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Research Article

A Modified Regularization Method for the Proximal Point Algorithm

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Under some weaker conditions, we prove the strong convergence of the sequence generated by a modified regularization method of finding a zero for a maximal monotone operator in a Hilbert space. In addition, an example is also given in order to illustrate the effectiveness of our generalizations. The results presented in this paper can be viewed as the improvement, supplement, and extension of the corresponding results.

1. Introduction

Let H be a real Hilbert space and C a nonempty closed convex subset of H, and let $F : H \to H$ be a nonlinear operator. The variational inequality problem is formulated as finding a point $x^* \in C$ such that

$$\langle Fx^*, v - x^* \rangle \geq 0, \quad \forall v \in C. \tag{1.1}$$

In 1964, Stampacchia [1] introduced and studied variational inequality initially. It is now well known that variational inequalities cover as diverse disciplines as partial differential equations, optimal control, optimization, mathematical programming, mechanics, and finance, see [1–5].

Let T be an operator with domain $D(T)$ and range $R(T)$ in H. A multivalued operator $T : H \to 2^H$ is called monotone if

$$\langle u - v, x - y \rangle \geq 0, \tag{1.2}$$

for any $u \in Tx,\ v \in Ty$, and maximal monotone if it is monotone and its graph

$$G(T) = \{(x, y) : x \in D(T), y \in Tx\} \tag{1.3}$$

is not properly contained in the graph of any other monotone operator.

One of the major problems in the theory of monotone operators is to find a point in the zero set, which can be formulated as finding a point x so that $x \in T^{-1}(0)$, where $T^{-1}(0)$ denotes the zero set of the operator T. A variety of problems, including convex programming and variational inequalities, can be formulated as finding a zero of maximal monotone operators. A classical way to solve such problem is Rockafellar's proximal point algorithm [6], which generates an iterative sequence as

$$x_{n+1} = J_c^T(x_n + e_n), \tag{1.4}$$

where, for $c > 0$, J_c^T denotes the resolvent of T given by $J_c^T = (I + cT)^{-1}$, with I being the identity map on the space H. If $T^{-1}(0) \neq \emptyset$, it is known that the sequence generated by (1.4) converges weakly to some point in $T^{-1}(0)$.

Motivated by Lehdili and Moudafi's prox-Tikhonov method [7], Xu [8] considered the following regularization iterative form: for a fixed point $u \in H$,

$$x_{n+1} = J_{c_n}^T((1 - t_n)x_n + t_n u + e_n), \quad n \geq 0, \tag{1.5}$$

where $t_n \in (0, 1)$ and $\{e_n\}$ is a sequence of errors. Then, the iterative sequence converges strongly to $P_{T^{-1}(0)}u$, provided that

(C1) $\lim_{n\to\infty} t_n = 0$,

(C2) $\sum_{n=0}^{\infty} |t_{n+1} - t_n| < \infty$,

(C3) $0 < \underline{c} \leq c_n \leq \overline{c}$,

(C4) $\sum_{n=0}^{\infty} |c_{n+1} - c_n| < \infty$,

(C5) $\sum_{n=0}^{\infty} t_n = \infty$, $\sum_{n=0}^{\infty} \|e_n\| < \infty$.

Recently, Song and Yang [9] removed some strict restrictions in Xu [8]. Under conditions (C1), (C2), (C4) (or $\sum_{n=0}^{\infty} |1 - (c_n/c_{n+1})| < +\infty$), (C5), and (C3′)

(C3′) $0 < \liminf_{n\to\infty} c_n$,

they proved that the sequence generated by (1.5) converges strongly to $P_{T^{-1}(0)}u$.

Very recently, under conditions (C1), (C3) (or C3′), (C5), and (C4′)

(C4′) $\lim_{n\to\infty} |1 - (c_n/c_{n+1})| = 0$.

Wang [10] proved the strong convergence of the sequence generated by (1.5). It is easy to see that conditions (C3′) and (C4′) are strictly weaker than conditions (C3) and (C4), respectively.

We remind the reader of the following fact: in order to guarantee the strong convergence of the iterative sequence $\{x_n\}$, there is at least one parameter sequence converging to zero (i.e., $t_n \to 0$) in the result of Xu [8], Song and Yang [9], and Wang [10]. So the above results bring us to the following natural questions.

Question 1. Can we obtain the strong convergence theorem without the parameter sequence $\{t_n\}$ converging to zero?

Question 2. Can we get that the sequence $\{x_n\}$ converges strongly to $x^* \in T^{-1}(0)$, which solves uniquely some variational inequalities?

