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Research Article

Rational Homotopy Perturbation Method

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The solution methods of nonlinear differential equations are very important because most of the physical phenomena are modelled by using such kind of equations. Therefore, this work presents a rational version of homotopy perturbation method (RHPM) as a novel tool with high potential to find approximate solutions for nonlinear differential equations. We present two case studies; for the first example, a comparison between the proposed method and the HPM method is presented; it will show how the RHPM generates highly accurate approximate solutions requiring less iteration, in comparison to results obtained by the HPM method. For the second example, which is a Van der Pol oscillator problem, we compare RHPM, HPM, and VIM, finding out that RHPM method generates the most accurate approximated solution.

1. Introduction

Solving nonlinear differential equations is an important issue in sciences because many physical phenomena are modelled using such equations. One of the most powerful methods to approximately solve nonlinear differential equations is the homotopy perturbation method (HPM) [1–28]. The HPM is based on the use of a power series, which transforms the original nonlinear differential equation into a series of linear differential equations. In this paper, we propose a generalization of the aforementioned concept by using a quotient of two power series of homotopy parameter, which will be called rational homotopy perturbation method (RHPM). In the same fashion, like HPM, the use of that quotient of power series transforms the nonlinear differential equation into a series of linear differential equations. We will present two case studies; for the first example, a comparison between the proposed method and the HPM method is presented; it will show how the RHPM generates highly accurate approximate solutions requiring less iteration steps, in comparison to results from HPM method. For the second example, the Van der Pol oscillator problem [3, 29], we compare

RHPM, HPM [3], and variational iteration method (VIM) [3], resulting that RHPM method generates the most accurate approximated solution.

This paper is organized as follows. In Section 2, we introduce the basic concept of the RHPM method. In Section 3, we present a study of convergence for the proposed method. In Sections 4 and 5, we present the solution of two nonlinear differential equations. In Section 6, numerical simulations and a discussion about the results are provided. Finally, a brief conclusion is given in Section 7.

2. Basic Concept of RHPM

The RHPM and HPM share common foundations. Thus, for both methods, it can be considered that a nonlinear differential equation can be expressed as

$$L(u) + N(u) - f(r) = 0, \quad \text{where } r \in \Omega, \tag{2.1}$$

having as boundary condition

$$B\left(u, \frac{\partial u}{\partial \eta}\right), \quad \text{where } r \in \Gamma, \tag{2.2}$$

where L and N are a linear and a nonlinear operator, respectively, $f(r)$ is a known analytic function, B is a boundary operator, Γ is the boundary of domain Ω, and $\partial u/\partial \eta$ denotes differentiation along the normal drawn outwards from Ω [27].

Now, a possible homotopy formulation is

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p\big(L(v) + N(v) - f(r)\big) = 0, \quad p \in [0,1], \tag{2.3}$$

where u_0 is the initial approximation for (2.1) which satisfies the boundary conditions and p is known as the perturbation homotopy parameter. Analysing (2.3), can be concluded that

$$\begin{aligned} H(v,0) &= L(v) - L(u_0) = 0, \\ H(v,1) &= L(v) + N(v) - f(r) = 0. \end{aligned} \tag{2.4}$$

For the HPM [8–11], we assume that the solution for (2.3) can be written as a power series of p:

$$v = p^0 v_0 + p^1 v_1 + p^2 v_2 + \cdots . \tag{2.5}$$

Considering that $p \to 1$, it results that the approximate solution for (2.1) is

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots . \tag{2.6}$$

The series (2.6) is convergent for most cases [1, 2, 8, 11].

For the RHPM, we assume that solution for (2.3) can be written as power series quotient of p:

$$v = \frac{p^0 v_0 + p^1 v_1 + p^2 v_2 + \cdots}{p^0 w_0 + p^1 w_1 + p^2 w_2 + \cdots}, \tag{2.7}$$

where $v_1, v_2, \ldots$ are unknown functions to be determined by the RHPM, and $w_1, w_2, \ldots$ are known analytic functions of the independent variable.

