Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 327434, 17 pages
doi:10.1155/2012/327434

Research Article

Common Fixed Points for Asymptotic Pointwise Nonexpansive Mappings in Metric and Banach Spaces

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Received 26 October 2011; Accepted 8 December 2011

Academic Editor: Rudong Chen

Let C be a nonempty bounded closed convex subset of a complete CAT(0) space X. We prove that the common fixed point set of any commuting family of asymptotic pointwise nonexpansive mappings on C is nonempty closed and convex. We also show that, under some suitable conditions, the sequence $\{x_k\}_{k=1}^{\infty}$ defined by $x_{k+1} = (1-t_{mk})x_k \oplus t_{mk}T_m^{n_k}y_{(m-1)k}$, $y_{(m-1)k} = (1-t_{(m-1)k})x_k \oplus t_{(m-1)k}T_{m-1}^{n_k}y_{(m-2)k}, y_{(m-2)k} = (1-t_{(m-2)k})x_k \oplus t_{(m-2)k}T_{m-2}^{n_k}y_{(m-3)k}, \ldots, y_{2k} = (1-t_{2k})x_k \oplus t_{2k}T_2^{n_k}y_{1k}, y_{1k} = (1-t_{1k})x_k \oplus t_{1k}T_1^{n_k}y_{0k}, y_{0k} = x_k$, $k \in \mathbb{N}$, converges to a common fixed point of $T_1, T_2, \ldots, T_m$ where they are asymptotic pointwise nonexpansive mappings on C, $\{t_{ik}\}_{k=1}^{\infty}$ are sequences in $[0,1]$ for all $i = 1, 2, \ldots, m$, and $\{n_k\}$ is an increasing sequence of natural numbers. The related results for uniformly convex Banach spaces are also included.

1. Introduction

A mapping T on a subset C of a Banach space X is said to be asymptotic pointwise nonexpansive if there exists a sequence of mappings $\alpha_n : C \to [0,\infty)$ such that

$$\|T^n x - T^n y\| \le \alpha_n(x)\|x - y\|, \tag{1.1}$$

where $\limsup_{n\to\infty}\alpha_n(x) \le 1$, for all $x, y \in C$. This class of mappings was introduced by Kirk and Xu [1], where it was shown that if C is a bounded closed convex subset of a uniformly convex Banach space X, then every asymptotic pointwise nonexpansive mapping $T : C \to C$ always has a fixed point. In 2009, Hussain and Khamsi [2] extended Kirk-Xu's result to the case of metric spaces, specifically to the so-called CAT(0) spaces. Recently, Kozlowski [3] defined an iterative sequence for an asymptotic pointwise nonexpansive mapping $T : C \to C$ by $x_1 \in C$ and

$$\begin{aligned} x_{k+1} &= (1-t_k)x_k + t_k T^{n_k} y_k, \\ y_k &= (1-s_k)x_k + s_k T^{n_k} x_k, \quad k \in \mathbb{N}, \end{aligned} \tag{1.2}$$

where $\{t_k\}$ and $\{s_k\}$ are sequences in $[0,1]$ and $\{n_k\}$ is an increasing sequence of natural numbers. He proved, under some suitable assumptions, that the sequence $\{x_k\}$ defined by (1.2) converges weakly to a fixed point of T where X is a uniformly convex Banach space which satisfies the Opial condition and $\{x_k\}$ converges strongly to a fixed point of T provided T^r is a compact mapping for some $r \in \mathbb{N}$. On the other hand, Khan et al. [4] studied the iterative process defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_{mn})x_n + \alpha_{mn} T_m^n y_{(m-1)n}, \\ y_{(m-1)n} &= \left(1-\alpha_{(m-1)n}\right)x_n + \alpha_{(m-1)n} T_{m-1}^n y_{(m-2)n}, \\ y_{(m-2)n} &= \left(1-\alpha_{(m-2)n}\right)x_n + \alpha_{(m-2)n} T_{m-2}^n y_{(m-3)n}, \\ &\vdots \\ y_{2n} &= (1-\alpha_{2n})x_n + \alpha_{2n} T_2^n y_{1n}, \\ y_{1n} &= (1-\alpha_{1n})x_n + \alpha_{1n} T_1^n y_{0n}, \\ y_{0n} &= x_n, \quad n \in \mathbb{N}, \end{aligned} \tag{1.3}$$

where $T_1, \dots, T_m$ are asymptotically quasi-nonexpansive mappings on C and $\{\alpha_{in}\}_{n=1}^{\infty}$ are sequences in $[0,1]$ for all $i = 1, 2, \dots, m$.

In this paper, motivated by the results mentioned above, we ensure the existence of common fixed points for a family of asymptotic pointwise nonexpansive mappings in a CAT(0) space. Furthermore, we obtain $\triangle$ and strong convergence theorems of a sequence defined by

$$\begin{aligned} x_{k+1} &= (1-t_{mk})x_k \oplus t_{mk} T_m^{n_k} y_{(m-1)k}, \\ y_{(m-1)k} &= \left(1-t_{(m-1)k}\right)x_k \oplus t_{(m-1)k} T_{m-1}^{n_k} y_{(m-2)k}, \\ y_{(m-2)k} &= \left(1-t_{(m-2)k}\right)x_k \oplus t_{(m-2)k} T_{m-2}^{n_k} y_{(m-3)k}, \\ &\vdots \\ y_{2k} &= (1-t_{2k})x_k \oplus t_{2k} T_2^{n_k} y_{1k}, \\ y_{1k} &= (1-t_{1k})x_k \oplus t_{1k} T_1^{n_k} y_{0k}, \\ y_{0k} &= x_k, \quad k \in \mathbb{N}, \end{aligned} \tag{1.4}$$

where $T_1, \dots, T_m$ are asymptotic pointwise nonexpansive mappings on a subset C of a complete CAT(0) space and $\{t_{ik}\}_{k=1}^{\infty}$ are sequences in $[0,1]$ for all $i = 1, 2, \dots, m$, and $\{n_k\}$ is an increasing sequence of natural numbers. We also note that our method can be used to prove the analogous results for uniformly convex Banach spaces.