In this work, motivated by the above results, we consider the following modified regularization method for the proximal point algorithm: for an arbitrary $x_0 \in H$,

$$\begin{aligned} z_n &= (I - t_n F)x_n + t_n u + e_n, \\ x_{n+1} &= J_{c_n}^T z_n, \quad n \geq 0, \end{aligned} \tag{1.6}$$

where F is a k-Lipschitzian and η-strongly monotone operator on H and u is a fixed point in H. Without the parameter sequence $\{t_n\}$ converging to zero, we prove that the sequence $\{x_n\}$ generated by the iterative algorithm (1.6) converges strongly to $x^* \in T^{-1}(0)$, which solves uniquely the variational inequality $\langle Fx^* - u, x^* - p\rangle \leq 0$, for all $p \in T^{-1}(0)$. In addition, an example is also given in order to illustrate the effectiveness of our generalizations. The results presented in this paper can be viewed as the improvement, supplement, and extension of the results obtained in [6–10].

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. For the sequence $\{x_n\}$ in H, we write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ means that $\{x_n\}$ converges strongly to x.

A mapping $F : H \to H$ is called k-Lipschitzian if there exists a positive constant k such that

$$\|Fx - Fy\| \leq k\|x - y\|, \quad \forall x, y \in H. \tag{2.1}$$

F is said to be η-strongly monotone if there exists a positive constant η such that

$$\langle Fx - Fy, x - y\rangle \geq \eta\|x - y\|^2, \quad \forall x, y \in H. \tag{2.2}$$

Let A be a strongly positive bounded linear operator on H, that is, there exists a constant $\widetilde{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \widetilde{\gamma}\|x\|^2, \quad \forall x \in H. \tag{2.3}$$

A typical problem is that of minimizing a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x \in \mathrm{Fix}(W)} \left\{ \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle \right\}, \tag{2.4}$$

where b is a given point in H and $\mathrm{Fix}(W)$ is the set of the fixed points of nonexpansive mapping W.

Remark 2.1 (see [11]). From the definition of A, we note that a strongly positive bounded linear operator A is a $\|A\|$-Lipschitzian and $\tilde{\gamma}$-strongly monotone operator.

Let T be a maximal monotone operator on a real Hilbert space H such that $S := T^{-1}(0) \neq \emptyset$. For $c > 0$, we use J_c^T to denote the resolvent of T, that is,

$$J_c^T = (I + cT)^{-1}. \tag{2.5}$$

It is well known that J_c^T is firmly nonexpansive and consequently nonexpansive; moreover, $S = \mathrm{Fix}(J_c^T) = \{x \in H : x = J_c^T x\}$.

The following lemma is known as the resolvent identity of maximal monotone operators.

Lemma 2.2 (see [8]). *Let $c, t > 0$. Then, for any $x \in H$,*

$$J_c^T x = J_t^T \left(\frac{t}{c} x + \left(1 - \frac{t}{c} \right) J_c^T x \right). \tag{2.6}$$

In order to prove our main results, we need the following lemmas.

Lemma 2.3 (see [11]). *Let F be a k-Lipschitzian and η-strongly monotone operator on a Hilbert space H with $0 < \eta \leq k$ and $0 < t < \eta/k^2$. Then, $S = (I - tF) : H \to H$ is a contraction with contraction coefficient $\tau_t = \sqrt{1 - t(2\eta - tk^2)}$.*

Lemma 2.4 (see [12]). *T is firmly nonexpansive if and only if $2T - I$ is nonexpansive.*

Lemma 2.5 (see [13]). *Let H be a Hilbert space, C a closed convex subset of H, and $T : C \to C$ a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$; if $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$.*

Lemma 2.6 (see [14]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in Banach space E and $\{\gamma_n\}$ a sequence in $[0, 1]$ which satisfies the following condition:*

$$0 < \liminf_{n \to \infty} \gamma_n \leq \limsup_{n \to \infty} \gamma_n < 1. \tag{2.7}$$

Suppose that $x_{n+1} = \gamma_n x_n + (1 - \gamma_n) z_n$, $n \geq 0$, and $\limsup_{n \to \infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n \to \infty} \|z_n - x_n\| = 0$.

Lemma 2.7 (see [15, 16]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \lambda_n) s_n + \lambda_n \delta_n + \gamma_n, \quad n \geq 0, \tag{2.8}$$

where $\{\lambda_n\}$, $\{\delta_n\}$, and $\{\gamma_n\}$ satisfy the following conditions: (i) $\{\lambda_n\} \subset [0, 1]$ and $\sum_{n=0}^{\infty} \lambda_n = \infty$, (ii) $\limsup_{n \to \infty} \delta_n \leq 0$ or $\sum_{n=0}^{\infty} \lambda_n \delta_n < \infty$, and (iii) $\gamma_n \geq 0$ $(n \geq 0)$, $\sum_{n=0}^{\infty} \gamma_n < \infty$. Then, $\lim_{n \to \infty} s_n = 0$.