For the HPM, the order of the approximation is determined by the highest power of p. Nevertheless, for the RHPM the order will be given as $[i, k]$, where i and k are the highest power of p employed in the numerator and denominator of (2.7). Here, the number of linear differential equations generated is $i + 1$.

The limit of (2.7), when $p \to 1$, provides an approximate solution for (2.1) in the form of

$$u = \lim_{p\to 1} v = \frac{v_0 + v_1 + v_2 + \cdots}{w_0 + w_1 + w_2 + \cdots}. \tag{2.8}$$

The above limit exists in the case that both limits

$$\begin{gathered} \lim_{p\to 1}\left(\sum_{i=0}^{\infty} v_i\right), \\ \lim_{p\to 1}\left(\sum_{i=0}^{\infty} w_i\right), \qquad \text{where } \sum_{i=0}^{\infty} w_i \neq 0, \end{gathered} \tag{2.9}$$

exist.

3. Convergence of RHPM

In order to analyse the convergence of RHPM, (2.3) is rewritten as

$$L(v) = L(u_0) + p\left[f(r) - N(v) - L(u_0)\right] = 0. \tag{3.1}$$

Applying the inverse operator, L^{-1}, to both sides of (3.1), we obtain

$$v = u_0 + p\left[L^{-1} f(r) - L^{-1} N(v) - u_0\right]. \tag{3.2}$$

Assuming that (see (2.7))

$$v = \frac{\sum_{i=0}^{\infty} p^i v_i}{\sum_{i=0}^{\infty} p^i w_i}, \tag{3.3}$$

substituting (3.3) in the right-hand side of (3.2) in the following form

$$v = u_0 + p\left[L^{-1}f(r) - \left(L^{-1}N\right)\left[\frac{\sum_{i=0}^{\infty} p^i v_i}{\sum_{i=0}^{\infty} p^i w_i}\right] - u_0\right], \tag{3.4}$$

the exact solution of (2.1) is obtained in the limit $p \to 1$ of (3.4), resulting in

$$\begin{aligned} u &= \lim_{p\to 1}\left(pL^{-1}f(r) - p\left(L^{-1}N\right)\left[\frac{\sum_{i=0}^{\infty} p^i v_i}{\sum_{i=0}^{\infty} p^i w_i}\right] + u_0 - pu_0\right), \\ &= L^{-1}f(r) - \left[\sum_{i=0}^{\infty}\left(L^{-1}N\right)\left(\frac{v_i}{\beta}\right)\right], \quad \beta = \sum_{i=0}^{\infty} w_i. \end{aligned} \tag{3.5}$$

In order to study the convergence of the RHPM, we use the Banach theorem as reported in [1, 2]. Such theorem relates the solution of (2.1) to the fixed point problem of the nonlinear operator N. Let us state the theorem as follows.

Theorem 3.1 (Sufficient Condition for Convergence). *Suppose that X and Y are Banach spaces and $N : X \to Y$ is a contractive nonlinear mapping, that is*

$$\forall w, w^* \in X; \quad \|N(w) - N(w^*)\| \leq \gamma\|w - w^*\|; \quad 0 < \gamma < 1. \tag{3.6}$$

Then, according to the banach fixed point theorem, N has a unique fixed point u; that is, $N(u) = u$. Assume that the sequence generated by the RHPM can be written as

$$W_n = N(W_{n-1}), \qquad W_{n-1} = \sum_{i=0}^{n-1}\left(\frac{v_i}{\beta}\right), \quad n = 1, 2, 3, \ldots, \tag{3.7}$$

and suppose that $W_0 = (v_0/\beta) \in B_r(u)$, where $B_r(u) = \{w^* \in X \mid \|w^* - u\| < r\}$; then, under these conditions,

(i) $W_n \in B_r(u)$,

(ii) $\lim_{n\to\infty} W_n = u$.

