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*Research Article*

# Convergence Analysis for the SMC-MeMBer and SMC-CBMeMBer Filters

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The convergence for the sequential Monte Carlo (SMC) implementations of the multitarget multi-Bernoulli (MeMBer) filter and cardinality-balanced MeMBer (CBMeMBer) filters is studied here. This paper proves that the SMC-MeMBer and SMC-CBMeMBer filters, respectively, converge to the true MeMBer and CBMeMBer filters in the mean-square sense and the corresponding bounds for the mean-square errors are given. The significance of this paper is in theory to present the convergence results of the SMC-MeMBer and SMC-CBMeMBer filters and the conditions under which the two filters satisfy mean-square convergence.

## 1. Introduction

Recently, the random finite-set- (RFS-) based multitarget tracking (MTT) approaches [1] have attracted extensive attention. Although theoretically solid, the RFS-based approaches usually involve intractable computations. By introducing the finite-set statistics (FISSTs) [2], Mahler developed the probability hypothesis density (PHD) [3], and cardinalized PHD (CPHD) [4] filters, which have been shown to be a computationally tractable alternative to full multitarget Bayes filters in the RFS framework. The sequential Monte Carlo (SMC) implementations for the PHD and CPHD filters were devised by Zajic and Mahler [5], Sidenbladh [6], and Vo et al. [7]. Vo et al. [8, 9] devised the Gaussian mixture (GM) implementation for the PHD and CPHD filters under the linear, Gaussian assumption on target dynamics, birth process, and sensor model. However, the SMC-PHD and SMC-CPHD approaches require clustering to extract state estimates from the particle population, which is expensive and unreliable [10, 11].

In 2007, Mahler proposed the multitarget multi-Bernoulli (MeMBer) [2] recursion, which is an approximation to the full multitarget Bayes recursion using multi-Bernoulli

RFSs under low clutter density scenarios. In 2009, Vo et al. showed that the MeMBer filter overestimates the number of targets and proposed a cardinality-balanced MeMBer (CBMeMBer) filter [12] to reduce the cardinality bias. Then, the SMC and GM implementations for the MeMBer and CBMeMBer filters were, respectively, proposed in nonlinear and linear-Gaussian dynamic and measurement models. The key advantage of this approach is that the multi-Bernoulli representation allows reliable and inexpensive extraction of state estimates. The Monte Carlo simulations given by Vo et al. showed that the SMC-CBMeMBer filter outperforms the SMC-CPHD (and hence SMC-PHD) filter despite having smaller complexity under certain range of signal settings.

Although the convergence results for the SMC-PHD and GM-PHD filters were established by Clark and Bell [13] in 2006 and by Clark and Vo [14] in 2007, respectively, there have been no results showing the asymptotic convergence for the SMC-MeMBer and SMC-CBMeMBer filters. This paper demonstrates the mean-square convergence of the errors [15–17] for the two filters. In other words, given simple sufficient conditions, the approximation error of the multi-Bernoulli parameter set comprised of a set of weighted samples is proved to converge to zero as the number of the samples tends to infinity at each stage of the two algorithms. In addition, the corresponding bounds for the mean-square errors are obtained.

## 2. MeMBer and CBMeMBer Filters

A Bernoulli RFS $Y^{(i)}$ has probability $1 - r^{(i)}$ of being empty, and probability $r^{(i)}$ $(0 \le r^{(i)} \le 1)$ of being a singleton whose only element is distributed according to a probability density $p^{(i)}$. The probability density of $Y^{(i)}$ is

$$\pi\left(Y^{(i)}\right) = \begin{cases} 1 - r^{(i)} & Y^{(i)} = \emptyset, \\ r^{(i)} p^{(i)}(\mathbf{y}_i) & Y^{(i)} = \{\mathbf{y}_i\}. \end{cases} \tag{2.1}$$

