Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 472935, 14 pages
doi:10.1155/2012/472935

Research Article

Implicit Iterative Method for Hierarchical Variational Inequalities

L.-C. Ceng,[1] Q. H. Ansari,[2,3] N.-C. Wong,[4] and J.-C. Yao[5]

[1] *Scientific Computing Key Laboratory of Shanghai Universities, Department of Mathematics, Shanghai Normal University, Shanghai 200234, China*
[2] *Department of Mathematics, Aligarh Muslim University, Aligarh 202 002, India*
[3] *Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia*
[4] *Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung 80424, Taiwan*
[5] *Center for General Education, Kaohsiung Medical University, Kaohsiung 80708, Taiwan*

Correspondence should be addressed to N.-C. Wong, wong@math.nsysu.edu.tw

Received 1 February 2012; Accepted 10 February 2012

Academic Editor: Yonghong Yao

We introduce a new implicit iterative scheme with perturbation for finding the approximate solutions of a hierarchical variational inequality, that is, a variational inequality over the common fixed point set of a finite family of nonexpansive mappings. We establish some convergence theorems for the sequence generated by the proposed implicit iterative scheme. In particular, necessary and sufficient conditions for the strong convergence of the sequence are obtained.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and C a nonempty closed convex subset of H. For a given nonlinear operator $A : C \to H$, the classical *variational inequality problem* (VIP) [1] is to find $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{1.1}$$

The set of solutions of VIP is denoted by $\mathrm{VI}(C, A)$. If the set C is replaced by the set $\mathrm{Fix}(T)$ of fixed points of a mapping T; then the VIP is called a *hierarchical variational inequality problem* (HVIP). The signal recovery [2], the power control problem [3], and the beamforming problem [4] can be written in the form of a hierarchical variational inequality problem. In the recent past, several authors paid their attention toward this kind of problem and developed different kinds of solution methods with applications; see [2, 5–11] and the references therein.

Let $F : H \to H$ be *η-strongly monotone* (i.e., if there exists a constant $\eta > 0$ such that $\langle Fx - Fy, x - y\rangle \geq \eta\|x - y\|^2$, for all $x, y \in H$) and *κ-Lipschitz continuous* (i.e., if there exists a constant $\kappa > 0$ such that $\|Fx - Fy\| \leq \kappa\|x - y\|$, for all $x, y \in H$). Assume that C is the intersection of the sets of fixed points of N nonexpansive mappings $T_i : H \to H$. For an arbitrary initial guess $x_0 \in H$, Yamada [10] proposed the following hybrid steepest-descent method:

$$x_{n+1} := T_{n+1}x_n - \lambda_{n+1}\mu F(T_{n+1}x_n), \quad \forall n \geq 0. \tag{1.2}$$

Here, $T_k := T_{k \bmod N}$, for every integer $k > N$, with the mod function taking values in the set $\{1, 2, \ldots, N\}$; that is, if $k = jN + q$ for some integers $j \geq 0$ and $0 \leq q < N$, then $T_k = T_N$ if $q = 0$ and $T_k = T_q$ if $1 < q < N$. Moreover, $\mu \in (0, 2\eta/\kappa^2)$ and the sequence $\{\lambda_n\} \subset (0, 1)$ of parameters satisfies the following conditions:

(i) $\lim_{n\to\infty}\lambda_n = 0$;

(ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$;

(iii) $\sum_{n=0}^{\infty} |\lambda_n - \lambda_{n+N}|$ is convergent.

Under these conditions, Yamada [10] proved the strong convergence of the sequence $\{x_n\}$ to the unique element of $\mathrm{VI}(C, F)$.

Xu and Kim [12] replaced the condition (iii) by the following condition:

(iii)′ $\lim_{n\to\infty}(\lambda_n/\lambda_{n+N}) = 1$, or equivalently, $\lim_{n\to\infty}((\lambda_n - \lambda_{n+N})/\lambda_{n+N}) = 0$

and proved the strong convergence of the sequence $\{x_n\}$ to the unique element of $\mathrm{VI}(C, F)$.

On the other hand, let K be a nonempty convex subset of H, and let $\{T_i\}_{i=1}^{N}$ be a finite family of nonexpansive self-maps on K. Xu and Ori [13] introduced the following implicit iteration process: for $x_0 \in K$ and $\{\alpha_n\}_{n=1}^{\infty} \subset (0, 1)$, the sequence $\{x_n\}_{n=1}^{\infty}$ is generated by the following process:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n)T_n x_n, \quad \forall n \geq 1, \tag{1.3}$$

where we use the convention $T_n := T_{n \bmod N}$. They also studied the weak convergence of the sequence generated by the above scheme to a common fixed point of the mappings $\{T_i\}_{i=1}^{N}$ under certain conditions. Subsequently, Zeng and Yao [14] introduced another implicit iterative scheme with perturbation for finding the approximate common fixed points of a finite family of nonexpansive self-maps on H.

Motivated and inspired by the above works, in this paper, we propose a new implicit iterative scheme with perturbation for finding the approximate solutions of the hierarchical variational inequalities, that is, variational inequality problem over the common fixed point set of a finite family of nonexpansive self-maps on H. We establish some convergence theorems for the sequence generated by the proposed implicit iterative scheme with perturbation. In particular, necessary and sufficient conditions for strong convergence of the sequence generated by the proposed implicit iterative scheme with perturbation are obtained.