Proof. (i) By inductive approach, for $n = 1$ we have

$$\|W_1 - u\| = \|N(W_0) - N(u)\| \leq \gamma\|w_0 - u\|. \tag{3.8}$$

Assuming that $\|W_{n-1} - u\| \leq \gamma^{n-1}\|w_0 - u\|$, as induction hypothesis, then

$$\|W_n - u\| = \|N(W_{n-1}) - N(u)\| \leq \gamma\|W_{n-1} - u\| \leq \gamma^n\|w_0 - u\|. \tag{3.9}$$

Using (i), we have

$$\|W_n - u\| \leq \gamma^n\|w_0 - u\| \leq \gamma^n r < r \Longrightarrow W_n \in B_r(u). \tag{3.10}$$

(ii) Because of $\|W_n - u\| \leq \gamma^n \|w_0 - u\|$ and $\lim_{p\to 1}\gamma^n = 0$, $\lim_{p\to 1}\|W_n - u\| = 0$; that is,

$$\lim_{n\to\infty} W_n = u. \tag{3.11}$$

□

4. Case Study 1

Consider the following nonlinear differential equation

$$y'(x) - y(x)^2 + 1 = 0, \quad y(0) = 0, \tag{4.1}$$

having exact solution

$$y(x) = -\tanh(x). \tag{4.2}$$

4.1. Solution Calculated by RHPM

We establish the following homotopy equations:

$$(1-p)\left(v'(x)+1\right) + p\left(v'(x) - v^2(x) + 1\right) = 0, \tag{4.3}$$

$$(1-p)\left(v'(x) + v(x) - v^2(x)\right) + p\left(v'(x) - v^2(x) + 1\right) = 0. \tag{4.4}$$

Equation (4.3) represents a standard homotopy with linear trial equation [11], and (4.4) is a homotopy with nonlinear trial equation [4].

Now, we suppose that solutions for (4.3) and (4.4) have approximations of order $[3,2]$ and $[2,1]$, which are expressed as follows:

$$v(x) = \frac{v_0(x) + v_1(x)p + v_2(x)p^2 + v_3(x)p^3}{1 + ax^2p + bx^4p^2}, \tag{4.5}$$

$$v(x) = \frac{v_0(x) + v_1(x)p + v_2(x)p^2}{1 + cx^2p}, \tag{4.6}$$

respectively. Besides, a, b, and c are adjustment parameters.

Substituting (4.5) into (4.3) and (4.6) into (4.4), regrouping and equating terms having the same p-powers, it can be solved for $v_0(x)$, $v_1(x)$, $v_2(x)$, and so on (in order to fulfil initial conditions from $v(0) = y(0) = 0$; it follows that $v_0(0) = 0, v_1(0) = 0, v_2(0) = 0$, and so on).

The results are the following two systems of differential equations:

$$\begin{aligned}
&p^0: v_0'(x) + 1 = 0, \qquad v_0(0) = 0,\\
&p^1: v_1'(x) + 2ax^2 - 2axv_0(x) - v_0(x)^2 + av_0'(x)x^2 = 0, \qquad v_1(0) = 0,\\
&p^2: v_2'(x) + \left(2b + a^2\right)x^4 - 2v_0(x)v_1(x) + bv_0'(x)x^4\\
&\quad - 4bx^3v_0(x) + av_1'(x)x^2 - 2axv_1(x) = 0, \qquad v_2(0) = 0,\\
&p^3: v_3'(x) - 2v_0(x)v_2(x) - 2axv_2(x) + av_2'(x)x^2\\
&\quad - v_1(x)^2 + 2abx^6 + bv_1'(x)x^4 - 4bx^3v_1(x) = 0, \qquad v_3(0) = 0,
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
&p^0: v_0'(x) + v_0(x) - v_0(x)^2 = 0, \qquad v_0(0) = 0,\\
&p^1: v_1'(x) + 1 - 2cxv_0(x) + cv_0(x)x^2 - v_0(x) + v_1(x)\\
&\quad + cv_0'(x)x^2 - 2v_0(x)v_1(x) = 0, \qquad v_1(0) = 0,\\
&p^2: v_2'(x) + cv_1'(x)x^2 + 2cx^2 - v_1(x)^2 + cv_1(x)x^2\\
&\quad - v_1(x) + v_2(x) - cv_0(x)x^2 - 2cxv_1(x) - 2v_0(x)v_2(x) = 0, \qquad v_2(0) = 0,
\end{aligned} \tag{4.8}$$

related to (4.3) and (4.4), respectively.