3. Main Results

Let F be a k-Lipschitzian and η-strongly monotone operator on H with $0 < \eta \leq k$ and J_c^T the resolvent of T. Let $t \in (0, \eta/k^2)$ and $\tau_t = \sqrt{1 - t(2\eta - tk^2)} \in (0,1)$, and consider a mapping V_t on H defined by

$$V_t x = J_c^T[(I - tF)x + tu], \quad x \in H, \tag{3.1}$$

where $c > 0$ is a fixed constant and $u \in H$ is a fixed point. It is easy to see that V_t is a contraction. Indeed, from Lemma 2.3, we have

$$\begin{aligned}\|V_t x - V_t y\| &= \left\| J_c^T[(I - tF)x + tu] - J_c^T[(I - tF)y + tu] \right\| \\ &\leq \|(I - tF)x - (I - tF)y\| \\ &\leq \tau_t \|x - y\|,\end{aligned} \tag{3.2}$$

for all $x, y \in H$. Hence, it has a unique fixed point, denoted by v_t, which uniquely solves the fixed point equation

$$v_t = J_c^T[(I - tF)v_t + tu], \quad v_t \in H. \tag{3.3}$$

Theorem 3.1. *For any $c > 0$ and $u \in H$, let the net $\{v_t\}$ be generated by* (3.3)*. Then, as $t \to 0$, the net $\{v_t\}$ converges strongly to v^* of S, which solves uniquely the variational inequality*

$$\langle Fv^* - u, v^* - p \rangle \leq 0, \quad \forall p \in S. \tag{3.4}$$

Proof. We first show the uniqueness of a solution of the variational inequality (3.4), which is indeed a consequence of the strong monotonicity of F. Suppose $v^* \in S$ and $\tilde{v} \in S$ both are solutions to (3.4); then,

$$\langle Fv^* - u, v^* - \tilde{v} \rangle \leq 0, \tag{3.5}$$

$$\langle F\tilde{v} - u, \tilde{v} - v^* \rangle \leq 0. \tag{3.6}$$

Adding (3.5) to (3.6), we get

$$\langle Fv^* - F\tilde{v}, v^* - \tilde{v} \rangle \leq 0. \tag{3.7}$$

The strong monotonicity of F implies that $v^* = \tilde{v}$ and the uniqueness is proved. Below we use $v^* \in S$ to denote the unique solution of (3.4). Next, we prove that $\{v_t\}$ is bounded. Taking $p \in S$, from (3.3) and using Lemma 2.3, we have

$$\begin{aligned} \|v_t - p\| &= \left\| J_c^T[(I - tF)v_t + tu] - p \right\| \\ &\leq \|(I - tF)v_t - (I - tF)p + t(u - Fp)\| \\ &\leq \tau_t\|v_t - p\| + t\|u - Fp\|, \end{aligned} \tag{3.8}$$

that is,

$$\|v_t - p\| \leq \frac{t}{1 - \tau_t}\|u - Fp\|. \tag{3.9}$$

Observe that

$$\lim_{t \to 0^+} \frac{t}{1 - \tau_t} = \frac{1}{\eta}. \tag{3.10}$$

From $t \to 0$, we may assume, without loss of generality, that $t \leq \eta/k^2 - \epsilon$, where ϵ is an arbitrarily small positive number. Thus, we have that $t/(1 - \tau_t)$ is continuous, for all $t \in [0, \eta/k^2 - \epsilon]$. Therefore, we obtain

$$\sup\left\{ \frac{t}{1 - \tau_t} : t \in \left(0, \frac{\eta}{k^2} - \epsilon\right] \right\} < +\infty. \tag{3.11}$$

From (3.9) and (3.11), we have $\{v_t\}$ bounded and so is $\{Fv_t\}$. On the other hand, from (3.3), we obtain

$$\begin{aligned} \left\| v_t - J_c^T v_t \right\| &= \left\| J_c^T[(I - tF)v_t + tu] - J_c^T v_t \right\| \\ &\leq \|(I - tF)v_t + tu - v_t\| \\ &= t\|u - Fv_t\| \longrightarrow 0 \quad (t \longrightarrow 0). \end{aligned} \tag{3.12}$$