A multi-Bernoulli RFS $Y$ is a union of a fixed number of independent Bernoulli RFSs $Y^{(i)}$, $i = 1, \ldots, M$, that is, $Y = \bigcup_{i=1}^{M} Y^{(i)}$. $Y$ is thus completely described by the multi-Bernoulli parameter set $\{(r^{(i)}, p^{(i)})\}_{i=1}^{M}$ with the mean cardinality $\sum_{i=1}^{M} r^{(i)}$ and the probability density [2]:

$$\pi(Y) = \prod_{j=1}^{M}\left(1 - r^{(j)}\right) \sum_{1 \le i_1 \ne \cdots \ne i_n \le M} \prod_{j=1}^{n} \frac{r^{(i_j)} p^{(i_j)}(\mathbf{y}_j)}{1 - r^{(i_j)}}. \tag{2.2}$$

Throughout this paper, we abbreviate a probability density of the form (2.2) by $\pi = \{(r^{(i)}, p^{(i)})\}_{i=1}^{M}$.

By approximating the multitarget RFS as a multi-Bernoulli RFS at each time step, Mahler proposed the MeMBer recursion, which propagated the multi-Bernoulli parameters of the posterior multitarget density forward in time [2]. The MeMBer filter is summarized as follows.

*MeMBer Prediction*

If at time $k-1$, the posterior multitarget density is a multi-Bernoulli of the form $\pi_{k-1} = \{(r_{k-1}^{(i)}, p_{k-1}^{(i)})\}_{i=1}^{M_{k-1}}$, then the predicted multitarget density is also a multi-Bernoulli and is given by

$$\pi_{k|k-1} = \left\{\left(r_{P,k|k-1}^{(i)}, p_{P,k|k-1}^{(i)}\right)\right\}_{i=1}^{M_{k-1}} \bigcup \left\{\left(r_{\Gamma,k}^{(i)}, p_{\Gamma,k}^{(i)}\right)\right\}_{i=1}^{M_{\Gamma,k}}, \tag{2.3}$$

where $\{(r_{\Gamma,k}^{(i)}, p_{\Gamma,k}^{(i)})\}_{i=1}^{M_{\Gamma,k}}$ are the parameters of the multi-Bernoulli RFS of births at time $k$:

$$r_{P,k|k-1}^{(i)} = r_{k-1}^{(i)} \left\langle p_{k-1}^{(i)}, p_{S,k} \right\rangle, \quad i = 1, \ldots, M_{k-1}, \tag{2.4}$$

$$p_{P,k|k-1}^{(i)}(\mathbf{x}_k) = \frac{\left\langle f_{k|k-1}(\mathbf{x}_k \mid \cdot), p_{k-1}^{(i)} p_{S,k} \right\rangle}{\left\langle p_{k-1}^{(i)}, p_{S,k} \right\rangle}, \quad i = 1, \ldots, M_{k-1}. \tag{2.5}$$

*MeMBer Update*

If at time $k$, the predicted multitarget density is a multi-Bernoulli of the form $\pi_{k|k-1} = \{(r_{k|k-1}^{(i)}, p_{k|k-1}^{(i)})\}_{i=1}^{M_{k|k-1}}$; then the posterior multitarget density can be approximated by a multi-Bernoulli as follows:

$$\pi_k \approx \left\{\left(r_{L,k}^{(i)}, p_{L,k}^{(i)}\right)\right\}_{i=1}^{M_{k|k-1}} \bigcup \left\{\left(r_{U,k}(\mathbf{z}_k), p_{U,k}(\cdot; \mathbf{z}_k)\right)\right\}_{\mathbf{z}_k \in Z_k}, \tag{2.6}$$

where,

$$r_{L,k}^{(i)} = r_{k|k-1}^{(i)} \frac{1 - \left\langle p_{k|k-1}^{(i)}, p_{D,k} \right\rangle}{1 - r_{k|k-1}^{(i)} \left\langle p_{k|k-1}^{(i)}, p_{D,k} \right\rangle}, \quad i = 1, \ldots, M_{k|k-1}, \tag{2.7}$$