Solving (4.7) results in

$$\begin{aligned}
&v_0(x) = -x, \qquad v_1(x) = -\frac{3a-1}{3}x^3, \qquad v_2(x) = \frac{-15b+5a-2}{15}x^5,\\
&v_3(x) = -\frac{-105b+42a-17}{315}x^7.
\end{aligned} \tag{4.9}$$

Substituting (4.9) into (4.5) and calculating the limit when $p \to 1$, we obtain

$$y(x) = \lim_{p\to 1} v = \frac{-x - ((3a-1)/3)x^3 + ((-15b+5a-2)/15)x^5 - ((-105b+42a-17)/315)x^7}{1 + ax^2 + bx^4}. \tag{4.10}$$

Choosing the adjustment parameters for (4.10) as $a = 74/165$ and $b = 26/1485$, results in

$$y(x) = \frac{-x - (19/165)x^3 - (2/1485)x^5 + (1/155925)x^7}{1 + (74/165)x^2 + (26/1485)x^4}. \tag{4.11}$$

Now, solving (4.8) results in

$$v_0(x) = 0, \qquad v_1(x) = -1 + \exp(-x),$$
$$v_2(x) = \left(-x + cx^2 + 1\right)\exp(-x) - cx^2 - \exp(-2x). \tag{4.12}$$

In the same manner, substituting (4.12) into (4.6), calculating the limit when $p \to 1$, and rearranging terms, we find

$$y(x) = \lim_{p\to 1} v = -1 + \exp(-x)\left(1 + \frac{-x + 1 - \exp(-x)}{1 + cx^2}\right). \tag{4.13}$$

Selecting the adjustment parameter as $c = 0.129677062$ with the procedure reported in [4, 5, 26], (4.13) shows good accuracy for positive values of x; thus, we propose the use of the odd symmetry of exact solution (4.2) to establish a solution with good accuracy throughout the range of x :

$$y(x) = \operatorname{sgn}(x)\left(-1 + \exp(-|x|)\left(1 + \frac{-|x| + 1 - \exp(-|x|)}{1 + 0.129677062|x|^2}\right)\right). \tag{4.14}$$

4.2. Solution Obtained by Using HPM

We apply the standard HPM using homotopies (4.3) and (4.4). Next, we suppose that solution for (4.3) and (4.4) has the form

$$v(x) = v_0(x) + v_1(x)p + v_2(x)p^2 + v_3(x)p^3 + \cdots . \tag{4.15}$$

Substituting (4.15) of order 10 and order 2 into (4.3) and (4.4), respectively, regrouping and equalling terms having the same order p-powers, it can be solved for $v_0(x)$, $v_1(x)$, $v_2(x)$, and so on (in order to fulfil initial conditions from $v(0) = y(0) = 0$, it follows that $v_0(0) = 0$, $v_1(0) = 0$, $v_2(0) = 0$ and so on).

The result is the following two sets of differential equations

$$p^0 : v_0'(x) + 1 = 0, \qquad v_0(0) = 0,$$
$$p^1 : v_1'(x) - v_0(x)^2 = 0, \qquad v_1(0) = 0,$$
$$p^2 : v_2'(x) - 2v_0(x)v_1(x) = 0, \qquad v_2(0) = 0,$$
$$p^3 : v_3'(x) - 2v_0(x)v_2(x) - v_1(x)^2 = 0, \qquad v_3(0) = 0,$$
$$p^4 : v_4'(x) - 2v_1(x)v_2(x) - 2v_0(x)v_3(x) = 0, \qquad v_4(0) = 0,$$
$$p^5 : v_5'(x) - 2v_1(x)v_3(x) - v_2(x)^2 - 2v_0(x)v_4(x) = 0, \qquad v_5(0) = 0,$$