2. Preliminaries

A metric space X is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as "thin" as its comparison triangle in the Euclidean plane. It is well-known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples include Pre-Hilbert spaces (see [5]), $\mathbb{R}$-trees (see [6]), Euclidean buildings (see [7]), and the complex Hilbert ball with a hyperbolic metric (see [8]). For a thorough discussion of these spaces and of the fundamental role they play in geometry, we refer the reader to Bridson and Haefliger [5].

Fixed point theory in CAT(0) spaces was first studied by Kirk (see [9, 10]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then the fixed point theory for single-valued and multivalued mappings in CAT(0) spaces has been rapidly developed, and many papers have appeared (see, e.g., [2, 11–22] and the references therein). It is worth mentioning that fixed point theorems in CAT(0) spaces (specially in $\mathbb{R}$-trees) can be applied to graph theory, biology, and computer science (see, e.g., [6, 23–26]).

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x,\ c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a *geodesic* (or *metric*) *segment* joining x and y. When it is unique, this geodesic is denoted by $[x, y]$. The space (X, d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subset X$ is said to be *convex* if Y includes every geodesic segment joining any two of its points.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic space (X, d) consists of three points x_1, x_2, x_3 in X (the *vertices* of $\triangle$) and a geodesic segment between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic space is said to be a CAT(0) space if all geodesic triangles satisfy the following comparison axiom.

CAT(0): Let $\triangle$ be a geodesic triangle in X, and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then, $\triangle$ is said to satisfy the CAT(0) *inequality* if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x, y) \le d_{\mathbb{E}^2}(\overline{x}, \overline{y}). \tag{2.1}$$

Let $x, y \in X$, by Lemma 2.1(iv) of [14] for each $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that

$$d(x, z) = t d(x, y), \qquad d(y, z) = (1 - t) d(x, y). \tag{2.2}$$

We will use the notation $(1 - t)x \oplus ty$ for the unique point z satisfying (2.2). We now collect some elementary facts about CAT(0) spaces.

Lemma 2.1. *Let X be a complete CAT(0) space.*

(i) *[5, Proposition 2.4] If C is a nonempty closed convex subset of X, then, for every $x \in X$, there exists a unique point $P(x) \in C$ such that $d(x, P(x)) = \inf\{d(x, y) : y \in C\}$. Moreover, the map $x \mapsto P(x)$ is a nonexpansive retract from X onto C.*

(ii) *[14, Lemma 2.4] For $x, y, z \in X$ and $t \in [0, 1]$, we have*

$$d\big((1-t)x \oplus ty, z\big) \leq (1-t)d(x, z) + td\big(y, z\big). \tag{2.3}$$

(iii) *[14, Lemma 2.5] For $x, y, z \in X$ and $t \in [0, 1]$, we have*

$$d\big((1-t)x \oplus ty, z\big)^2 \leq (1-t)d(x, z)^2 + td\big(y, z\big)^2 - t(1-t)d\big(x, y\big)^2. \tag{2.4}$$

We now give the concept of Δ-convergence and collect some of its basic properties. Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n). \tag{2.5}$$

The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}, \tag{2.6}$$

and the *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}. \tag{2.7}$$

It is known from Proposition 7 of [27] that, in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point.

Definition 2.2 (see [28, 29]). A sequence $\{x_n\}$ in a CAT(0) space X is said to Δ-*converge* to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_n x_n = x$ and call x the Δ-*limit* of $\{x_n\}$.

Lemma 2.3. *Let X be a complete CAT(0) space.*

(i) *[28, page 3690] Every bounded sequence in X has a Δ-convergent subsequence.*

(ii) *[30, Proposition 2.1] If C is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.*

(iii) *[14, Lemma 2.8] If $\{x_n\}$ is a bounded sequence in a complete CAT(0) space with $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

Recall that a mapping $T : X \to X$ is said to be *nonexpansive* [31] if

$$d(Tx, Ty) \leq d(x, y), \quad \forall x, y \in X, \tag{2.8}$$

where T is called *asymptotically nonexpansive* [32] if there is a sequence $\{k_n\}$ of positive numbers with the property $\lim_{n\to\infty} k_n = 1$ and such that

$$d(T^n x, T^n y) \leq k_n d(x,y), \quad \forall n \geq 1,\ x,y \in X, \tag{2.9}$$

where T is called an *asymptotic pointwise nonexpansive mapping* [1] if there exists a sequence of functions $\alpha_n : X \to [0,\infty)$ such that

$$d(T^n x, T^n y) \leq \alpha_n(x) d(x,y), \quad \forall n \geq 1,\ x,y \in X, \tag{2.10}$$

where $\limsup_{n\to\infty} \alpha_n(x) \leq 1$. The following implications hold.

$$\begin{aligned} T \ \text{ is nonexpansive} &\Longrightarrow T \ \text{ is asymptotically nonexpansive} \\ &\Longrightarrow T \ \text{ is asymptotic pointwise nonexpansive.} \end{aligned} \tag{2.11}$$

A point $x \in X$ is called a fixed point of T if $x = Tx$. We shall denote by $F(T)$ the set of fixed points of T. The existence of fixed points for asymptotic pointwise nonexpansive mappings in CAT(0) spaces was proved by Hussain and Khamsi [2] as the following result.

Theorem 2.4. *Let C be a nonempty bounded closed convex subset of a complete CAT(0) space X. Suppose that $T : C \to C$ is an asymptotic pointwise nonexpansive mapping. Then, $F(T)$ is nonempty closed and convex.*

3. Existence Theorems

Let M be a metric space and $\mathcal{F}$ a family of subsets of M. Then, we say that $\mathcal{F}$ defines a *convexity structure* on M if it contains the closed balls and is stable by intersection.