$$p_{L,k}^{(i)}(\mathbf{x}_k) = p_{k|k-1}^{(i)}(\mathbf{x}_k) \frac{1 - p_{D,k}(\mathbf{x}_k)}{1 - \left\langle p_{k|k-1}^{(i)}, p_{D,k} \right\rangle}, \quad i = 1, \ldots, M_{k|k-1}, \tag{2.8}$$

$$r_{U,k}(\mathbf{z}_k) = \frac{1}{\kappa_k(\mathbf{z}_k) + \sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i)}(\mathbf{z}_k)} \sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i)}(\mathbf{z}_k), \quad \mathbf{z}_k \in Z_k, \tag{2.9}$$

$$p_{U,k}(\mathbf{x}_k; \mathbf{z}_k) = \frac{1}{\sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i)}(\mathbf{z}_k)} \sum_{i=1}^{M_{k|k-1}} p_{\alpha,k}^{(i)}(\mathbf{x}_k; \mathbf{z}_k), \quad \mathbf{z}_k \in Z_k, \tag{2.10}$$

$$p_{\alpha,k}^{(i)}(\mathbf{x}_k; \mathbf{z}_k) = \frac{r_{k|k-1}^{(i)} p_{k|k-1}^{(i)}(\mathbf{x}_k) \psi_{k,\mathbf{z}_k}(\mathbf{x}_k)}{1 - r_{k|k-1}^{(i)} \left\langle p_{k|k-1}^{(i)}, p_{D,k} \right\rangle}, \quad i = 1, \ldots, M_{k|k-1}, \tag{2.11}$$

$$r_{\alpha,k}^{(i)}(\mathbf{z}_k) = \left\langle p_{\alpha,k}^{(i)}(\mathbf{x}_k; \mathbf{z}_k), 1 \right\rangle = \frac{r_{k|k-1}^{(i)} \left\langle p_{k|k-1}^{(i)}, \psi_{k,\mathbf{z}_k} \right\rangle}{1 - r_{k|k-1}^{(i)} \left\langle p_{k|k-1}^{(i)}, p_{D,k} \right\rangle}, \quad i = 1, \ldots, M_{k|k-1}. \tag{2.12}$$

By correcting the cardinality bias in the $r_{U,k}(\mathbf{z}_k)$ of the MeMBer update step, Vo et al. proposed the CBMeMBer filter [12]. The CBMeMBer recursions are the same as the MeMBer recursions except the update of $r_{U,k}(\mathbf{z}_k)$, which is revised as

$$r^*_{U,k}(\mathbf{z}_k) = \frac{1}{\kappa_k(\mathbf{z}_k) + \sum_{i=1}^{M_{k|k-1}} r^{(i)}_{\alpha,k}(\mathbf{z}_k)} \sum_{i=1}^{M_{k|k-1}} \frac{\left(1 - r^{(i)}_{k|k-1}\right) r^{(i)}_{\alpha,k}(\mathbf{z}_k)}{1 - r^{(i)}_{k|k-1}\left\langle p^{(i)}_{k|k-1}, p_{D,k}\right\rangle}. \tag{2.13}$$

Note that not (38) in [12] but (2.10) in our paper is used in the CBMeMBer update step here. The reasons are (1) the (38) in [12] and the (2.10) in our paper are both the approximations of (36) in [12] under the same assumption $\langle p^{(i)}_{k|k-1}, p_{D,k}\rangle \approx 1$, but the latter is more precise than former; (2) the (38) in [12] is unbounded at $r^{(i)}_{k|k-1} = 1$ while (2.10) in our paper is bounded at $r^{(i)}_{k|k-1} = 1$ as long as $p_{D,k}(\mathbf{x}_k) \neq 1$.