$$\begin{aligned}
p^6 &: v_6'(x) - 2v_2(x)v_3(x) - 2v_1(x)v_4(x) - 2v_0(x)v_5(x) = 0, \qquad v_6(0) = 0,\\
p^7 &: v_7'(x) - 2v_2(x)v_4(x) - 2v_1(x)v_5(x) - v_3(x)^2\\
&\quad - 2v_0(x)v_6(x) = 0, \qquad v_7(0) = 0,\\
p^8 &: v_8'(x) - 2v_1(x)v_6(x) - 2v_3(x)v_4(x)\\
&\quad - 2v_2(x)v_5(x) - 2v_0(x)v_7(x) = 0, \qquad v_8(0) = 0,\\
p^9 &: v_9'(x) - v_4(x)^2 - 2v_0(x)v_8(x) - 2v_2(x)v_6(x)\\
&\quad - 2v_3(x)v_5(x) - 2v_1(x)v_7(x) = 0, \qquad v_9(0) = 0,\\
p^{10} &: v_{10}'(x) - 2v_3(x)v_6(x) - 2v_4(x)v_5(x) - 2v_0(x)v_9(x)\\
&\quad - 2v_1(x)v_8(x) - 2v_2(x)v_7(x) = 0, \qquad v_{10}(0) = 0,
\end{aligned} \tag{4.16}$$

$$\begin{aligned}
p^0 &: v_0'(x) + v_0(x) - v_0(x)^2 = 0, \qquad v_0(0) = 0,\\
p^1 &: v_1'(x) + 1 - v_0(x) + v_1(x) - 2v_0(x)v_1(x) = 0, \qquad v_1(0) = 0,\\
p^2 &: v_2'(x) - v_1(x)^2 - v_1(x) + v_2(x) - 2v_0(x)v_2(x) = 0, \qquad v_2(0) = 0,
\end{aligned} \tag{4.17}$$

related to (4.3) and (4.4), respectively.

By solving (4.2), we obtain

$$\begin{aligned}
&v_0(x) = -x, \qquad v_1(x) = \frac{1}{3}x^3, \qquad v_2(x) = -\frac{2}{15}x^5, \qquad v_3(x) = \frac{17}{315}x^7,\\
&v_4(x) = -\frac{62}{2835}x^9, \qquad v_5(x) = \frac{1382}{155925}x^{11}, \qquad v_6(x) = -\frac{21844}{6081075}x^{13},\\
&v_7(x) = \frac{929569}{638512875}x^{15}, \qquad v_8(x) = -\frac{6404582}{10854718875}x^{17},\\
&v_9(x) = \frac{443861162}{1856156927625}x^{19}, \qquad v_{10}(x) = -\frac{18888466084}{194896477400625}x^{21}.
\end{aligned} \tag{4.18}$$

Substituting solutions (4.18) into (4.15) and calculating the limit when $p \to 1$, it results that

$$\begin{aligned}
y(x) = \lim_{p\to 1} v = &-x + \frac{1}{3}x^3 - \frac{2}{15}x^5 + \frac{17}{315}x^7 - \frac{62}{2835}x^9\\
&+ \frac{1382}{155925}x^{11} - \frac{21844}{6081075}x^{13} + \frac{929569}{638512875}x^{15}\\
&- \frac{6404582}{10854718875}x^{17} + \frac{443861162}{1856156927625}x^{19} - \frac{18888466084}{194896477400625}x^{21}.
\end{aligned} \tag{4.19}$$

Solving (4.17), we obtain

$$v_0(x) = 0, \qquad v_1(x) = -1 + \exp(-x),$$
$$v_2(x) = -(x + \exp(-x) - 1)\exp(-x). \tag{4.20}$$