To prove that $v_t \to v^*$, for a given $p \in S$, using Lemma 2.3, we have

$$\begin{aligned} \|v_t - p\|^2 &= \left\| J_c^T[(I - tF)v_t + tu] - p \right\|^2 \\ &\leq \|(I - tF)v_t - (I - tF)p + t(u - Fp)\|^2 \\ &\leq \tau_t^2\|v_t - p\|^2 + t^2\|u - Fp\|^2 + 2t\langle (I - tF)v_t - (I - tF)p, u - Fp \rangle \end{aligned}$$

$$\begin{aligned}&\le \tau_t\left\|v_t-p\right\|^2+t^2\left\|u-Fp\right\|^2+2t\langle v_t-p,u-Fp\rangle+2t^2\langle Fp-Fv_t,u-Fp\rangle\\&\le \tau_t\left\|v_t-p\right\|^2+t^2\left\|u-Fp\right\|^2+2t\langle v_t-p,u-Fp\rangle+2t^2k\|p-v_t\|\|u-Fp\|\\&\le \tau_t\left\|v_t-p\right\|^2+2t^2M+2t\langle v_t-p,u-Fp\rangle,\end{aligned} \tag{3.13}$$

where $M=\max\{\|u-Fp\|^2,2k\|p-v_t\|\|u-Fp\|\}$. Therefore,

$$\left\|v_t-p\right\|^2\le \frac{2t^2}{1-\tau_t}M+\frac{2t}{1-\tau_t}\langle v_t-p,u-Fp\rangle. \tag{3.14}$$

From $\tau_t=\sqrt{1-t(2\eta-tk^2)}$, we have $\lim_{t\to 0}(t^2/(1-\tau_t))=0$. Moreover, if $v_t\rightharpoonup p$, we have $\lim_{t\to 0}((2t/(1-\tau_t))\langle v_t-p,u-Fp\rangle)=0$.

Since $\{v_t\}$ is bounded, we see that if $\{t_n\}$ is a sequence in $(0,\eta/k^2-\epsilon]$ such that $t_n\to 0$ and $v_{t_n}\rightharpoonup\tilde v$, then, by (3.14), we see that $v_{t_n}\to\tilde v$. Moreover, by (3.12) and using Lemma 2.5, we have $\tilde v\in S$. We next prove that $\tilde v$ solves the variational inequality (3.4). From (3.3) and $p\in S$, we have

$$\begin{aligned}\left\|v_t-p\right\|^2&\le\left\|(I-tF)v_t+tu-p\right\|^2\\&=\left\|v_t-p\right\|^2+t^2\|u-Fv_t\|^2+2t\langle v_t-p,u-Fv_t\rangle,\end{aligned} \tag{3.15}$$

that is,

$$\langle Fv_t-u,v_t-p\rangle\le\frac{t}{2}\|u-Fv_t\|^2. \tag{3.16}$$

Now replacing t in (3.16) with t_n and letting $n\to\infty$, we have

$$\langle F\tilde v-u,\tilde v-p\rangle\le 0. \tag{3.17}$$

That is, $\tilde v\in S$ is a solution of (3.4), and hence $\tilde v=v^*$ by uniqueness. In a summary, we have shown that each cluster point of $\{v_t\}$ (at $t\to 0$) equals v^*. Therefore, $v_t\to v^*$ as $t\to 0$. □

Setting $F=A$ in Theorem 3.1, we can obtain the following result.

Corollary 3.2. *For any $c>0$ and $u\in H$, let A be a strongly positive bounded linear operator with coefficient $0<\tilde\gamma\le\|A\|$. For each $t\in(0,\tilde\gamma/\|A\|^2)$, let the net $\{v_t\}$ be generated by $v_t=J_c^T[(I-tA)v_t+tu]$. Then, as $t\to 0$, the net $\{v_t\}$ converges strongly to v^* of S which solves uniquely the variational inequality*

$$\langle Av^*-u,v^*-p\rangle\le 0,\quad\forall p\in S. \tag{3.18}$$

Setting $F=I$ and $v^*=P_Su$ in Theorem 3.1, we can obtain the following result.

Corollary 3.3 (Xu [8, Theorem 3.1]). *For any $c > 0$ and $u \in H$. For each $t \in (0,1)$, let the net $\{v_t\}$ be generated by $v_t = J_c^T[(1-t)v_t + tu]$. Then, as $t \to 0$, $\{v_t\}$ converges strongly to the projection of u onto S; that is, $\lim_{t\to 0} v_t = P_S u$. Moreover, this limit is attained uniformly for $c > 0$.*

The next result gives a strong convergence theorem on algorithm (1.6) with a weaker restriction on the sequence $\{t_n\}$.