For the multi-Bernoulli representation $\pi_k = \{(r^{(i)}_k, p^{(i)}_k)\}_{i=1}^{M_k}$, the probability $r^{(i)}_k$ indicates how likely the $i$th hypothesized track is a true track, and the posterior density $p^{(i)}_k$ describes the distribution of the estimated current state of the track. Hence, $\sum_{i=1}^{M_k} r^{(i)}_k$ denotes the multitarget number and the multitarget state estimate can be obtained by choosing the means or modes from the posterior densities of the hypothesized tracks with existence probabilities exceeding a given threshold.

## 3. SMC-MeMBer and SMC-CBMeMBer Filters

The SMC implementations of the MeMBer and CBMeMBer recursions are summarized as follows.

*SMC-MeMBer and SMC-CBMeMBer Predictions*

Suppose that at time $k-1$ the (multi-Bernoulli) posterior multitarget density $\tilde{\pi}_{k-1} = \{(r^{(i),L^{(i)}_{k-1}}_{k-1}, p^{(i),L^{(i)}_{k-1}}_{k-1})\}_{i=1}^{M_{k-1}}$ is given and each $p^{(i),L^{(i)}_{k-1}}_{k-1}$, $i = 1, \ldots, M_{k-1}$, is comprised of a set of weighted samples $\{\omega^{(i,j)}_{k-1}, \mathbf{x}^{(i,j)}_{k-1}\}_{j=1}^{L^{(i)}_{k-1}}$:

$$p^{(i),L^{(i)}_{k-1}}_{k-1}(\mathbf{x}_k) = \sum_{j=1}^{L^{(i)}_{k-1}} \omega^{(i,j)}_{k-1} \delta_{\mathbf{x}^{(i,j)}_{k-1}}(\mathbf{x}_k), \quad i = 1, \ldots, M_{k-1}. \tag{3.1}$$

Then, given proposal densities $q^{(i)}_k(\cdot \mid \mathbf{x}^{(i,j)}_{k-1}, Z_k)$ and $b^{(i)}_k(\cdot \mid Z_k)$, the predicted (multi-Bernoulli) multitarget density $\tilde{\pi}_{k|k-1} = \{(r^{(i),L^{(i)}_{k-1}}_{P,k|k-1}, p^{(i),L^{(i)}_{k-1}}_{P,k|k-1})\}_{i=1}^{M_{k-1}} \cup \{(r^{(i),L^{(i)}_{\Gamma,k}}_{\Gamma,k}, p^{(i),L^{(i)}_{\Gamma,k}}_{\Gamma,k})\}_{i=1}^{M_{\Gamma,k}}$ can be computed as follows:

$$r^{(i),L^{(i)}_{k-1}}_{P,k|k-1} = r^{(i),L^{(i)}_{k-1}}_{k-1} \left\langle p^{(i),L^{(i)}_{k-1}}_{k-1}, p_{S,k}\right\rangle, \quad i = 1, \ldots, M_{k-1}, \tag{3.2}$$

$$p_{P,k|k-1}^{(i),L_{k-1}^{(i)}}(\mathbf{x}_k) = \sum_{j=1}^{L_{k-1}^{(i)}} \widetilde{\omega}_{P,k|k-1}^{(i,j)} \delta_{\mathbf{x}_{P,k|k-1}^{(i,j)}}(\mathbf{x}_k), \quad i = 1, \ldots, M_{k-1},$$
$$p_{\Gamma,k}^{(i),L_{\Gamma,k}^{(i)}}(\mathbf{x}_k) = \sum_{j=1}^{L_{\Gamma,k}^{(i)}} \widetilde{\omega}_{\Gamma,k}^{(i,j)} \delta_{\mathbf{x}_{\Gamma,k}^{(i,j)}}(\mathbf{x}_k), \quad i = 1, \ldots, M_{\Gamma,k}, \tag{3.3}$$

where $r_{\Gamma,k}^{(i),L_{\Gamma,k}^{(i)}}$ $(i = 1, \ldots, M_{\Gamma,k})$ is given by birth model; $\mathbf{x}_{P,k|k-1}^{(i,j)}$, $\widetilde{\omega}_{P,k|k-1}^{(i,j)}$ $(i = 1, \ldots, M_{k-1})$ and $\mathbf{x}_{\Gamma,k}^{(i,j)}$, $\widetilde{\omega}_{\Gamma,k}^{(i,j)}$ $(i = 1, \ldots, M_{\Gamma,k})$ are, respectively, given by