Definition 3.1 (see [2]). Let $\mathcal{F}$ be a convexity structure on M. We will say that $\mathcal{F}$ is *compact* if any family $\{A_\alpha\}_{\alpha\in\Gamma}$ of elements of $\mathcal{F}$ has a nonempty intersection provided $\bigcap_{\alpha\in F} A_\alpha \neq \emptyset$ for any finite subset $F \subset \Gamma$.

Let X be a complete CAT(0) space. We denote by $\mathcal{C}(X)$ the family of all closed convex subsets of X. Then, $\mathcal{C}(X)$ is a compact convexity structure on X (see, e.g., [2]).

The following theorem is an extension of Theorem 4.3 in [33]. For an analog of this result in uniformly convex Banach spaces, see [34].

Theorem 3.2. *Let C be a nonempty bounded closed and convex subset of a complete CAT(0) space X. Then, for any commuting family $\mathcal{S}$ of asymptotic pointwise nonexpansive mappings on C, the set $\mathcal{F}(\mathcal{S})$ of common fixed points of $\mathcal{S}$ is nonempty closed and convex.*

Proof. Let $\mathcal{T}$ be the family of all finite intersections of the fixed point sets of mappings in the commutative family $\mathcal{S}$. We first show that $\mathcal{T}$ has the finite intersection property. Let $T_1, T_2, \ldots, T_n \in \mathcal{S}$. By Theorem 2.4, $F(T_1)$ is a nonempty closed and convex subset of C. We assume that $A := \bigcap_{j=1}^{k-1} F(T_j)$ is nonempty closed and convex for some $k \in \mathbb{N}$ with $1 < k \le n$. For $x \in A$ and $j \in \mathbb{N}$ with $1 \le j < k$, we have

$$T_k(x) = T_k \circ T_j(x) = T_j \circ T_k(x). \tag{3.1}$$

Thus, $T_k(x)$ is a fixed point of T_j, which implies that $T_k(x) \in A$; therefore, A is invariant under T_k. Again, by Theorem 2.4, T_k has a fixed point in A, that is,

$$\bigcap_{j=1}^{k} F(T_j) = F(T_k) \bigcap A \neq \emptyset. \tag{3.2}$$

By induction, $\bigcap_{j=1}^{n} F(T_j) \neq \emptyset$. Hence, $\mathcal{T}$ has the finite intersection property. Since $\mathcal{C}(X)$ is compact,

$$\mathcal{F}(\mathcal{S}) = \bigcap_{T \in \mathcal{T}} T \neq \emptyset. \tag{3.3}$$

Obviously, the set is closed and convex. □

As a consequence of Lemma 2.1(i) and Theorem 3.2, we obtain an analog of Bruck's theorem [35].

Corollary 3.3. *Let C be a nonempty bounded closed and convex subset of a complete CAT(0) space X. Then, for any commuting family $\mathcal{S}$ of nonexpansive mappings on C, the set $\mathcal{F}(\mathcal{S})$ of common fixed points of $\mathcal{S}$ is a nonempty nonexpansive retract of C.*

4. Convergence Theorems

Throughout this section, X stands for a complete CAT(0) space. Let C be a closed convex subset of X. We shall denote by $\mathcal{T}(C)$ the class of all asymptotic pointwise nonexpansive mappings from C into C. Let $T_1, \ldots, T_m \in \mathcal{T}(C)$, without loss of generality, we can assume that there exists a sequence of mappings $\alpha_n : C \to [0, \infty)$ such that for all $x, y \in C$, $i = 1, \ldots, m$, and $n \in \mathbb{N}$, we have

$$d(T_i^n x, T_i^n y) \le \alpha_n(x) d(x, y), \qquad \limsup_{n \to \infty} \alpha_n(x) \le 1. \tag{4.1}$$

Let $a_n(x) = \max\{\alpha_n(x), 1\}$. Again, without loss of generality, we can assume that

$$d(T_i^n x, T_i^n y) \le a_n(x) d(x, y), \quad \lim_{n \to \infty} a_n(x) = 1, \quad a_n(x) \ge 1, \tag{4.2}$$

for all $x, y \in C$, $i = 1, \ldots, m$, and $n \in \mathbb{N}$. We define $b_n(x) = a_n(x) - 1$, then, for each $x \in C$, we have $\lim_{n \to \infty} b_n(x) = 0$.

The following definition is a mild modification of [3, Definition 2.3].

Definition 4.1. Define $\mathfrak{T}_r(C)$ as a class of all $T \in \mathfrak{T}(C)$ such that

$$\sum_{n=1}^{\infty} \sup_{x\in C} b_n(x) < \infty, \tag{4.3}$$

a_n is a bounded function for every $n \in \mathbb{N}$.

Let $T_1, \ldots, T_m \in \mathfrak{T}_r(C)$, and let $\{t_{ik}\}_{k=1}^{\infty} \subset (0,1)$ be bounded away from 0 and 1 for all $i = 1, 2, \ldots, m$, and $\{n_k\}$ an increasing sequence of natural numbers. Let $x_1 \in C$, and define a sequence $\{x_k\}$ in C as

$$\begin{aligned}
x_{k+1} &= (1 - t_{mk})x_k \oplus t_{mk} T_m^{n_k} y_{(m-1)k}, \\
y_{(m-1)k} &= \left(1 - t_{(m-1)k}\right)x_k \oplus t_{(m-1)k} T_{m-1}^{n_k} y_{(m-2)k}, \\
y_{(m-2)k} &= \left(1 - t_{(m-2)k}\right)x_k \oplus t_{(m-2)k} T_{m-2}^{n_k} y_{(m-3)k}, \\
&\vdots \\
y_{2k} &= (1 - t_{2k})x_k \oplus t_{2k} T_2^{n_k} y_{1k}, \\
y_{1k} &= (1 - t_{1k})x_k \oplus t_{1k} T_1^{n_k} y_{0k}, \\
y_{0k} &= x_k, \quad k \in \mathbb{N}.
\end{aligned} \tag{4.4}$$

We say that the sequence $\{x_k\}$ in (4.4) is well defined if $\limsup_{k\to\infty} a_{n_k}(x_k) = 1$. As in [3], we observe that $\lim_{k\to\infty} a_k(x) = 1$ for every $x \in C$. Hence, we can always choose a subsequence $\{a_{n_k}\}$ which makes $\{x_k\}$ well defined.