2. Preliminaries

Throughout the paper, we write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x in a Banach space E. Meanwhile, $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. For a given sequence $\{x_n\} \subset E$, $\omega_w(x_n)$ denotes the weak ω-limit set of $\{x_n\}$, that is,

$$\omega_w(x_n) := \left\{x \in E : x_{n_j} \rightharpoonup x \text{ for some subsequence } \{n_j\} \text{ of } \{n\}\right\}. \tag{2.1}$$

A Banach space E is said to satisfy *Opial's property* if

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \quad \forall y \in E,\ y \neq x, \tag{2.2}$$

whenever a sequence $x_n \rightharpoonup x$ in E. It is well known that every Hilbert space H satisfies Opial's property; see for example [15].

A mapping $A : H \to H$ is said to be *hemicontinuous* if for any $x, y \in H$, the mapping $g : [0,1] \to H$, defined by $g(t) := A(tx+(1-t)y)$ (for all $t \in [0,1]$), is continuous in the weak topology of the Hilbert space H. The *metric projection* onto a nonempty, closed and convex set $C \subseteq H$, denoted by P_C, is defined by, for all $x \in H$, $P_C x \in C$ and $\|x - P_C x\| = \inf_{y\in C}\|x - y\|$.

Proposition 2.1. *Let $C \subseteq H$ be a nonempty closed and convex set and $A : H \to H$ monotone and hemicontinuous. Then,*

(a) *[1]* $\mathrm{VI}(C, A) = \{x^* \in C : \langle Ay, y - x^*\rangle \geq 0,$ *for all* $y \in C\}$,

(b) *[1]* $\mathrm{VI}(C, A) \neq \emptyset$ *when C is bounded,*

(c) *[16, Lemma 2.24]* $\mathrm{VI}(C, A) = \mathrm{Fix}(P_C(I - \lambda A))$ *for all* $\lambda > 0$, *where I stands for the identity mapping on* H,

(d) *[16, Theorem 2.31]* $\mathrm{VI}(C, A)$ *consists of one point if A is strongly monotone and Lipschitz continuous.*

On the other hand, it is well known that the metric projection P_C onto a given nonempty closed and convex set $C \subseteq H$ is nonexpansive with $\mathrm{Fix}(P_C) = C$ [17, Theorem 3.1.4 (i)]. The fixed point set of a nonexpansive mapping has the following properties.

Proposition 2.2. *Let $C \subseteq H$ be a nonempty closed and convex subset and $T : C \to C$ a nonexpansive map.*

(a) *[18, Proposition 5.3]* $\mathrm{Fix}(T)$ *is closed and convex.*

(b) *[18, Theorem 5.1]* $\mathrm{Fix}(T) \neq \emptyset$ *when C is bounded.*

The following proposition provides an example of a nonexpansive mapping in which the set of fixed points is equal to the solution set of a monotone variational inequality.

Proposition 2.3 (see [6, Proposition 2.3]). *Let $C \subseteq H$ be a nonempty closed and convex set and $A : H \to H$ an α-inverse-strongly monotone operator. Then, for any given $\lambda \in (0, 2\alpha]$, the mapping $S_\lambda : H \to H$, defined by*

$$S_\lambda x := P_C(I - \lambda A)x, \quad \forall x \in H, \tag{2.3}$$

is nonexpansive and $\mathrm{Fix}(S_\lambda) = \mathrm{VI}(C, A)$.

The following lemmas will be used in the proof of the main results of this paper.

Lemma 2.4 (see [18, Demiclosedness Principle]). *Assume that T is a nonexpansive self-mapping on a closed convex subset K of a Hilbert space H. If T has a fixed point, then $I - T$ is demiclosed, that is, whenever $\{x_n\}$ is a sequence in K weakly converging to some $x \in K$ and the sequence $\{(I - T)x_n\}$ strongly converges to some y, it follows that $(I - T)x = y$, where I is the identity operator of H.*

Lemma 2.5 (see [19, page 80]). *Let $\{a_n\}_{n=1}^\infty$, $\{b_n\}_{n=1}^\infty$, and $\{\delta_n\}_{n=1}^\infty$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \le (1 + \delta_n)a_n + b_n, \quad \forall n \ge 1. \tag{2.4}$$

If $\sum_{n=1}^\infty \delta_n < \infty$ and $\sum_{n=1}^\infty b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. If in addition $\{a_n\}_{n=1}^\infty$ has a subsequence, which converges to zero, then $\lim_{n\to\infty} a_n = 0$.

Let $T : H \to H$ be a nonexpansive mapping and $F : H \to H$ κ-Lipschitz continuous and η-strongly monotone for some constants $\kappa > 0$, $\eta > 0$. For any given numbers $\lambda \in [0, 1)$ and $\mu \in (0, 2\eta/\kappa^2)$, we define the mapping $T^\lambda : H \to H$ by

$$T^\lambda x := Tx - \lambda\mu F(Tx), \quad \forall x \in H. \tag{2.5}$$

Lemma 2.6 (see [12]). *If $0 \le \lambda < 1$ and $0 < \mu < 2\eta/\kappa^2$, then*

$$\left\| T^\lambda x - T^\lambda y \right\| \le (1 - \lambda\tau)\|x - y\|, \quad \forall x, y \in H, \tag{2.6}$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0, 1)$.