Substituting solutions (4.20) into (4.15) and calculating the limit when $p \to 1$, it results that

$$y(x) = -\exp(-2x) - 1 + (2 - x)\exp(-x). \tag{4.21}$$

Equation (4.21) shows good accuracy for positive values of x. Therefore, we propose the use of the odd symmetry of exact solution (4.2) to establish a fairly accurate solution throughout the range of x

$$y(x) = \operatorname{sgn}(x)\left(-\exp(-2|x|) - 1 + (2 - |x|)\exp(-|x|)\right). \tag{4.22}$$

5. Case Study 2

Consider the Van der Pol oscillator problem [3, 29]

$$\frac{d^2u}{dt^2} + \frac{du}{dt} + u + u^2\frac{du}{dt} = 2\cos(t) - \cos^3(t), \qquad u(0) = 0, \qquad u'(0) = 1, \tag{5.1}$$

with exact solution

$$u(t) = \sin(t). \tag{5.2}$$

To solve (5.1) by means of RHPM, we establish the following homotopy equation:

$$(1-p)\left(\frac{d^2v}{dt^2}\right) + p\left(\frac{d^2v}{dt^2} + \frac{dv}{dt} + v + v^2\frac{dv}{dt} - 2\cos(t) + \cos^3(t)\right) = 0. \tag{5.3}$$

We suppose that solution for (5.3) has the following rational form:

$$v = \frac{v_0 + v_1 p}{1 + at^2 p}. \tag{5.4}$$

Substituting (5.4) into (5.3), rearranging and equating terms having the same p-powers,

$$
\begin{aligned}
p^0 : \frac{d^2 v_0}{dt^2} &= 0, \qquad v_0(0) = 0, \qquad v_0'(0) = 1, \\
p^1 : \frac{d^2 v_1}{dt^2} &+ \left(-4at + 1 + v_0^2\right)\frac{dv_0}{dt} + 3at^2 \frac{d^2 v_0}{dt^2} \\
&+ (-2a+1)v_0 - 2\cos(t) + \cos^3(t) = 0, \qquad v_1(0) = 0, \qquad v_1'(0) = 0.
\end{aligned}
\tag{5.5}
$$

By solving (5.5), we obtain

$$
\begin{aligned}
v_0(t) &= t, \\
v_1(t) &= \frac{1}{9}\cos^3(t) - \frac{4}{3}\cos(t) + at^3 - \frac{1}{6}t^3 - \frac{1}{12}t^4 - \frac{1}{2}t^2 + \frac{11}{9}.
\end{aligned}
\tag{5.6}
$$

Substituting solutions (5.6) into (5.4) and calculating the limit when $p \to 1$, we obtain the first-order RHPM approximation:

$$
u(t) = \lim_{p\to 1} v = \frac{1}{36}\frac{36t + 4\cos^3(t) - 48\cos(t) + (36a-6)t^3 - 3t^4 - 18t^2 + 44}{1 + at^2}. \tag{5.7}
$$

Finally, we select the adjustment parameter as $a = 0.407946126513$ using the procedure reported in [4, 5, 26].

6. Numerical Simulation and Discussion

Figure 1 and Table 1 show a comparison between the exact solution (4.2) for the nonlinear differential equation (4.1) and the analytic approximations (4.11), (4.14), (4.19), and (4.22). Considering the odd symmetry from the exact solution and approximations, Table 1 presents the relative error only for positive values of x. In the range of $x \in [0,5]$, the maximum relative error for (4.11) is 0.0022083, while the maximum error for (4.19) in the same range is −4.206E10 (see Figure 1). Besides, the table also shows the relative error for (4.11) at $x = 8$, which is 0.0372487, that is, fifteen orders of magnitude lower than the relative error obtained for (4.19). Also, the RHPM is required to solve (4.7) using just three iterations to obtain (4.11); while in order to obtain (4.19), HPM required to solve the system (4.2) containing ten differential equations. Therefore, for this case study, RHPM reached results having higher precision and wider range requiring less iteration than HPM.

If we perform the Padé [30, 31] approximant of order [7/4] to the exact solution (4.2), the result is exactly the same to the approximate solution (4.11) calculated by using the RHPM. This result is interesting and deserves deeper study in a future work.