Lemma 4.2 (see [36, Lemma 2.2]). *Let $\{a_n\}$ and $\{u_n\}$ be sequences of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 + u_n)a_n, \quad \forall n \in \mathbb{N}, \ \sum_{n=1}^{\infty} u_n < \infty. \tag{4.5}$$

Then, (i) $\lim_n a_n$ *exists, (ii) if* $\liminf_n a_n = 0$*, then* $\lim_n a_n = 0$*.*

Lemma 4.3 (see [37, 38]). *Suppose $\{t_n\}$ is a sequence in $[b, c]$ for some $b, c \in (0,1)$ and $\{u_n\}$, $\{v_n\}$ are sequences in X such that* $\limsup_n d(u_n, w) \leq r$, $\limsup_n d(v_n, w) \leq r$, *and* $\lim_n d((1-t_n)u_n \oplus t_n v_n, w) = r$ *for some $r \geq 0$. Then,*

$$\lim_{n\to\infty} d(u_n, v_n) = 0. \tag{4.6}$$

Lemma 4.4. *Let C be a nonempty closed convex subset of X and $T_1, \ldots, T_m \in \mathfrak{T}_r(C)$. Let $\{t_{ik}\}_{k=1}^{\infty} \subset [a, b] \subset (0,1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (4.4) is well defined. Assume that $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Then,*

(a) *there exists a sequence $\{v_k\}$ in $[0,\infty)$ such that $\sum_{k=1}^{\infty} v_k < \infty$ and $d(x_{k+1}, p) \leq (1 + v_k)^m d(x_k, p)$, for all $p \in F$ and all $k \in \mathbb{N}$,*

(b) *there exists a constant* $M > 0$ *such that* $d(x_{k+l}, p) \leq Md(x_k, p)$, *for all* $p \in F$ *and* $k, l \in \mathbb{N}$.

Proof. (a) Let $p \in F$ and $v_k = \sup_{x\in C} b_{n_k}(x)$ for all $k \in \mathbb{N}$. Since $\sum_{k=1}^{\infty} \sup_{x\in C} b_{n_k}(x) < \infty$, we have $\sum_{k=1}^{\infty} v_k < \infty$. Now,

$$\begin{aligned} d(y_{1k}, p) &\leq (1-t_{1k})d(x_k, p) + t_{1k} d(T_1^{n_k} x_k, p) \\ &\leq (1-t_{1k})d(x_k, p) + t_{1k}(1 + b_{n_k}(p))d(x_k, p) \\ &= (1 + t_{1k} b_{n_k}(p))d(x_k, p) \\ &\leq (1+v_k)d(x_k, p). \end{aligned} \tag{4.7}$$

Suppose that $d(y_{jk}, p) \leq (1+v_k)^j d(x_k, p)$ holds for some $1 \leq j \leq m-2$. Then,

$$\begin{aligned} d(y_{(j+1)k}, p) &\leq (1-t_{(j+1)k})d(x_k, p) + t_{(j+1)k} d\left(T_{j+1}^{n_k} y_{jk}, p\right) \\ &\leq (1-t_{(j+1)k})d(x_k, p) + t_{(j+1)k}(1 + b_{n_k}(p))d(y_{jk}, p) \\ &\leq (1-t_{(j+1)k})d(x_k, p) + t_{(j+1)k}(1+v_k)^{j+1} d(x_k, p) \\ &= \left[1 - t_{(j+1)k} + t_{(j+1)k}\left(1 + \sum_{r=1}^{j+1} \frac{(j+1)j\cdots(j+2-r)}{r!} v_k^r\right)\right] d(x_k, p) \\ &= \left[1 + t_{(j+1)k} \sum_{r=1}^{j+1} \frac{(j+1)j\cdots(j+2-r)}{r!} v_k^r\right] d(x_k, p) \\ &\leq (1+v_k)^{j+1} d(x_k, p). \end{aligned} \tag{4.8}$$

By induction, we have

$$d(y_{ik}, p) \leq (1+v_k)^i d(x_k, p), \quad \forall i = 1, 2, \ldots, m-1. \tag{4.9}$$

This implies

$$\begin{aligned} d(x_{k+1}, p) &\leq (1-t_{mk})d(x_k, p) + t_{mk} d(T_m^{n_k} y_{(m-1)k}, p) \\ &\leq (1-t_{mk})d(x_k, p) + t_{mk}(1 + b_{n_k}(p))d(y_{(m-1)k}, p) \\ &\leq (1-t_{mk})d(x_k, p) + t_{mk}(1+v_k)(1+v_k)^{m-1} d(x_k, p) \\ &\leq (1-t_{mk})d(x_k, p) + t_{mk}(1+v_k)^m d(x_k, p) \\ &= \left[1 - t_{mk} + t_{mk}\left(1 + \sum_{r=1}^{m} \frac{m(m-1)\cdots(m-r+1)}{r!} v_k^r\right)\right] d(x_k, p) \\ &= \left[1 + t_{mk} \sum_{r=1}^{m} \frac{m(m-1)\cdots(m-r+1)}{r!} v_k^r\right] d(x_k, p) \\ &\leq (1+v_k)^m d(x_k, p). \end{aligned} \tag{4.10}$$

This completes the proof of (a).