Theorem 3.4. *Let T be a maximal monotone operator on a Hilbert space H with $S \neq \emptyset$. Let F be a k-Lipschitzian and η-strongly monotone operator on H with $0 < \eta \le k$. Let $\{t_n\}$ be a sequence in $(0,1)$, $\{c_n\}$ a sequence in $(0,+\infty)$, and ϵ an arbitrarily small positive number. Assume that the control conditions* (C1′), (C3′), (C4′), *and* (C5) *hold for $\{t_n\}$, $\{c_n\}$, and $\{e_n\}$*

(C1′) $0 < t_n \le \eta/k^2 - \epsilon$, *for all $n \ge n_0$ for some integer $n_0 \ge 0$.*

For an arbitrary point $x_0 \in H$, let the sequence $\{x_n\}$ be generated by (1.6). *Then,*

$$z_n \longrightarrow x^* \Longleftrightarrow t_n(u - Fx_n) \longrightarrow 0 \quad (n \longrightarrow \infty), \tag{3.19}$$

where $x^* \in S$ solves the variational inequality

$$\langle Fx^* - u, x^* - p\rangle \le 0, \quad \forall p \in S. \tag{3.20}$$

Proof. On the one hand, suppose that $t_n(u - Fx_n) \to 0\,(n \to \infty)$. We proceed with the following steps.

Step 1. We claim that $\{x_n\}$ is bounded. In fact, taking $p \in S$, from (1.6) and (C1′) and using Lemma 2.3, we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \left\|J_{c_n}^T z_n - p\right\| \\
&\le \|(I - t_nF)x_n + t_nu + e_n - p\| \\
&\le \|(I - t_nF)x_n - (I - t_nF)p + t_n(u - Fp) + e_n\| \\
&\le \tau_{t_n}\|x_n - p\| + t_n\|u - Fp\| + \|e_n\| \\
&\le [1 - (1 - \tau_{t_n})]\|x_n - p\| + (1 - \tau_{t_n})\frac{t_n}{1 - \tau_{t_n}}\|u - Fp\| + \|e_n\| \\
&\le \max\left\{\|x_n - p\|, \frac{t_n}{1 - \tau_{t_n}}\|u - Fp\|\right\} + \|e_n\|,
\end{aligned} \tag{3.21}$$

for all $n \ge n_0$ for some integer $n_0 \ge 0$, where $\tau_{t_n} = \sqrt{1 - t_n(2\eta - t_nk^2)} \in (0,1)$. By induction, we have

$$\|x_n - p\| \le \max\{\|x_0 - p\|, \|u - Fp\|M_1\} + \sum_{j=0}^{n-1}\|e_j\|, \tag{3.22}$$

for all $n \geq n_0$ for some integer $n_0 \geq 0$, where $M_1 = \sup\{t_n/(1-\tau_{t_n}) : 0 < t_n \leq \eta/k^2 - \epsilon\} < +\infty$. Therefore, $\{x_n\}$ is bounded. We also obtain that $\{z_n\}$ and $\{Fx_n\}$ are bounded.

Step 2. We claim that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. In fact, write $J_n = J_{c_n}^T$ and $T_n = 2J_n - I$. Then, J_n is firmly nonexpansive and T_n is nonexpansive (see Lemma 2.4).

Observe that

$$\begin{aligned} x_{n+1} = J_n z_n &= \left(\frac{I+T_n}{2}\right) z_n \\ &= \frac{1}{2} z_n + \frac{1}{2} T_n z_n \\ &= \frac{1}{2} x_n + \frac{1}{2}\left[t_n(u - Fx_n) + e_n + T_n z_n\right] \\ &= \frac{1}{2} x_n + \frac{1}{2} y_n, \end{aligned} \tag{3.23}$$

where $y_n = t_n(u - Fx_n) + e_n + T_n z_n$. Therefore,

$$\begin{aligned} \left\|y_{n+1} - y_n\right\| &= \|t_{n+1}(u - Fx_{n+1}) + e_{n+1} + T_{n+1} z_{n+1} - t_n(u - Fx_n) - e_n - T_n z_n\| \\ &\leq \|t_{n+1}(u - Fx_{n+1})\| + \|t_n(u - Fx_n)\| + \|e_{n+1}\| + \|e_n\| + \|T_{n+1} z_{n+1} - T_n z_n\|. \end{aligned} \tag{3.24}$$

It follows from the resolvent identity that

$$\begin{aligned} \|T_{n+1} x - T_n x\| &= 2\|J_{n+1} x - J_n x\| \\ &= 2\left\| J_n\left(\frac{c_n}{c_{n+1}} x + \left(1 - \frac{c_n}{c_{n+1}}\right) J_{n+1} x\right) - J_n x\right\| \\ &\leq 2\left|1 - \frac{c_n}{c_{n+1}}\right| \|J_{n+1} x - x\| \\ &\leq \left|1 - \frac{c_n}{c_{n+1}}\right| \|T_{n+1} x - x\| \end{aligned} \tag{3.25}$$

for any $x \in H$. From (1.6), we get

$$\begin{aligned} \|z_{n+1} - z_n\| &= \|(I - t_{n+1}F)x_{n+1} + t_{n+1}u + e_{n+1} - (I - t_n F)x_n - t_n u - e_n\| \\ &\leq \|x_{n+1} - x_n\| + \|t_{n+1}(u - Fx_{n+1})\| + \|t_n(u - Fx_n)\| + \|e_{n+1}\| + \|e_n\|. \end{aligned} \tag{3.26}$$