$$\mathbf{x}_{P,k|k-1}^{(i,j)} \sim q_k^{(i)}\left(\cdot \mid \mathbf{x}_{k-1}^{(i,j)}, Z_k\right), \quad j = 1, \ldots, L_{k-1}^{(i)}, \qquad \mathbf{x}_{\Gamma,k}^{(i,j)} \sim b_k^{(i)}(\cdot \mid Z_k), \quad j = 1, \ldots, L_{\Gamma,k}^{(i)}, \tag{3.4}$$

$$\widetilde{\omega}_{P,k|k-1}^{(i,j)} = \frac{1}{\sum_{j=1}^{L_{k-1}^{(i)}} \omega_{P,k|k-1}^{(i,j)}} \omega_{P,k|k-1}^{(i,j)},$$
$$\omega_{P,k|k-1}^{(i,j)} = \frac{\omega_{k-1}^{(i,j)} f_{k|k-1}\left(\mathbf{x}_{P,k|k-1}^{(i,j)} \mid \mathbf{x}_{k-1}^{(i,j)}\right) p_{S,k}\left(\mathbf{x}_{k-1}^{(i,j)}\right)}{q_k^{(i)}\left(\mathbf{x}_{P,k|k-1}^{(i,j)} \mid \mathbf{x}_{k-1}^{(i,j)}, Z_k\right)}, \quad j = 1, \ldots, L_{k-1}^{(i)}, \tag{3.5}$$

$$\widetilde{\omega}_{\Gamma,k}^{(i,j)} = \frac{1}{\sum_{j=1}^{L_{\Gamma,k}^{(i)}} \omega_{\Gamma,k}^{(i,j)}} \omega_{\Gamma,k}^{(i,j)}, \qquad \omega_{\Gamma,k}^{(i,j)} = \frac{p_{\Gamma,k}\left(\mathbf{x}_{\Gamma,k}^{(i,j)}\right)}{b_k^{(i)}\left(\mathbf{x}_{\Gamma,k}^{(i,j)} \mid Z_k\right)}, \quad j = 1, \ldots, L_{\Gamma,k}^{(i)}. \tag{3.6}$$

*SMC-MeMBer and SMC-CBMeMBer Updates*

Suppose that at time $k$ the predicted (multi-Bernoulli) multitarget density $\widetilde{\pi}_{k|k-1} = \{(r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}, p_{k|k-1}^{(i),L_{k|k-1}^{(i)}})\}_{i=1}^{M_{k|k-1}}$ is given and each $p_{k|k-1}^{(i),L_{k|k-1}^{(i)}}$, $i = 1, \ldots, M_{k|k-1}$, is comprised of a set of weighted samples $\{\omega_{k|k-1}^{(i,j)}, \mathbf{x}_{k|k-1}^{(i,j)}\}_{j=1}^{L_{k|k-1}^{(i)}}$:

$$p_{k|k-1}^{(i),L_{k|k-1}^{(i)}}(\mathbf{x}_k) = \sum_{j=1}^{L_{k|k-1}^{(i)}} \omega_{k|k-1}^{(i,j)} \delta_{\mathbf{x}_{k|k-1}^{(i,j)}}(\mathbf{x}_k), \quad i = 1, \ldots, M_{k|k-1}. \tag{3.7}$$