(b) We observe that $(1+\alpha)^n \le e^{n\alpha}$ holds for all $n \in \mathbb{N}$ and $\alpha \ge 0$. Thus, by (a), for $k, l \in \mathbb{N}$, we have

$$\begin{aligned} d(x_{k+l}, p) &\le (1+v_{k+l-1})^m d(x_{k+l-1}, p) \\ &\le \exp\{m v_{k+l-1}\} d(x_{k+l-1}, p) \le \cdots \le \exp\left\{ m \sum_{i=1}^{k+l-1} v_i \right\} d(x_k, p) \\ &\le \exp\left\{ m \sum_{i=1}^{\infty} v_i \right\} d(x_k, p). \end{aligned} \tag{4.11}$$

The proof is complete by setting $M = \exp\{m \sum_{i=1}^{\infty} v_i\}$. □

Theorem 4.5. *Let C be a nonempty closed convex subset of X and $T_1, \ldots, T_m \in \mathfrak{T}_r(C)$. Let $\{t_{ik}\}_{k=1}^{\infty} \subset [a,b] \subset (0,1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (4.4) is well defined. Assume that $F \neq \emptyset$. Then, $\{x_k\}$ converges to some point in F if and only if $\liminf_{k\to\infty} d(x_k, F) = 0$, where $d(x, F) = \inf_{p\in F} d(x, p)$.*

Proof. The necessity is obvious. Now, we prove the sufficiency. From Lemma 4.4(a), we have

$$d(x_{k+1}, p) \le (1+v_k)^m d(x_k, p), \quad \forall p \in F,\ \forall k \in \mathbb{N}. \tag{4.12}$$

This implies

$$d(x_{k+1}, F) \le (1+v_k)^m d(x_k, F) = \left(1 + \sum_{r=1}^{m} \frac{m(m-1)\cdots(m-r+1)}{r!} v_k^r \right) d(x_k, F). \tag{4.13}$$

Since $\sum_{k=1}^{\infty} v_k < \infty$, then $\sum_{k=1}^{\infty} \sum_{r=1}^{m} (m(m-1)\cdots(m-r+1)/r!) v_k^r < \infty$. By Lemma 4.2(ii), we get $\lim_{k\to\infty} d(x_k, F) = 0$. Next, we show that $\{x_k\}$ is a Cauchy sequence. From Lemma 4.4(b), there exists $M > 0$ such that

$$d(x_{k+l}, p) \le M d(x_k, p), \quad \forall p \in F,\ k, l \in \mathbb{N}. \tag{4.14}$$

Since $\lim_{k\to\infty} d(x_k, F) = 0$, for each $\varepsilon > 0$, there exists $k_1 \in \mathbb{N}$ such that

$$d(x_k, F) < \frac{\varepsilon}{2M}, \quad \forall k \ge k_1. \tag{4.15}$$

Hence, there exists $z_1 \in F$ such that

$$d(x_{k_1}, z_1) < \frac{\varepsilon}{2M}. \tag{4.16}$$

By (4.14) and (4.16), for $k \geq k_1$, we have

$$
\begin{aligned}
d(x_{k+l}, x_k) &\leq d(x_{k+l}, z_1) + d(x_k, z_1) \\
&\leq M d(x_{k_1}, z_1) + M d(x_{k_1}, z_1) \\
&< 2M\left(\frac{\varepsilon}{2M}\right) \\
&= \varepsilon.
\end{aligned}
\tag{4.17}
$$

This shows that $\{x_k\}$ is a Cauchy sequence and so converges to some $q \in C$. We next show that $q \in F$. Let $L = \sup\{a_1(x) : x \in C\}$. Then, for each $\epsilon > 0$, there exists $k_2 \in \mathbb{N}$ such that

$$
d(x_k, q) < \frac{\epsilon}{2(1+L)}, \quad \forall k \geq k_2. \tag{4.18}
$$

Since $\lim_{k\to\infty} d(x_k, F) = 0$, there exists $k_3 \geq k_2$ such that

$$
d(x_k, F) < \frac{\epsilon}{2(1+L)}, \quad \forall k \geq k_3. \tag{4.19}
$$

Thus, there exists $z_2 \in F$ such that

$$
d(x_{k_3}, z_2) < \frac{\epsilon}{2(1+L)}. \tag{4.20}
$$

By (4.18) and (4.20), for each $i = 1, 2, \ldots, m$, we have

$$
\begin{aligned}
d(T_i q, q) &\leq d(T_i q, T_i x_{k_3}) + d(T_i x_{k_3}, z_2) + d(z_2, x_{k_3}) + d(x_{k_3}, q) \\
&\leq L d(x_{k_3}, q) + L d(x_{k_3}, z_2) + d(x_{k_3}, z_2) + d(x_{k_3}, q) \\
&\leq (1+L) d(x_{k_3}, q) + (1+L) d(x_{k_3}, z_2) \\
&< (1+L)\frac{\epsilon}{2(1+L)} + (1+L)\frac{\epsilon}{2(1+L)} \\
&= \epsilon.
\end{aligned}
\tag{4.21}
$$

Since ϵ is arbitrary, we have $T_i q = q$ for all $i = 1, 2, \ldots, m$. Hence, $q \in F$. □

As an immediate consequence of Theorem 4.5, we obtain the following.

Corollary 4.6. *Let C be a nonempty closed convex subset of X and $T_1, \ldots, T_m \in \mathfrak{T}_r(C)$. Let $\{t_{ik}\}_{k=1}^{\infty} \subset [a, b] \subset (0, 1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (4.4) is well defined. Assume that $F \neq \emptyset$. Then, $\{x_k\}$ converges to a point $p \in F$ if and only if there exists a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ which converges to p.*

Definition 4.7. A strictly increasing sequence $\{n_k\} \subset \mathbb{N}$ is called *quasiperiodic* [39] if the sequence $\{n_{k+1} - n_k\}$ is bounded or equivalently if there exists a number $p \in \mathbb{N}$ such that any block of p consecutive natural numbers must contain a term of the sequence $\{n_k\}$. The smallest of such numbers p will be called a quasiperiod of $\{n_k\}$.