By (3.25) and (3.26), we have

$$\begin{aligned} \|T_{n+1} z_{n+1} - T_n z_n\| &\leq \|T_{n+1} z_{n+1} - T_n z_{n+1}\| + \|T_n z_{n+1} - T_n z_n\| \\ &\leq \left|1 - \frac{c_n}{c_{n+1}}\right| \|T_{n+1} z_{n+1} - z_{n+1}\| + \|z_{n+1} - z_n\| \end{aligned}$$

$$\leq \left|1 - \frac{c_n}{c_{n+1}}\right| \|T_{n+1}z_{n+1} - z_{n+1}\| + \|x_{n+1} - x_n\| \\ + \|t_{n+1}(u - Fx_{n+1})\| + \|t_n(u - Fx_n)\| + \|e_{n+1}\| + \|e_n\|. \tag{3.27}$$

Substituting (3.27) into (3.24) at once gives

$$\begin{aligned} \|y_{n+1} - y_n\| &\leq 2\|t_{n+1}(u - Fx_{n+1})\| + 2\|t_n(u - Fx_n)\| + 2\|e_{n+1}\| \\ &\quad + 2\|e_n\| + \left|1 - \frac{c_n}{c_{n+1}}\right| M_2 + \|x_{n+1} - x_n\|, \end{aligned} \tag{3.28}$$

that is,

$$\begin{aligned} \|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| &\leq 2\|t_{n+1}(u - Fx_{n+1})\| + 2\|t_n(u - Fx_n)\| + 2\|e_{n+1}\| \\ &\quad + 2\|e_n\| + \left|1 - \frac{c_n}{c_{n+1}}\right| M_2, \end{aligned} \tag{3.29}$$

where $M_2 = \sup\{\|T_{n+1}z_{n+1} - z_{n+1}\|, n \geq 0\}$. Observing $t_n(u - Fx_n) \to 0$, $e_n \to 0$, and $|1 - (c_n/c_{n+1})| \to 0\,(n \to \infty)$, it follows that

$$\limsup_{n\to\infty}\left(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|\right) \leq 0. \tag{3.30}$$

From (3.23) and using Lemma 2.6, we have $\lim_{n\to\infty}\|y_n - x_n\| = 0$. Therefore,

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}\frac{1}{2}\|y_n - x_n\| = 0. \tag{3.31}$$

Step 3. We claim that $\lim_{n\to\infty}\|x_n - J_c^T x_n\| = 0$. Since $\liminf_{n\to\infty}c_n > 0$, there exist $\alpha > 0$ and a positive integer N such that for all $n \geq N$, $c_n \geq \alpha$. From Lemma 2.2, for each $c \in (0, \alpha)$, we have

$$\begin{aligned} \left\|J_n x_n - J_c^T x_n\right\| &= \left\|J_c^T\left(\frac{c}{c_n}x_n + \left(1 - \frac{c}{c_n}\right)J_n x_n\right) - J_c^T x_n\right\| \\ &\leq \left\|\frac{c}{c_n}x_n + \left(1 - \frac{c}{c_n}\right)J_n x_n - x_n\right\| \\ &= \left|1 - \frac{c}{c_n}\right| \|J_n x_n - x_n\| \\ &\leq \|J_n x_n - x_{n+1}\| + \|x_{n+1} - x_n\|. \end{aligned} \tag{3.32}$$

Observe that

$$\|J_n x_n - x_{n+1}\| \leq \|x_n - z_n\| \leq \|t_n(u - Fx_n)\| + \|e_n\| \longrightarrow 0. \tag{3.33}$$