3. An Iterative Scheme and Convergence Results

Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive self-maps on H. Let $A : H \to H$ be an *α-inverse-strongly monotone* mapping (i.e., if there exists a constant $\alpha > 0$ such that $\langle Ax - Ay, x - y\rangle \ge \alpha\|Ax - Ay\|^2$, for all $x, y \in H$). Let $F : H \to H$ be κ-Lipschitz continuous and η-strongly monotone for some constants $\kappa > 0$, $\eta > 0$. Let $\{\alpha_n\}_{n=1}^\infty \subset (0, 1)$, $\{\beta_n\}_{n=1}^\infty \subset (0, 2\alpha]$, $\{\lambda_n\}_{n=1}^\infty \subset [0, 1)$, and take a fixed number $\mu \in (0, 2\eta/\kappa^2)$. We introduce the following implicit iterative scheme with perturbation F. For an arbitrary initial point $x_0 \in H$, the sequence $\{x_n\}_{n=1}^{\infty}$ is generated by the following process:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)\left[T_n(x_n - \beta_n A x_n) - \lambda_n \mu F \circ T_n(x_n - \beta_n A x_n)\right], \quad \forall n \geq 1. \tag{3.1}$$

Here, we use the convention $T_n := T_{n \bmod N}$. If $A \equiv 0$, then the implicit iterative scheme (3.1) reduces to the implicit iterative scheme studied in [14].

Let $A : H \to H$ be an α-inverse-strongly monotone mapping and $s \in (0, 2\alpha]$. By Lemma 2.6, for every $u \in H$ and $t \in (0,1)$, the mapping $S_t : H \to H$ defined by

$$S_t x := tu + (1-t)T^{\lambda}\widetilde{x} \quad \text{with } \widetilde{x} = x - sAx, \tag{3.2}$$

satisfies

$$\begin{aligned}
\|S_t x - S_t y\| &= (1-t)\left\|T^{\lambda}\widetilde{x} - T^{\lambda}\widetilde{y}\right\| \\
&\leq (1-t)(1-\lambda\tau)\|\widetilde{x} - \widetilde{y}\| \\
&\leq (1-t)\|\widetilde{x} - \widetilde{y}\| \\
&= (1-t)\|(x - sAx) - (y - sAy)\| \\
&= (1-t)\|(x-y) - s(Ax - Ay)\| \\
&\leq (1-t)\sqrt{\|x-y\|^2 - s(2\alpha - s)\|Ax - Ay\|^2} \\
&\leq (1-t)\|x - y\|, \quad \forall x, y \in H,
\end{aligned} \tag{3.3}$$

where $0 \leq \lambda < 1$, $0 < \mu < 2\eta/\kappa^2$, and $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0,1)$. By Banach's contraction principle, there exists a unique $x_t \in H$ such that

$$x_t = tu + (1-t)T^{\lambda}\widetilde{x}_t = tu + (1-t)\left[T(x_t - sAx_t) - \lambda\mu F \circ T(x_t - sAx_t)\right]. \tag{3.4}$$

This shows that the implicit iterative scheme (3.1) with perturbation F is well defined and can be employed for finding the approximate solutions of the variational inequality problem over the common fixed point set of a finite family of nonexpansive self-maps on H.

We now state and prove the main results of this paper.

Theorem 3.1. *Let H be a real Hilbert space, A an α-inverse-strongly monotone mapping, and $F : H \to H$ a κ-Lipschitz continuous and η-strongly monotone mapping for some constants κ, $\eta > 0$. Let $\{T_i\}_{i=1}^{N}$ be N nonexpansive self-maps on H with a nonempty common fixed point set $\bigcap_{i=1}^{N} \mathrm{Fix}(T_i)$. Suppose $\mathrm{VI}(\bigcap_{i=1}^{N} \mathrm{Fix}(T_i), A) \neq \emptyset$. Denote by $T_n := T_{n \bmod N}$ for $n > N$. Let $\mu \in (0, 2\eta/\kappa^2)$, $x_0 \in H$, $\{\lambda_n\}_{n=1}^{\infty} \subset [0,1)$, $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$, and $\{\beta_n\}_{n=1}^{\infty} \subset (0, 2\alpha]$ be such that $\sum_{n=1}^{\infty} \lambda_n < \infty$, $\beta_n \leq \lambda_n$ and $a \leq \alpha_n \leq b$, for all $n \geq 1$, for some $a, b \in (0,1)$. Then, the sequence $\{x_n\}_{n=1}^{\infty}$, defined by*

$$\begin{aligned}
x_n &:= \alpha_n x_{n-1} + (1-\alpha_n)T_n^{\lambda_n}\widetilde{x}_n \\
&= \alpha_n x_{n-1} + (1-\alpha_n)\left[T_n(x_n - \beta_n A x_n) - \lambda_n \mu F \circ T_n(x_n - \beta_n A x_n)\right], \quad \forall n \geq 1,
\end{aligned} \tag{3.5}$$

converges weakly to an element of $\bigcap_{i=1}^{N} \mathrm{Fix}(T_i)$.

If, in addition, $\|x_n - T_n\widetilde{x}_n\| = o(\beta_n)$*, then* $\{x_n\}$ *converges weakly to an element of* VI $(\bigcap_{i=1}^N \mathrm{Fix}(T_i), A)$.

Proof. Notice first that the following identity:

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2 \tag{3.6}$$

holds for all $x, y \in H$ and all $t \in [0,1]$. Let $\widehat{x}$ be an arbitrary element of $\bigcap_{i=1}^N \mathrm{Fix}(T_i)$. Observe that

$$\begin{aligned}\|x_n - \widehat{x}\|^2 &= \left\|\alpha_n x_{n-1} + (1-\alpha_n)T_n^{\lambda_n}\widetilde{x}_n - \widehat{x}\right\|^2 \\ &= \alpha_n\|x_{n-1} - \widehat{x}\|^2 + (1-\alpha_n)\left\|T_n^{\lambda_n}\widetilde{x}_n - \widehat{x}\right\|^2 - \alpha_n(1-\alpha_n)\left\|x_{n-1} - T_n^{\lambda_n}\widetilde{x}_n\right\|^2.\end{aligned} \tag{3.7}$$