Lemma 4.8. *Let C be a nonempty closed convex subset of X and $T_1, \dots, T_m \in \mathfrak{T}_r(C)$. Let $\{t_{ik}\}_{k=1}^{\infty} \subset [\delta, 1-\delta]$ for some $\delta \in (0, 1/2)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (4.4) is well defined. Then,*

(i) $\lim_{k\to\infty} d(x_k, p)$ *exists for all $p \in F$,*

(ii) $\lim_{k\to\infty} d(x_k, T_j^{n_k} y_{(j-1)k}) = 0$, *for all $j = 1, 2, \dots, m$,*

(iii) *if the set $\mathcal{J} = \{k \in \mathbb{N} : n_{k+1} = 1 + n_k\}$ is quasiperiodic, then* $\lim_{k\to\infty} d(x_k, T_j x_k) = 0$, *for all $j = 1, 2, \dots, m$.*

Proof. (i) Follows from Lemmas 4.2(i) and 4.4(a).

(ii) Let $p \in F$, then, by (i), we have $\lim_{k\to\infty} d(x_k, p)$ exists. Let

$$\lim_{k\to\infty} d(x_k, p) = c. \tag{4.22}$$

By (4.9) and (4.22), we get that

$$\limsup_{k\to\infty} d(y_{jk}, p) \le c, \quad \text{for } 1 \le j \le m-1. \tag{4.23}$$

Note that

$$\begin{aligned}
d(x_{k+1}, p) &\le (1 - t_{mk}) d(x_k, p) + t_{mk} d(T_m^{n_k} y_{(m-1)k}, p) \\
&\le (1 - t_{mk}) d(x_k, p) + t_{mk}(1 + v_k) d(y_{(m-1)k}, p) \\
&\ \vdots \\
&\le (1 - t_{mk} t_{(m-1)k} \cdots t_{(j+1)k})(1 + v_k)^{m-j} d(x_k, p) \\
&\quad + t_{mk} t_{(m-1)k} \cdots t_{(j+1)k} (1 + v_k)^{m-j} d(y_{jk}, p).
\end{aligned} \tag{4.24}$$

Thus,

$$d(x_k, p) \le \frac{d(x_k, p)}{\delta^{m-j}} - \frac{d(x_{k+1}, p)}{\delta^{m-j}(1 + v_k)^{m-j}} + d(y_{jk}, p), \tag{4.25}$$

so that

$$c \le \liminf_{k\to\infty} d(y_{jk}, p), \quad \text{for } 1 \le j \le m-1. \tag{4.26}$$

From (4.23) and (4.26), we have

$$\lim_{k\to\infty} d(y_{jk}, p) = c, \quad \text{for each } j = 1, 2, \dots, m-1. \tag{4.27}$$

That is

$$\lim_{k\to\infty} d\Big((1 - t_{jk}) x_k \oplus t_{jk} T_j^{n_k} y_{(j-1)k}, p\Big) = c, \tag{4.28}$$

for each $j = 1, 2, \dots, m-1$.

We also obtain from (4.23) that

$$\limsup_{k\to\infty} d\left(T_j^{n_k} y_{(j-1)k}, p\right) \le c, \quad \text{for each } j = 1, 2, \dots, m-1. \tag{4.29}$$

By Lemma 4.3, we get that

$$\lim_{k\to\infty} d\left(T_j^{n_k} y_{(j-1)k}, x_k\right) = 0, \quad \text{for each } j = 1, 2, \dots, m-1. \tag{4.30}$$

For the case $j = m$, by (4.1), we have

$$d\left(T_m^{n_k} y_{(m-1)k}, p\right) \le \left(1 + b_{n_k}(p)\right) d\left(y_{(m-1)k}, p\right) \le \left(1 + b_{n_k}(p)\right)\left(1 + v_{n_k}\right)^{m-1} d\left(x_k, p\right). \tag{4.31}$$

But since $\lim_{k\to\infty} d(x_k, p) = c$, then

$$\limsup_{k\to\infty} d\left(T_m^{n_k} y_{(m-1)k}, p\right) \le c. \tag{4.32}$$

Moreover,

$$\lim_{k\to\infty} d\left((1 - t_{mk})x_k \oplus t_{mk} T_m^{n_k} y_{(m-1)k}, p\right) = \lim_{k\to\infty} d\left(x_{k+1}, p\right) = c. \tag{4.33}$$

Again, by Lemma 4.3, we get that

$$\lim_{k\to\infty} d\left(T_m^{n_k} y_{(m-1)k}, x_k\right) = 0. \tag{4.34}$$

Thus, (4.30) and (4.34) imply that

$$\lim_{k\to\infty} d\left(T_j^{n_k} y_{(j-1)k}, x_k\right) = 0, \quad \text{for each } j = 1, 2, \dots, m. \tag{4.35}$$

(iii) For $j = 1$, from (ii), we have

$$\lim_{k\to\infty} d\left(T_1^{n_k} x_k, x_k\right) = 0. \tag{4.36}$$

If $j = 2, 3, \dots, m$, then we have

$$\begin{aligned} d\left(T_j^{n_k} x_k, x_k\right) &\le d\left(T_j^{n_k} x_k, T_j^{n_k} y_{(j-1)k}\right) + d\left(T_j^{n_k} y_{(j-1)k}, x_k\right) \\ &\le a_{n_k}(x_k) d\left(x_k, y_{(j-1)k}\right) + d\left(T_j^{n_k} y_{(j-1)k}, x_k\right) \\ &\le a_{n_k}(x_k) t_{(j-1)k} d\left(x_k, T_{j-1}^{n_k} y_{(j-2)k}\right) + d\left(T_j^{n_k} y_{(j-1)k}, x_k\right). \end{aligned} \tag{4.37}$$