Thus, it follows from (3.32), (3.33), and Step 2 that

$$\lim_{n\to\infty}\left\|J_n x_n - J_c^T x_n\right\| = 0. \tag{3.34}$$

Since $\|x_n - J_c^T x_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - J_n x_n\| + \|J_n x_n - J_c^T x_n\|$, then

$$\lim_{n\to\infty}\left\|x_n - J_c^T x_n\right\| = 0. \tag{3.35}$$

Step 4. We claim that $\limsup_{n\to\infty}\langle x_n - x^*, u - Fx^*\rangle \le 0$, where $x^* = \lim_{t\to 0} v_t$ and v_t is defined by (3.3). Since x_n is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to ω. From Step 3, we obtain $J_c^T x_n \rightharpoonup \omega$. From Lemma 2.5, we have $\omega \in S$. Hence, by Theorem 3.1, we have

$$\limsup_{n\to\infty}\langle x_n - x^*, u - Fx^*\rangle = \lim_{k\to\infty}\langle x_{n_k} - x^*, u - Fx^*\rangle = \langle \omega - x^*, u - Fx^*\rangle \le 0. \tag{3.36}$$

Step 5. We claim that $\{z_n\}$ converges strongly to $x^* \in S$. From (1.6), for an appropriate constant $\gamma > 0$, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|J_n z_n - x^*\|^2 \\
&\le \|z_n - x^*\|^2 \\
&= \|(I - t_n F)x_n + t_n u + e_n - x^*\|^2 \\
&\le \|(I - t_n F)x_n + t_n u - x^*\|^2 + \gamma\|e_n\| \\
&\le \|(I - t_n F)x_n - (I - t_n F)x^* + t_n(u - Fx^*)\|^2 + \gamma\|e_n\| \\
&\le \tau_{t_n}^2\|x_n - x^*\|^2 + t_n^2\|u - Fx^*\|^2 + 2t_n\langle (I - t_n F)x_n - (I - t_n F)x^*, u - Fx^*\rangle + \gamma\|e_n\| \\
&\le \tau_{t_n}\|x_n - x^*\|^2 + t_n^2\|u - Fx^*\|^2 + 2t_n\langle x_n - x^* - t_n F x_n + t_n u, u - Fx^*\rangle \\
&\quad + 2t_n^2\langle Fx^* - u, u - Fx^*\rangle + \gamma\|e_n\| \\
&\le \tau_{t_n}\|x_n - x^*\|^2 + 2t_n\langle x_n - x^*, u - Fx^*\rangle + 2t_n\|t_n(u - Fx_n)\|\|u - Fx^*\| + \gamma\|e_n\| \\
&\le [1 - (1 - \tau_{t_n})]\|x_n - x^*\|^2 \\
&\quad + (1 - \tau_{t_n})[2M_1\langle x_n - x^*, u - Fx^*\rangle + 2M_1\|t_n(u - Fx_n)\|\|u - Fx^*\|] + \gamma\|e_n\|,
\end{aligned} \tag{3.37}$$

for all $n \ge n_0$ for some integer $n_0 \ge 0$. For every $n \ge n_0$, put $\mu_n = 1 - \tau_{t_n}$ and $\delta_n = 2M_1\langle x_n - x^*, u - Fx^*\rangle + 2M_1\|t_n(u - Fx_n)\|\|u - Fx^*\|$. It follows that

$$\|x_{n+1} - x^*\|^2 \le (1 - \mu_n)\|x_n - x^*\|^2 + \mu_n\delta_n + \gamma\|e_n\|, \quad \forall n \ge n_0. \tag{3.38}$$

It is easy to see that $\sum_{n=1}^{\infty} \mu_n = \infty$ and $\limsup_{n\to\infty} \delta_n \le 0$. Hence, by Lemma 2.7, the sequence $\{x_n\}$ converges strongly to $x^* \in S$. Observe that

$$\begin{aligned} \|z_n - x^*\| &= \|(I - t_n F)x_n + t_n u + e_n - x^*\| \\ &\le \|x_n - x^*\| + \|t_n(u - Fx_n)\| + \|e_n\|. \end{aligned} \tag{3.39}$$

Thus, it follows that the sequence $\{z_n\}$ converges strongly to $x^* \in S$.

On the other hand, suppose that $z_n \to x^* \in S$ as $n \to \infty$. From (1.6), we have

$$\|x_{n+1} - x^*\| = \|J_n z_n - x^*\| \le \|z_n - x^*\| \longrightarrow 0. \tag{3.40}$$

Therefore,

$$\begin{aligned} \|t_n(u - Fx_n)\| &= \|z_n - x_n - e_n\| \\ &\le \|z_n - x_n\| + \|e_n\| \\ &\le \|z_n - x^*\| + \|x_n - x^*\| + \|e_n\| \longrightarrow 0. \end{aligned} \tag{3.41}$$

□

Setting $F = I$ and $x^* = P_S u$ in Theorem 3.4, we can obtain the following result.

