 8$, $m_1 = 3.301$, $m_2 = 4.1$. (b) $\tau = 0.03$, $T = 10$, $m_1 = 3.231$, $m_2 = 4.01$.

Hence,

$$\lim_{t \to +\infty} |x_2(t) - x_1(t)| = 0, \quad \lim_{t \to +\infty} |y_2(t) - y_1(t)| = 0. \tag{3.45}$$

This result implies that $e_1(t)$, $e_2(t)$ converge to zero as time t tends to infinity, that is, the drive system (3.27) and the response system (3.28) asymptotically synchronize. The proof is complete. ☐

4. Numerical Simulations

In this section, simulation results are given below to illustrate the validity of the proposed method. Numerical simulations are carried out using MATLAB. DDE23 solver is adopted to solve delay differential equations in Section 3.

The initial values of the drive and response systems in simulations are chosen as follows:

(1) the initial values of systems (3.5) and (3.6): $x_1(s) = 0.01$, $y_1(s) = -0.01$, $z_1(s) = -0.018$ and $x_2(s) = 0.02$, $y_2(s) = -0.03$, $z_2(s) = -0.01$, for all $s \in [-\tau, 0]$;

(2) the initial values of systems (3.27) and (3.28): (i) when $T = 8$, $x_1(s) = 0.85$, $y_1(s) = 0.5$ and $x_2(s) = -0.65$, $y_2(s) = -0.5$, for all $s \in [-\tau, 0]$. (ii) when $T = 10$, $x_1(s) = 0.85$, $y_1(s) = -0.5$ and $x_2(s) = -0.65$, $y_2(s) = 0.5$, for all $s \in [-\tau, 0]$.

The dynamics of synchronization errors for the drive and response systems are shown in Figures 3 and 4. It can be seen in these figures that the synchronization errors will rapidly converge to zero, that is to say, with the passage of time, by means of the linear control method, rank-one chaotic systems without and with delay are ultimately synchronous.

5. Conclusions

In this paper, the presence of chaos in rank-one chaotic systems with delay are confirmed via a binary test (called 0-1 test) for chaos. The synchronization problems for rank-one chaotic systems are discussed. The linear delayed feedback control is used to achieve the synchronization between two rank-one chaotic systems without and with delay. The numerical simulations are in excellent agreement with the analytical results.

Acknowledgments

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