By (ii) and $\limsup_{k\to\infty} a_{n_k}(x_k) = 1$, we get

$$\limsup_{k\to\infty} d\left(T_j^{n_k} x_k, x_k\right) = 0, \quad \text{for } j = 2, 3, \ldots, m. \tag{4.38}$$

By (4.36) and (4.38), we have

$$\lim_{k\to\infty} d\left(T_j^{n_k} x_k, x_k\right) = 0, \quad \forall j = 1, 2, \ldots, m. \tag{4.39}$$

By the construction of the sequence $\{x_k\}$, we have from (4.35) that

$$\lim_{k\to\infty} d(x_{k+1}, x_k) = 0. \tag{4.40}$$

Next, we show that

$$\lim_{k\to\infty} d(T_j x_k, x_k) = 0, \quad \forall j = 1, 2, \ldots, m. \tag{4.41}$$

It is enough to prove that $d(T_j x_k, x_k) \to 0$ as $k \to \infty$ though $\mathcal{Q}$. Indeed, let p be a quasiperiod of $\mathcal{Q}$, and let $\varepsilon > 0$ be given. Then, there exists $N_1 \in \mathbb{N}$ such that

$$\lim_{k\to\infty} d(T_j x_k, x_k) < \frac{\varepsilon}{3}, \quad \forall k \in \mathcal{Q} \text{ such that } k \geq N_1. \tag{4.42}$$

By the quasiperiodicity of $\mathcal{Q}$, for each $l \in \mathbb{N}$, there exists $i_l \in \mathcal{Q}$ such that $|l - i_l| \leq p$. Without loss of generality, we can assume that $l \leq i_l \leq l + p$ (the proof for the other case is identical). Let $M = \sup\{a_1(x) : x \in C\}$. Then, $M \geq 1$. Since $\lim_{l\to\infty} d(x_{l+1}, x_l) = 0$ by (4.40), there exists $N_2 \in \mathbb{N}$ such that

$$d(x_{l+1}, x_l) < \frac{\varepsilon}{3pM}, \quad \forall l \geq N_2. \tag{4.43}$$

This implies that

$$d(x_{i_l}, x_l) \leq d(x_{i_l}, x_{i_l - 1}) + \cdots + d(x_{l+1}, x_l) \leq p\left(\frac{\varepsilon}{3pM}\right) = \frac{\varepsilon}{3M}. \tag{4.44}$$

By the definition of T, we have

$$d(T_j x_{i_l}, T_j x_l) \leq M d(x_{i_l}, x_l) \leq M\left(\frac{\varepsilon}{3M}\right) = \frac{\varepsilon}{3}. \tag{4.45}$$

Let $N = \max\{N_1, N_2\}$. Then, for $l \geq N$, we have from (4.42), (4.44), and (4.45) that

$$d(x_l, T_j x_l) \leq d(x_l, x_{i_l}) + d(x_{i_l}, T_j x_{i_l}) + d(T_j x_{i_l}, T_j x_l) < \frac{\varepsilon}{3M} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \leq \varepsilon. \tag{4.46}$$

To prove that $d(T_j x_k, x_k) \to 0$ as $k \to \infty$ though $\mathcal{J}$. Since $\mathcal{J} = \{k \in \mathbb{N} : n_{k+1} = n_k + 1\}$ is quasiperiodic, for each $k \in \mathcal{J}$, we have

$$\begin{aligned} d(x_k, T_j x_k) &\leq d(x_k, x_{k+1}) + d\left(x_{k+1}, T_j^{n_{k+1}} x_{k+1}\right) + d\left(T_j^{n_{k+1}} x_{k+1}, T_j^{n_{k+1}} x_k\right) + d\left(T_j^{n_k+1} x_k, T_j x_k\right) \\ &\leq d(x_k, x_{k+1}) + d\left(x_{k+1}, T_j^{n_{k+1}} x_{k+1}\right) + a_{n_{k+1}}(x_{k+1}) d(x_{k+1}, x_k) + a_1(x_k) d\left(T_j^{n_k} x_k, x_k\right). \end{aligned} \tag{4.47}$$

From this, together with (4.39) and (4.40), we can obtain that $d(T_j x_k, x_k) \to 0$ as $k \to \infty$ through $\mathcal{J}$. □

The following lemmas can be found in [3] (see also [2]).

Lemma 4.9. *Let C be a nonempty closed convex subset of X, and let $T \in \mathcal{T}_r(C)$. If $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then $\lim_{n\to\infty} d(x_n, T^l x_n) = 0$ for every $l \in \mathbb{N}$.*

Lemma 4.10. *Let C be a nonempty closed convex subset of X, and let $T \in \mathcal{T}_r(C)$. Suppose $\{x_n\}$ is a bounded sequence in C such that $\lim_n d(x_n, Tx_n) = 0$ and $\Delta\text{-}\lim_n x_n = w$. Then, $Tw = w$.*

By using Lemmas 2.3 and 4.10, we can obtain the following result. We omit the proof because it is similar to the one given in [38].

Lemma 4.11. *Let C be a closed convex subset of X, and let $T : C \to C$ be an asymptotic pointwise nonexpansive mapping. Suppose $\{x_n\}$ is a bounded sequence in C such that $\lim_n d(x_n, T(x_n)) = 0$ and $d(x_n, v)$ converges for each $v \in F(T)$, then $\omega_w(x_n) \subset F(T)$. Here, $\omega_w(x_n) = \bigcup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. Moreover, $\omega_w(x_n)$ consists of exactly one point.*

Now, we are ready to prove our Δ-convergence theorem.