Corollary 3.5. *Let T be a maximal monotone operator on a Hilbert space H with $S \neq \emptyset$. Let $\{t_n\}$ be a sequence in $(0,1)$, $\{c_n\}$ a sequence in $(0,+\infty)$, and ϵ an arbitrarily small positive number. Assume that the control conditions* (C1″), (C3′), (C4′), *and* (C5) *hold for $\{t_n\}$, $\{c_n\}$, and $\{e_n\}$.*

(C1″) $0 < t_n \le 1 - \epsilon$, *for all $n \ge n_0$ for some integer $n_0 \ge 0$.*

For an arbitrary point $x_0 \in H$, let the sequence $\{x_n\}$ be generated by

$$\begin{aligned} z_n &= (1 - t_n)x_n + t_n u + e_n, \\ x_{n+1} &= J_{c_n}^T z_n, \quad n \ge 0. \end{aligned} \tag{3.42}$$

Then,

$$z_n \longrightarrow P_S u \Longleftrightarrow t_n(u - x_n) \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{3.43}$$

Corollary 3.6 ([Wang [10], Theorem 4]). *Let $\{c_n\}$, $\{t_n\}$, and $\{e_n\}$ satisfy* (C1), (C3), (*or* (C3′)), (C4′) *and* (C5). *In addition, if $S \neq \emptyset$, then the sequence generated by* (1.5) *converges strongly to $P_S u$.*

Proof. Since $\lim_{n\to\infty} t_n = 0$, it is easy to see that $t_n \le \eta/k^2 - \epsilon$, for all $n \ge n_0$ for some integer $n_0 \ge 0$. Without loss of generality, we assume that $0 < t_n \le \eta/k^2 - \epsilon$, for all $n \ge n_0$ for some integer $n_0 \ge 0$. Repeating the same argument as in the proof of Theorem 4 in Wang [10], we know that $\{x_n\}$ is bounded. Thus, we have that $t_n(u - x_n) \to 0$. Therefore, all conditions of Corollary 3.5 are satisfied. Using Corollary 3.5, we have that $\{z_n\}$ converges strongly to $P_S u \in S$, with $z_n = (1-t_n)x_n + t_n u + e_n$. Therefore,

$$\|x_{n+1} - P_S u\| \leq \|z_n - P_S u\| \longrightarrow 0. \tag{3.44}$$

□

Remark 3.7. Corollary 3.5 is more general than Theorem 4 of Wang [10]. The following example is given in order to illustrate the effectiveness of our generalizations.

Example 3.8. Let $H = R$ be the set of real numbers, $u = 0$ and $c_n = 1/2$ for all $n \geq 0$. Define a maximal monotone operator T as follows: $Tx = 2x$, for all $x \in R$. It is easy to see that $J_{c_n}^T = (1/2)I$ and $S = \{0\}$. Given sequences $\{t_n\}$ and $\{e_n\}$, $t_n = 1/2$ and $e_n = 0$, for all $n \geq 0$. For an arbitrary $x_0 \in R$, let $\{x_n\}$ be defined by (3.42), that is,

$$\begin{aligned} z_n &= \frac{1}{2}x_n, \\ x_{n+1} &= \frac{1}{2}z_n = \frac{1}{4}x_n, \quad n \geq 0. \end{aligned} \tag{3.45}$$

Observe that

$$\|x_{n+1} - 0\| = \left\|\frac{1}{4}x_n - 0\right\| = \frac{1}{4}\|x_n - 0\|. \tag{3.46}$$

Hence, we have $\|x_{n+1} - 0\| = (1/4)^{n+1}\|x_0 - 0\|$ for all $n \geq 0$. This implies that $\{x_n\}$ converges strongly to $0 = P_S 0$. Thus,

$$\|t_n(u - x_n)\| = \frac{1}{2}\|x_n\| \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{3.47}$$

Furthermore, it is easy to see that there hold the following:

(B1) $0 < t_n = 1/2 \leq 1 - \epsilon$, for all $n \geq n_0$ for some integer $n_0 \geq 0$,

(B2) $\sum_{n=0}^{\infty} t_n = \sum_{n=0}^{\infty}(1/2) = \infty$,

(B3) $\liminf_{n\to\infty} c_n = 1/2 > 0$ and $\lim_{n\to\infty}|1 - (c_n/c_{n+1})| = 0$,

(B4) $\sum_{n=0}^{\infty}\|e_n\| = \sum_{n=0}^{\infty} 0 = 0 < \infty$.

Hence there is no doubt that all conditions of Corollary 3.5 are satisfied. Since $t_n = 1/2 \nrightarrow 0$, the condition $t_n \to 0$ of Wang [10, Theorem 4] is not satisfied. So, by Corollary 3.5, we obtain that the sequence $\{x_n\}$ and $\{z_n\}$ converges strongly to zero but Theorem 4 of Wang [10] cannot be applied to $\{x_n\}$ and $\{z_n\}$ in this example.

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