The differential equation (4.1) was solved using homotopy (4.4) in its RHPM and HPM versions, resulting in approximations (4.14) and (4.22), respectively. From Table 1, it is possible to observe that the lowest relative error in the range $x \in [0,5]$ for (4.14) is 0.000282807, while the minimum relative error for (4.22) is −0.0203519. In fact, there is a

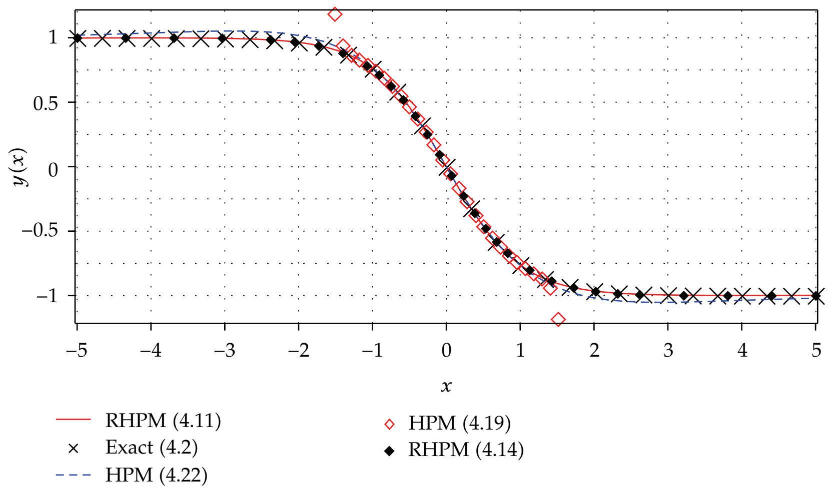

Figure 1: Exact solution (4.2) (diagonal cross) for (4.1) and its approximate solutions (4.11) (solid line), (4.14) (solid diamond), (4.19) (empty diamonds), and (4.22) (dashed).

Table 1: Comparison of the relative error between exact solution (4.2) for (4.1) to the results of approximations (4.11), (4.14), (4.19), and (4.22).

		Homotopy (4.3)	Homotopy (4.3)	Homotopy (4.4)	Homotopy (4.4)
Order		$[3, 2]$	10	$[2, 1]$	2
x	Exact (4.2)	RHPM (4.11)	HPM (4.19)	RHPM (4.14)	HPM (4.22)
0.50	−0.462117	$2.82139e-13$	$-9.2001e-12$	0.0131193	0.00872875
1.00	−0.761594	$7.99295e-10$	$-3.6699e-05$	0.0127018	−0.0076966
1.50	−0.905148	$6.27205e-08$	−0.254724	0.00372419	−0.0365396
2.00	−0.964028	$1.12495e-06$	−130.396	−0.00187676	−0.0563138
2.50	−0.986614	$9.09157e-06$	−16013.2	−0.0030716	−0.061996
3.00	−0.995055	$4.49154e-05$	−799594	−0.00226142	−0.0574954
3.50	−0.998178	0.000159567	$-2.152e+07$	−0.00114353	−0.0481177
4.00	−0.999329	0.000448827	$-3.695e+08$	−0.000334271	−0.0376627
4.50	−0.999753	0.00106284	$-4.508e+09$	0.000105112	−0.0281496
5.00	−0.999909	0.0022083	$-4.206e+10$	0.000282807	−0.0203519
8.00	−1.000000	0.0372487	$-8.607e+14$	$8.27086e-05$	−0.00201311
100	−1.000000	358.626	$-9.689e+37$	$3.43629e-44$	$-3.64567e-42$

difference of one or two orders of magnitude throughout the domain of solutions favouring the RHPM. Furthermore, for $x = 100$, (4.14) has the lower relative error of all approximations with a value of −3.43629E-44. In case that a better approximation is required, it would be necessary to perform more iteration for both methods. This shows that using a nonlinear trial equation may generate highly accurate results for both HPM and RHPM.