Theorem 4.12. *Let C be a nonempty closed convex subset of X and $T_1, \ldots, T_m \in \mathcal{T}_r(C)$. Let $\{t_{ik}\}_{k=1}^{\infty} \subset [\delta, 1-\delta]$ for some $\delta \in (0, 1/2)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (4.4) is well defined. Suppose that $F := \bigcap_{i=1}^m F(T_i) \neq \emptyset$ and the set $\mathcal{J} = \{k \in \mathbb{N} : n_{k+1} = 1 + n_k\}$ is quasiperiodic. Then, $\{x_k\}$ Δ-converges to a common fixed point of the family $\{T_i : i = 1, 2, \ldots, m\}$.*

Proof. Let $p \in F$, by Lemma 4.8, $\lim_{k\to\infty} d(x_k, p)$ existsm and hence $\{x_k\}$ is bounded. Since $\lim_{k\to\infty} d(x_k, T_j x_k) = 0$ for all $j = 1, 2, \ldots, m$, then by Lemma 4.11 $\omega_w(x_k) \subset F(T_j)$ for all $j = 1, 2, \ldots, m$, and hence $\omega_w(x_k) \subset \bigcap_{j=1}^m F(T_j) = F$. Since $\omega_w(x_n)$ consists of exactly one point, then $\{x_k\}$ Δ-converges to an element of F. □

Before proving our strong convergence theorem, we recall that a mapping $T : C \to C$ is said to be *semicompact* if C is closed and, for any bounded sequence $\{x_n\}$ in C with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $x \in C$ such that $\lim_{k\to\infty} x_{n_k} = x$.

Theorem 4.13. *Let C be a nonempty closed convex subset of X and $T_1, \ldots, T_m \in \mathcal{T}_r(C)$ such that T_i^l is semicompact for some $i \in \{1, \ldots, m\}$ and $l \in \mathbb{N}$. Let $\{t_{ik}\}_{k=1}^{\infty} \subset [\delta, 1-\delta]$ for some $\delta \in (0, 1/2)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (4.4) is well defined. Suppose that $F := \bigcap_{i=1}^m F(T_i) \neq \emptyset$ and the set $\mathcal{J} = \{k \in \mathbb{N} : n_{k+1} = 1 + n_k\}$ is quasiperiodic. Then, $\{x_k\}$ converges to a common fixed point of the family $\{T_i : i = 1, 2, \ldots, m\}$.*

Proof. By Lemma 4.8, we have

$$\lim_{k\to\infty} d(x_k, T_i x_k) = 0, \quad \text{for } i = 1, \ldots, m. \tag{4.48}$$

Let $i \in \{1, \ldots, m\}$ be such that T_i^l is semicompact. Thus, by Lemma 4.9,

$$\lim_{k\to\infty} d\left(x_k, T_i^l x_k\right) = 0. \tag{4.49}$$

We can also find a subsequence $\{x_{n_j}\}$ of $\{x_k\}$ such that $\lim_{j\to\infty} x_{k_j} = q \in C$. Hence, from (4.48), we have

$$d(q, T_i q) = \lim_{j\to\infty} d\left(x_{k_j}, T_i x_{k_j}\right) = 0, \quad \forall i = 1, \ldots, m. \tag{4.50}$$

Thus, $q \in F$, and, by Corollary 4.6, $\{x_k\}$ converges to q. This completes the proof. □

5. Concluding Remarks

One may observe that our method can be used to obtain the analogous results for uniformly convex Banach spaces. Let C be a nonempty closed convex subset of a Banach space X and fix $x_1 \in C$. Define a sequence $\{x_k\}$ in C as

$$\begin{aligned}
x_{k+1} &= (1 - t_{mk})x_k + t_{mk} T_m^{n_k} y_{(m-1)k}, \\
y_{(m-1)k} &= \left(1 - t_{(m-1)k}\right)x_k + t_{(m-1)k} T_{m-1}^{n_k} y_{(m-2)k}, \\
y_{(m-2)k} &= \left(1 - t_{(m-2)k}\right)x_k + t_{(m-2)k} T_{m-2}^{n_k} y_{(m-3)k}, \\
&\vdots \\
y_{2k} &= (1 - t_{2k})x_k + t_{2k} T_2^{n_k} y_{1k}, \\
y_{1k} &= (1 - t_{1k})x_k + t_{1k} T_1^{n_k} y_{0k}, \\
y_{0k} &= x_k, \quad k \in \mathbb{N},
\end{aligned} \tag{5.1}$$

where $T_1, \ldots, T_m \in \mathcal{T}_r(C)$, $\{t_{ik}\}_{k=1}^{\infty}$ are sequences in $[0,1]$ for all $i = 1, 2, \ldots, m$, and $\{n_k\}$ is an increasing sequence of natural numbers.

Theorem 5.1. *Let X be a uniformly convex Banach space with the Opial property, and let C be a nonempty closed convex subset of X. Let $T_1, \ldots, T_m \in \mathcal{T}_r(C)$, $\{t_{ik}\}_{k=1}^{\infty} \subset [\delta, 1-\delta]$ for some $\delta \in (0, 1/2)$, and let $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (5.1) is well defined. Suppose that $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$ and the set $\mathcal{J} = \{k \in \mathbb{N} : n_{k+1} = 1 + n_k\}$ is quasiperiodic. Then, $\{x_k\}$ converges weakly to a common fixed point of the family $\{T_i : i = 1, 2, \ldots, m\}$.*

Theorem 5.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $T_1, \ldots, T_m \in \mathcal{T}_r(C)$ such that T_i^l is semicompact for some $i \in \{1, \ldots, m\}$ and $l \in \mathbb{N}$. Let $\{t_{ik}\}_{k=1}^{\infty} \subset [\delta, 1-\delta]$ for some $\delta \in (0, 1/2)$, and let $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (5.1) is well defined. Suppose that $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$ and the set $\mathcal{J} = \{k \in \mathbb{N} : n_{k+1} = 1 + n_k\}$ is quasiperiodic. Then, $\{x_k\}$ converges strongly to a common fixed point of the family $\{T_i : i = 1, 2, \ldots, m\}$.*

Acknowledgments

This research is (partially) supported by the Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand. The first author also thanks the Graduate School of Chiang Mai University, Thailand.

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