Since A is α-inverse strongly monotone and $\{\beta_n\}_{n=1}^\infty \subset (0, 2\alpha]$, we have

$$\begin{aligned}\|x_n - \widehat{x} - \beta_n(Ax_n - A\widehat{x})\|^2 &= \|x_n - \widehat{x}\|^2 - 2\beta_n\langle Ax_n - A\widehat{x}, x_n - \widehat{x}\rangle + \beta_n^2\|Ax_n - A\widehat{x}\|^2 \\ &\le \|x_n - \widehat{x}\|^2 - \beta_n(2\alpha - \beta_n)\|Ax_n - A\widehat{x}\|^2 \\ &\le \|x_n - \widehat{x}\|^2.\end{aligned} \tag{3.8}$$

By Lemma 2.6, we have

$$\begin{aligned}\left\|T_n^{\lambda_n}\widetilde{x}_n - \widehat{x}\right\| &= \left\|T_n^{\lambda_n}\widetilde{x}_n - T_n^{\lambda_n}\widehat{x} + T_n^{\lambda_n}\widehat{x} - \widehat{x}\right\| \\ &\le \left\|T_n^{\lambda_n}\widetilde{x}_n - T_n^{\lambda_n}\widehat{x}\right\| + \left\|T_n^{\lambda_n}\widehat{x} - \widehat{x}\right\| \\ &\le (1-\lambda_n\tau)\|\widetilde{x}_n - \widehat{x}\| + \lambda_n\mu\|F(\widehat{x})\| \\ &\le (1-\lambda_n\tau)\left[\|x_n - \widehat{x} - \beta_n(Ax_n - A\widehat{x})\| + \beta_n\|A\widehat{x}\|\right] + \lambda_n\mu\|F(\widehat{x})\| \\ &\le (1-\lambda_n\tau)\left[\|x_n - \widehat{x}\| + \beta_n\|A\widehat{x}\|\right] + \lambda_n\mu\|F(\widehat{x})\| \\ &\le (1-\lambda_n\tau)\|x_n - \widehat{x}\| + \beta_n\|A\widehat{x}\| + \lambda_n\mu\|F(\widehat{x})\| \\ &\le (1-\lambda_n\tau)\|x_n - \widehat{x}\| + \lambda_n\left(\|A\widehat{x}\| + \mu\|F(\widehat{x})\|\right) \\ &= (1-\lambda_n\tau)\|x_n - \widehat{x}\| + \lambda_n\tau \cdot \left[\frac{\|A\widehat{x}\| + \mu\|F(\widehat{x})\|}{\tau}\right].\end{aligned} \tag{3.9}$$

It follows

$$\left\|T_n^{\lambda_n}\widetilde{x}_n - \widehat{x}\right\|^2 \le (1-\lambda_n\tau)\|x_n - \widehat{x}\|^2 + \lambda_n \cdot \frac{\left(\|A\widehat{x}\| + \mu\|F(\widehat{x})\|\right)^2}{\tau}. \tag{3.10}$$

This together with (3.7) yields

$$\begin{aligned}\|x_n-\hat{x}\|^2 &\leq \alpha_n\|x_{n-1}-\hat{x}\|^2+(1-\alpha_n)\left[(1-\lambda_n\tau)\|x_n-\hat{x}\|^2+\lambda_n\cdot\frac{(\|A\hat{x}\|+\mu\|F(\hat{x})\|)^2}{\tau}\right]\\ &\quad-\alpha_n(1-\alpha_n)\left\|x_{n-1}-T_n^{\lambda_n}\tilde{x}_n\right\|^2\\ &\leq \alpha_n\|x_{n-1}-\hat{x}\|^2+(1-\alpha_n)\|x_n-\hat{x}\|^2+(1-\alpha_n)\lambda_n\cdot\frac{(\|A\hat{x}\|+\mu\|F(\hat{x})\|)^2}{\tau}\\ &\quad-\alpha_n(1-\alpha_n)\left\|x_{n-1}-T_n^{\lambda_n}\tilde{x}_n\right\|^2,\end{aligned} \tag{3.11}$$

and so,

$$\begin{aligned}\|x_n-\hat{x}\|^2 &\leq \|x_{n-1}-\hat{x}\|^2+(1-\alpha_n)\frac{\lambda_n}{\alpha_n}\cdot\frac{(\|A\hat{x}\|+\mu\|F(\hat{x})\|)^2}{\tau}-(1-\alpha_n)\left\|x_{n-1}-T_n^{\lambda_n}\tilde{x}_n\right\|^2\\ &\leq \|x_{n-1}-\hat{x}\|^2+\lambda_n\cdot\frac{(\|A\hat{x}\|+\mu\|F(\hat{x})\|)^2}{\tau a}-\|x_n-x_{n-1}\|^2.\end{aligned} \tag{3.12}$$

Since $\sum_{n=1}^{\infty}\lambda_n\cdot[(\|A\hat{x}\|+\mu\|F(\hat{x})\|)^2/\tau a]$ converges, by Lemma 2.5, $\lim_{n\to\infty}\|x_n-\hat{x}\|$ exists. As a consequence, the sequence $\{x_n\}$ is bounded. Moreover, we have

$$\|x_n-x_{n-1}\|^2 \leq \|x_{n-1}-\hat{x}\|^2-\|x_n-\hat{x}\|^2+\lambda_n\cdot\frac{(\|A\hat{x}\|+\mu\|F(\hat{x})\|)^2}{\tau a}\longrightarrow 0 \quad \text{as } n\longrightarrow\infty. \tag{3.13}$$