Then, the multi-Bernoulli approximation of the SMC-MeMBer-updated multitarget density $\widetilde{\pi}_k \approx \{(r_{L,k}^{(i),L_{k|k-1}^{(i)}}, p_{L,k}^{(i),L_{k|k-1}^{(i)}})\}_{i=1}^{M_{k|k-1}} \cup \{(r_{U,k}^{L_{k|k-1}^{(i)}}(\mathbf{z}_k), p_{U,k}^{L_{k|k-1}^{(i)}}(\cdot;\mathbf{z}_k))\}_{\mathbf{z}_k \in Z_k}$ and SMC-CBMeMBer-updated multitarget density $\widetilde{\pi}_k^* \approx \{(r_{L,k}^{(i),L_{k|k-1}^{(i)}}, p_{L,k}^{(i),L_{k|k-1}^{(i)}})\}_{i=1}^{M_{k|k-1}} \cup \{(r_{U,k}^{*,L_{k|k-1}^{(i)}}(\mathbf{z}_k), p_{U,k}^{L_{k|k-1}^{(i)}}(\cdot;\mathbf{z}_k))\}_{\mathbf{z}_k \in Z_k}$ can be computed as follows:

$$r_{L,k}^{(i),L_{k|k-1}^{(i)}} = r_{k|k-1}^{(i),L_{k|k-1}^{(i)}} \frac{1 - \left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}}, p_{D,k} \right\rangle}{1 - r_{k|k-1}^{(i),L_{k|k-1}^{(i)}} \left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}}, p_{D,k} \right\rangle}, \quad i = 1, \ldots, M_{k|k-1}, \tag{3.8}$$

$$p_{L,k}^{(i),L_{k|k-1}^{(i)}}(\mathbf{x}_k) = \sum_{j=1}^{L_{k|k-1}^{(i)}} \omega_{k|k-1}^{(i,j)} \frac{1 - p_{D,k}\left(\mathbf{x}_{k|k-1}^{(i,j)}\right)}{1 - \left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}}, p_{D,k} \right\rangle} \delta_{\mathbf{x}_{k|k-1}^{(i,j)}}(\mathbf{x}_k), \quad i = 1, \ldots, M_{k|k-1}, \tag{3.9}$$

$$r_{U,k}^{L_{k|k-1}^{(i)}}(\mathbf{z}_k) = \frac{1}{\kappa_k(\mathbf{z}_k) + \sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i),L_{k|k-1}^{(i)}}(\mathbf{z}_k)} \sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i),L_{k|k-1}^{(i)}}(\mathbf{z}_k), \quad \mathbf{z}_k \in Z_k, \tag{3.10}$$

$$\begin{aligned} r_{U,k}^{*,L_{k|k-1}^{(i)}}(\mathbf{z}_k) = {} & \frac{1}{\kappa_k(\mathbf{z}_k) + \sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i),L_{k|k-1}^{(i)}}(\mathbf{z}_k)} \\ & \times \sum_{i=1}^{M_{k|k-1}} \frac{\left(1 - r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\right) r_{\alpha,k}^{(i),L_{k|k-1}^{(i)}}(\mathbf{z}_k)}{1 - r_{k|k-1}^{(i),L_{k|k-1}^{(i)}} \left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}}, p_{D,k} \right\rangle}, \quad \mathbf{z}_k \in Z_k, \end{aligned} \tag{3.11}$$

$$p_{U,k}^{L_{k|k-1}^{(i)}}(\mathbf{x}_k; \mathbf{z}_k) = \frac{1}{\sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i),L_{k|k-1}^{(i)}}(\mathbf{z}_k)} \sum_{i=1}^{M_{k|k-1}} p_{\alpha,k}^{(i),L_{k|k-1}^{(i)}}(\mathbf{x}_k; \mathbf{z}_k), \quad \mathbf{z}_k \in Z_k, \tag{3.12}$$

where,

$$p_{\alpha,k}^{(i),L_{k|k-1}^{(i)}}(\mathbf{x}_k; \mathbf{z}_k) = \sum_{j=1}^{L_{k|k-1}^{(i)}} \frac{\omega_{k|k-1}^{(i,j)} r_{k|k-1}^{(i),L_{k|k-1}^{(i)}} \psi_{k,\mathbf{z}_k}\left(\mathbf{x}_{k|k-1}^{(i,j)}\right)}{1 - r_{k|k-1}^{(i),L_{k|k-1}^{(i)}} \left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}}, p_{D,k} \right\rangle} \delta_{\mathbf{x}_{k|k-1}^{(i,j)}}(\mathbf{x}_k), \quad i = 1, \ldots, M_{k|k-1}, \tag{3.13}$$