In Table 2, the relative error for the exact solution (5.2) of the Van der Pol oscillator (5.1), RHPM solution (5.7), and approximations obtained by HPM and VIM reported in [3] are shown. It can be seen that the approximated solution (5.7) shows the lowest relative error

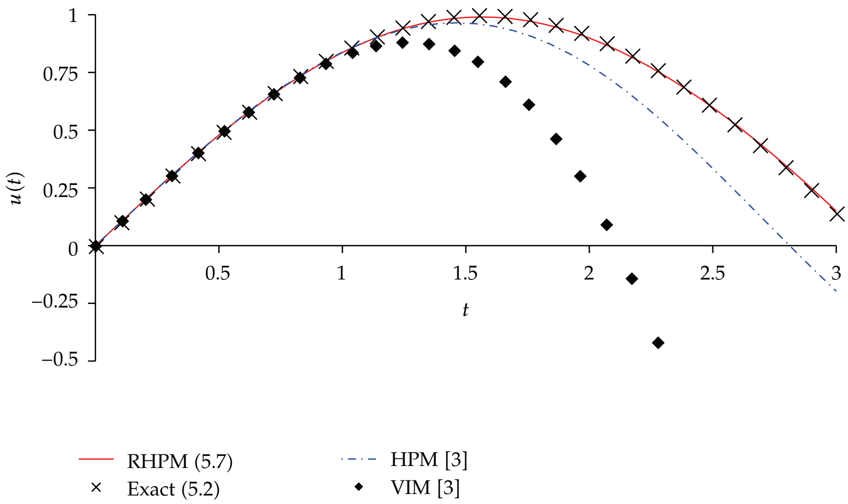

Figure 2: Exact solution (5.2) (diagonal cross) for Van der Pol oscillator problem (5.1) and its first order approximate solutions obtained by RHPM (5.7) (solid line), HPM [3] (dash-dot), and VIM [3] (solid diamond).

Table 2: Comparison of the relative error between exact solution (5.2) for (5.1) and its first order approximate solutions given by RHPM (5.7), HPM [3], and VIM [3].

Order		[1, 1]	1	1
t	Exact (4.2)	RHPM (5.7)	HPM [3]	VIM [3]
0.5	0.47942554	0.00219553	0.00007743	0.00080698
1.0	0.84147098	0.00629261	0.00389778	0.02403725
1.5	0.99749499	0.00924481	0.03354518	0.17347474
2.0	0.90929743	0.01132241	0.14119662	0.74571902
2.5	0.59847214	0.00061706	0.44219762	2.85423607
3.0	0.14112001	−0.06022780	2.40024981	23.28655437

in the range $t \in [0, 3]$ (see Figure 2). Besides, the [1, 1] order solution (5.7) has the lowest number of terms compared to the first-order solutions obtained by HPM [3] and VIM [3].

For both case studies, w polynomial functions were employed. Nevertheless, w is arbitrary and may contain exponentials, trigonometric functions, among others. Likewise, w terms play a significant role in the accuracy of the resultant approximation. Therefore, thorough study is required to propose a methodology leading to select w functions to obtain more accurate solutions using RHPM method.

In this work, by using two case studies, the RHPM is presented as a novel tool with high potential to solve nonlinear differential equations. Given that HPM and RHPM are closely related, it is highly possible that differential equations solved by HPM can be solved by RHPM in order to find more accurate solutions.

7. Conclusions

This paper presented the rational homotopy perturbation method as a novel tool with high potential to solve nonlinear differential equations. Also, a comparison between the results of applying the proposed method and HPM was shown. Likewise, for the first example, a comparison between the proposed method and the HPM was presented, showing how the RHPM generates highly accurate approximate solutions using less iteration steps, in comparison to results obtained using the HPM. Besides, a Van der Pol oscillator problem was solved by the proposed method and compared to solutions obtained by HPM and VIM; the result was that the RHPM generated the most accurate approximated solution. Finally, there is a possible connection between the Padé approximant and the RHPM, which will be studied in future works.

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