Therefore,

$$\lim_{n\to\infty}\|x_n-x_{n-1}\|=0. \tag{3.14}$$

Obviously, it is easy to see that $\lim_{n\to\infty}\|x_n-x_{n+i}\|=0$ for each $i=1,2,\ldots,N$. Now observe that

$$(1-b)\left\|x_{n-1}-T_n^{\lambda_n}\tilde{x}_n\right\| \leq (1-\alpha_n)\left\|x_{n-1}-T_n^{\lambda_n}\tilde{x}_n\right\|=\|x_n-x_{n-1}\|\longrightarrow 0 \quad \text{as } n\longrightarrow\infty. \tag{3.15}$$

Also note that the boundedness of $\{x_n\}$ implies that $\{T_n\tilde{x}_n\}$ and $\{F(T_n\tilde{x}_n)\}$ are both bounded. Thus, we have

$$\begin{aligned}\|x_{n-1}-T_n\tilde{x}_n\| &\leq \left\|x_{n-1}-T_n^{\lambda_n}\tilde{x}_n\right\|+\left\|T_n^{\lambda_n}\tilde{x}_n-T_n\tilde{x}_n\right\|\\ &\leq \left\|x_{n-1}-T_n^{\lambda_n}\tilde{x}_n\right\|+\lambda_n\mu\|F(T_n\tilde{x}_n)\|\longrightarrow 0 \quad \text{as } n\longrightarrow\infty.\end{aligned} \tag{3.16}$$

This implies

$$\|x_n-T_n\tilde{x}_n\| \leq \|x_n-x_{n-1}\|+\|x_{n-1}-T_n\tilde{x}_n\|\longrightarrow 0 \quad \text{as } n\longrightarrow\infty. \tag{3.17}$$

Consequently,

$$\begin{aligned}\|x_n - T_n x_n\| &\le \|x_n - T_n\tilde{x}_n\| + \|T_n\tilde{x}_n - T_n x_n\| \\ &\le \|x_n - T_n\tilde{x}_n\| + \|\tilde{x}_n - x_n\| \\ &= \|x_n - T_n\tilde{x}_n\| + \beta_n\|Ax_n\| \\ &\le \|x_n - T_n\tilde{x}_n\| + \lambda_n\|Ax_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,\end{aligned} \tag{3.18}$$

and hence, for each $i = 1, 2, \ldots, N$,

$$\begin{aligned}\|x_n - T_{n+i}x_n\| &\le \|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i}x_{n+i}\| + \|T_{n+i}x_{n+i} - T_{n+i}x_n\| \\ &\le 2\|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i}x_{n+i}\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{3.19}$$

This shows that $\lim_{n\to\infty}\|x_n - T_{n+i}x_n\| = 0$ for each $i = 1, 2, \ldots, N$. Therefore,

$$\lim_{n\to\infty}\|x_n - T_l x_n\| = 0, \quad \text{for each } l = 1, 2, \ldots, N. \tag{3.20}$$

On the other hand, since $\{x_n\}$ is bounded, it has a subsequence $\{x_{n_j}\}$, which converges weakly to some $\overline{x} \in H$, and so, we have $\lim_{j\to\infty}\|x_{n_j} - T_l x_{n_j}\| = 0$. From Lemma 2.4, it follows that $I - T_l$ is demiclosed at zero. Thus, $\overline{x} \in \operatorname{Fix}(T_l)$. Since l is an arbitrary element in the finite set $\{1, 2, \ldots, N\}$, we get $\overline{x} \in \bigcap_{i=1}^{N}\operatorname{Fix}(T_i)$.

Now, let x^* be an arbitrary element of $\omega_w(x_n)$. Then, there exists another subsequence $\{x_{n_k}\}$ of $\{x_n\}$, which converges weakly to $x^* \in H$. Clearly, by repeating the same argument, we get $x^* \in \bigcap_{i=1}^{N}\operatorname{Fix}(T_i)$. We claim that $x^* = \overline{x}$. Indeed, if $x^* \ne \overline{x}$, then by the Opial's property of H, we conclude that

$$\begin{aligned}\lim_{n\to\infty}\|x_n - x^*\| &= \liminf_{k\to\infty}\|x_{n_k} - x^*\| \\ &< \liminf_{k\to\infty}\|x_{n_k} - \overline{x}\| = \lim_{n\to\infty}\|x_n - \overline{x}\| = \liminf_{j\to\infty}\left\|x_{n_j} - \overline{x}\right\| \\ &< \liminf_{j\to\infty}\left\|x_{n_j} - x^*\right\| = \lim_{n\to\infty}\|x_n - x^*\|.\end{aligned} \tag{3.21}$$

This leads to a contradiction, and so, we get $x^* = \overline{x}$. Therefore, $\omega_w(x_n)$ is a singleton set. Hence, $\{x_n\}$ converges weakly to a common fixed point of the mappings $\{T_i\}_{i=1}^{N}$, denoted still by x^*.

Assume that $\|x_n - T_n\tilde{x}_n\| = o(\beta_n)$. Let $y \in \bigcap_{i=1}^{N}\operatorname{Fix}(T_i)$ be arbitrary but fixed. Then, it follows from the nonexpansiveness of each T_i and the monotonicity of A that

$$\begin{aligned}\left\|T_n\tilde{x}_n - y\right\|^2 &= \left\|T_n(x_n - \beta_n Ax_n) - T_n(y)\right\|^2 \\ &\le \left\|(x_n - y) - \beta_n Ax_n\right\|^2 \\ &= \left\|x_n - y\right\|^2 + 2\beta_n\langle Ax_n, y - x_n\rangle + \beta_n^2\|Ax_n\|^2 \\ &= \left\|x_n - y\right\|^2 + 2\beta_n(\langle Ay, y - x_n\rangle + \langle Ax_n - Ay, y - x_n\rangle) + \beta_n^2\|Ax_n\|^2 \\ &\le \left\|x_n - y\right\|^2 + 2\beta_n\langle Ay, y - x_n\rangle + \beta_n^2\|Ax_n\|^2,\end{aligned} \tag{3.22}$$

which implies that

$$
\begin{aligned}
0 &\le \frac{1}{\beta_n}\left(\|x_n - y\|^2 - \|T_n\tilde{x}_n - y\|^2\right) + 2\langle Ay, y - x_n\rangle + \beta_n\|Ax_n\|^2 \\
&= \left(\|x_n - y\| + \|T_n\tilde{x}_n - y\|\right)\frac{\|x_n - y\| - \|T_n\tilde{x}_n - y\|}{\beta_n} + 2\langle Ay, y - x_n\rangle + \beta_n\|Ax_n\|^2 \\
&\le M\frac{\|x_n - T_n\tilde{x}_n\|}{\beta_n} + 2\langle Ay, y - x_n\rangle + M^2\beta_n,
\end{aligned}
\tag{3.23}
$$

where $M := \sup\{\|x_n - y\| + \|T_n\tilde{x}_n - y\| + \|Ax_n\| : n \ge 1\} < \infty$. Note that $\lambda_n \to 0$, $\beta_n \le \lambda_n$, for all $n \ge 1$, and $\|x_n - T_n\tilde{x}_n\| = o(\beta_n)$. Thus, for any $\varepsilon > 0$, there exists an integer $m_0 \ge 1$ such that $M\|x_n - T_n\tilde{x}_n\|/\beta_n + M^2\beta_n \le \varepsilon$ for all $n \ge m_0$. Consequently, $0 \le \varepsilon + 2\langle Ay, y - x_n\rangle$ for all $n \ge m_0$. Since $x_n \rightharpoonup x^*$, we have $\varepsilon + 2\langle Ay, y - x^*\rangle \ge 0$ as $n \to \infty$. Therefore, from the arbitrariness of $\varepsilon > 0$, we deduce that $\langle Ay, y - x^*\rangle \ge 0$ for all $y \in \bigcap_{i=1}^N \operatorname{Fix}(T_i)$. Proposition 2.1 (a) ensures that

$$
\langle Ax^*, y - x^*\rangle \ge 0, \quad \forall y \in \bigcap_{i=1}^N \operatorname{Fix}(T_i); \tag{3.24}
$$

that is, $x^* \in \operatorname{VI}(\bigcap_{i=1}^N \operatorname{Fix}(T_i), A)$. □

Corollary 3.2 (see [14], Theorem 2.1). *Let H be a real Hilbert space and $F : H \to H$ κ-Lipschitz continuous and η-strongly monotone for some constants $\kappa > 0$, $\eta > 0$. Let $\{T_i\}_{i=1}^N$ be N nonexpansive self-maps on H such that $C = \bigcap_{i=1}^N \operatorname{Fix}(T_i) \ne \emptyset$. Let $\mu \in (0, 2\eta/\kappa^2)$, $x_0 \in H$, $\{\lambda_n\}_{n=1}^\infty \subset [0,1)$, and $\{\alpha_n\}_{n=1}^\infty \subset (0,1)$ be such that $\sum_{n=1}^\infty \lambda_n < \infty$ and $a \le \alpha_n \le b$, for all $n \ge 1$, for some $a, b \in (0,1)$. Then, the sequence $\{x_n\}_{n=1}^\infty$, defined by*

$$
x_n := \alpha_n x_{n-1} + (1-\alpha_n)T_n^{\lambda_n} x_n = \alpha_n x_{n-1} + (1-\alpha_n)\left[T_n x_n - \lambda_n \mu F(T_n x_n)\right], \quad \forall n \ge 1, \tag{3.25}
$$

converges weakly to a common fixed point of the mappings $\{T_i\}_{i=1}^N$.

Proof. In Theorem 3.1, putting $A \equiv 0$, we can see readily that for any given positive number $\alpha \in (0,\infty)$, $A : H \to H$ is an α-inverse-strongly monotone mapping. In this case, we have

$$
\operatorname{VI}\left(\bigcap_{i=1}^N \operatorname{Fix}(T_i), A\right) = \bigcap_{i=1}^N \operatorname{Fix}(T_i). \tag{3.26}
$$

Hence, for any given sequence $\{\beta_n\}_{n=1}^\infty \subset (0, 2\alpha]$ with $\beta_n \le \lambda_n (\forall n \ge 1)$, the implicit iterative scheme (3.5) reduces to (3.25). Therefore, by Theorem 3.1, we obtain the desired result. □

Lemma 3.3. *In the setting of Theorem 3.1, we have*

(a) $\lim_{n\to\infty}\|x_n - \hat{x}\|$ *exists for each* $\hat{x} \in C$,

(b) $\lim_{n\to\infty} d(x_n, C)$ *exists, where* $d(x_n, C) = \inf_{p\in C}\|x_n - p\|$,

(c) $\liminf_{n\to\infty}\|x_n - T_n x_n\| = 0$,

where $C := \mathrm{VI}(\bigcap_{i=1}^{N}\mathrm{Fix}(T_i), A)$.

Proof. Conclusion (a) follows from (3.12), and conclusion (c) follows from (3.18). We prove conclusion (b). Indeed, for each $\widehat{x} \in C$,

$$\begin{aligned}
\left(\|A\widehat{x}\| + \mu\|F(\widehat{x})\|\right)^2 &\leq \left(\|A\widehat{x} - Ax_{n-1}\| + \|Ax_{n-1}\| + \mu\|F(\widehat{x}) - F(x_{n-1})\| + \mu\|F(x_{n-1})\|\right)^2 \\
&\leq \left(\frac{1}{\alpha}\|x_{n-1} - \widehat{x}\| + \|Ax_{n-1}\| + \mu\kappa\|x_{n-1} - \widehat{x}\| + \mu\|F(x_{n-1})\|\right)^2 \\
&= \left[\left(\frac{1}{\alpha} + \mu\kappa\right)\|x_{n-1} - \widehat{x}\| + \left(\|Ax_{n-1}\| + \mu\|F(x_{n-1})\|\right)\right]^2 \\
&\leq 2\left(\frac{1}{\alpha} + \mu\kappa\right)^2\|x_{n-1} - \widehat{x}\|^2 + 2\left(\|Ax_{n-1}\| + \mu\|F(x_{n-1})\|\right)^2.
\end{aligned} \tag{3.27}$$

This together with (3.12) implies that

$$\begin{aligned}
\|x_n - \widehat{x}\|^2 &\leq \|x_{n-1} - \widehat{x}\|^2 + \lambda_n\cdot\frac{\left(\|A\widehat{x}\| + \mu\|F(\widehat{x})\|\right)^2}{\tau a} \\
&\leq \|x_{n-1} - \widehat{x}\|^2 + \lambda_n\cdot\frac{1}{\tau a}\left[2\left(\frac{1}{\alpha} + \mu\kappa\right)^2\|x_{n-1} - \widehat{x}\|^2 + 2\left(\|Ax_{n-1}\| + \mu\|F(x_{n-1})\|\right)^2\right] \\
&\leq \left(1 + \lambda_n\cdot\frac{2\left((1/\alpha) + \mu\kappa\right)^2}{\tau a}\right)\|x_{n-1} - \widehat{x}\|^2 + \lambda_n\cdot\frac{2}{\tau a}\left(\|Ax_{n-1}\| + \mu\|F(x_{n-1})\|\right)^2 \\
&\leq (1 + \gamma_n)\|x_{n-1} - \widehat{x}\|^2 + \gamma_n,
\end{aligned} \tag{3.28}$$

and hence,

$$[d(x_n, C)]^2 \leq (1 + \gamma_n)[d(x_{n-1}, C)]^2 + \gamma_n, \tag{3.29}$$

where

$$\gamma_n = \lambda_n\cdot\max\left\{\frac{2\left((1/\alpha) + \mu\kappa\right)^2}{\tau a}, \frac{2}{\tau a}\left(\|Ax_{n-1}\| + \mu\|F(x_{n-1})\|\right)^2\right\}, \quad \forall n \geq 1. \tag{3.30}$$

Since $\sum_{n=1}^{\infty}\lambda_n < \infty$ and both $\{Ax_{n-1}\}$ and $\{F(x_{n-1})\}$ are bounded, it is known that $\sum_{n=1}^{\infty}\gamma_n < \infty$. On account of Lemma 2.5, we deduce that $\lim_{n\to\infty} d(x_n, C)$ exists, that is, conclusion (b) holds. $\square$

Finally, we give necessary and sufficient conditions for the strong convergence of the sequence generated by the implicit iterative scheme (3.5) with perturbation F.

Theorem 3.4. *In the setting of Theorem 3.1, the sequence* $\{x_n\}$ *converges strongly to an element of* $\mathrm{VI}(\bigcap_{i=1}^N \mathrm{Fix}(T_i), A)$ *if and only if* $\liminf_{n\to\infty} d(x_n, C) = 0$ *where* $C := \mathrm{VI}(\bigcap_{i=1}^N \mathrm{Fix}(T_i), A)$.

Proof. From (3.28), we derive for each $n \geq 1$

$$\|x_n - \hat{x}\|^2 \leq (1+\gamma_n)\|x_{n-1} - \hat{x}\|^2 + \gamma_n, \quad \forall \hat{x} \in C, \tag{3.31}$$

where $\sum_{n=1}^{\infty} \gamma_n < \infty$. Put $\widetilde{M} = \prod_{n=1}^{\infty}(1+\gamma_n)$. Then $1 \leq \widetilde{M} < \infty$.

Suppose that the sequence $\{x_n\}$ converges strongly to a common fixed point p of the family $\{T_i\}_{i=1}^N$. Then, $\lim_{n\to\infty}\|x_n - p\| = 0$. Since

$$0 \leq d(x_n, C) \leq \|x_n - p\|, \tag{3.32}$$

we have $\liminf_{n\to\infty} d(x_n, C) = 0$.

Conversely, suppose that $\liminf_{n\to\infty} d(x_n, C) = 0$. Then, by Lemma 3.3 (b), we deduce that $\lim_{n\to\infty} d(x_n, C) = 0$. Thus, for arbitrary $\varepsilon > 0$, there exists a positive integer N_0 such that

$$d(x_n, C) < \frac{\varepsilon}{\sqrt{8\widetilde{M}}}, \quad \forall n \geq N_0. \tag{3.33}$$

Furthermore, the condition $\sum_{n=1}^{\infty} \gamma_n < \infty$ implies that there exists a positive integer N_1 such that

$$\sum_{j=n}^{\infty} \gamma_j < \frac{\varepsilon^2}{\left(8\widetilde{M}\right)}, \quad \forall n \geq N_1. \tag{3.34}$$

Choose $N_* = \max\{N_0, N_1\}$. Observe that (3.31) yields

$$\begin{aligned}
\|x_n - \hat{x}\|^2 &\leq (1+\gamma_n)(1+\gamma_{n-1})\|x_{n-2} - \hat{x}\|^2 + (1+\gamma_n)\gamma_{n-1} + \gamma_n \\
&\leq \prod_{j=N_*+1}^{n}(1+\gamma_j)\|x_{N_*} - \hat{x}\|^2 + \sum_{j=N_*+1}^{n-1} \gamma_j \prod_{i=j+1}^{n}(1+\gamma_j) + \gamma_n \\
&\leq \widetilde{M}\left[\|x_{N_*} - \hat{x}\|^2 + \sum_{j=N_*+1}^{n} \gamma_j\right].
\end{aligned} \tag{3.35}$$

Note that $d(x_{N_*}, C) < \varepsilon/\sqrt{8\widetilde{M}}$ and $\sum_{j=N_*}^{\infty} \gamma_j < \varepsilon^2/(8\widetilde{M})$. Thus, for all $n, m \geq N_*$ and all $\widehat{x} \in C$, we have from (3.35) that

$$\begin{aligned}
\|x_n - x_m\|^2 &\leq (\|x_n - \widehat{x}\| + \|x_m - \widehat{x}\|)^2 \\
&\leq 2\|x_n - \widehat{x}\|^2 + 2\|x_m - \widehat{x}\|^2 \\
&\leq 2\widetilde{M}\left[\|x_{N_*} - \widehat{x}\|^2 + \sum_{j=N_*+1}^{n} \gamma_j\right] + 2\widetilde{M}\left[\|x_{N_*} - \widehat{x}\|^2 + \sum_{j=N_*+1}^{m} \gamma_j\right] \\
&\leq 4\widetilde{M}\left[\|x_{N_*} - \widehat{x}\|^2 + \sum_{j=N_*}^{\infty} \gamma_j\right] \\
&< 4\widetilde{M}\left(\|x_{N_*} - \widehat{x}\|^2 + \frac{\varepsilon^2}{8\widetilde{M}}\right).
\end{aligned} \tag{3.36}$$

Taking the infimum over all $\widehat{x} \in C$, we obtain

$$\|x_n - x_m\|^2 \leq 4\widetilde{M}\left([d(x_{N_*}, C)]^2 + \frac{\varepsilon^2}{8\widetilde{M}}\right) \leq 4\widetilde{M}\left(\frac{\varepsilon^2}{8\widetilde{M}} + \frac{\varepsilon^2}{8\widetilde{M}}\right) = \varepsilon^2, \tag{3.37}$$

and hence, $\|x_n - x_m\| \leq \varepsilon$. This shows that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in H. Let $x_n \to p \in H$ as $n \to \infty$. Then, we derive from (3.20) that for each $l = 1, 2, \ldots, N$,

$$\begin{aligned}
\|p - T_l p\| &\leq \|p - x_n\| + \|x_n - T_l x_n\| + \|T_l x_n - T_l p\| \\
&\leq 2\|x_n - p\| + \|x_n - T_l x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{3.38}$$

Therefore, $p \in \operatorname{Fix}(T_l)$ for each $l = 1, 2, \ldots, N$, and hence, $p \in \bigcap_{i=1}^{N} \operatorname{Fix}(T_i)$.

On the other hand, choose a positive sequence $\{\varepsilon_n\}_{n=1}^{\infty} \subset (0, \infty)$ such that $\varepsilon_n \to 0$ as $n \to \infty$. For each $n \geq 1$, from the definition of $d(x_n, C)$, it follows that there exists a point $p_n \in C$ such that

$$\|x_n - p_n\| \leq d(x_n, C) + \varepsilon_n. \tag{3.39}$$

Since $d(x_n, C) \to 0$ and $\varepsilon_n \to 0$ as $n \to \infty$, it is clear that $\|x_n - p_n\| \to 0$ as $n \to \infty$. Note that

$$\|p_n - p\| \leq \|p_n - x_n\| + \|x_n - p\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.40}$$

Hence, we get

$$\lim_{n \to \infty} \|p_n - p\| = 0. \tag{3.41}$$

Furthermore, for each $\beta_n \in (0, 2\alpha]$, the mapping $S_{\beta_n} : H \to H$ is defined as follows:

$$S_{\beta_n} x := P_{\bigcap_{i=1}^{N} \operatorname{Fix}(T_i)}(I - \beta_n A)x, \quad \forall x \in H. \tag{3.42}$$

From Proposition 2.3, we deduce that S_{β_n} is nonexpansive and

$$\mathrm{Fix}(S_{\beta_n}) = \mathrm{VI}\left(\bigcap_{i=1}^{N} \mathrm{Fix}(T_i), A\right)(= C). \tag{3.43}$$

From Proposition 2.2 (a), we conclude that $\mathrm{Fix}(S_{\beta_n})$ is closed and convex. Thus, from the condition $\mathrm{VI}(\bigcap_{i=1}^{N} \mathrm{Fix}(T_i), A) \neq \emptyset$, it is known that C is a nonempty closed and convex set. Since $\{p_n\}$ lies in C and converges strongly to p, we must have $p \in C$. □

Remark 3.5. Setting $A = 0$ in Lemma 3.3 and Theorem 3.4 above, we shall derive Lemma 2.1 and Theorem 2.2 in [14] as direct consequences, respectively.

Acknowledgment

This paper is partially supported by the Taiwan NSC Grants 99-2115-M-110-007-MY3 and 99-2221-E-037-007-MY3.

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