$$\begin{aligned} r_{\alpha,k}^{(i),L_{k|k-1}^{(i)}}(\mathbf{z}_k) &= \left\langle p_{\alpha,k}^{(i),L_{k|k-1}^{(i)}}(\mathbf{x}_k; \mathbf{z}_k), 1 \right\rangle \\ &= \frac{r_{k|k-1}^{(i),L_{k|k-1}^{(i)}} \left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}}, \psi_{k,\mathbf{z}_k} \right\rangle}{1 - r_{k|k-1}^{(i),L_{k|k-1}^{(i)}} \left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}}, p_{D,k} \right\rangle}, \quad i = 1, \ldots, M_{k|k-1}. \end{aligned} \tag{3.14}$$

*Resampling*

To reduce the effect of degeneracy, we resample the particles for the multi-Bernoulli parameter set after the update step.

## 4. Convergence of the Mean-Square Errors for the SMC-MeMBer and SMC-CBMeMBer Filters

To show the convergence results for the SMC-MeMBer and SMC-CBMeMBer filters, certain conditions on the functions need to be met:

(1) the transition kernel $f_{k|k-1}(\mathbf{x}_k \mid \mathbf{x}_{k-1})$ satisfies the Feller property [18], that is, for all $\varphi \in C_b(\mathbb{R}^d)$, $\int \varphi(\mathbf{x}_{k-1}) \phi_{k|k-1}(\mathbf{x}_k \mid \mathbf{x}_{k-1}) d\mathbf{x}_{k-1} \in C_b(\mathbb{R}^d)$;

(2) single-sensor/target likelihood density $\varphi_{k,\mathbf{z}_k}(\mathbf{x}_k) \in B(\mathbb{R}^d)$;

(3) $Q_k^{(i)}$ are rational-valued random variables such that there exists $p > 1$, some constant $C$, and $\alpha < p - 1$ so that

$$E\left[\left|\sum_{i=1}^{N}\left(Q_k^{(i)} - N\omega_k^{(i)}\right)q^{(i)}\right|^p\right] \leq CN^{\alpha}\|q\|^p, \quad \text{with } \sum_{i=1}^{N} Q_k^{(i)} = N \tag{4.1}$$

for all vectors $q = (q^{(1)}, \ldots, q^{(N)})$;

(4) the importance sampling ratios are bounded, that is, there exists constants $B_1$ and $B_2$ such that $\|p_{\Gamma,k}^{(i)}/b_k^{(i)}\| \leq B_1$, $i = 1, \ldots, M_{\Gamma,k}$, and $\|f_{k|k-1}/q_k^{(i)}\| \leq B_2$, $i = 1, \ldots, M_{k-1}$;

(5) the resampling strategy is multinomial and hence unbiased [19].

First, the convergence of the mean-square errors for the initialization steps of the two filters can easily be established by Lemma 0 in [13]. Assuming that at time $k = 0$, we can sample exactly from the initial distribution $p_0^{(i)}$ $(i = 1, \ldots, M_0)$. Then, for all $\varphi \in B(\mathbb{R}^d)$,

$$\begin{aligned} E\left[\left(r_0^{(i),L_0^{(i)}} - r_0^{(i)}\right)^2\right] &\leq \frac{c_0}{L_0^{(i)}}, \quad i = 1, \ldots, M_0, \\ E\left[\left(\left\langle p_0^{(i),L_0^{(i)}}, \varphi\right\rangle - \left\langle p_0^{(i)}, \varphi\right\rangle\right)^2\right] &\leq \|\varphi\|^2 \frac{d_0}{L_0^{(i)}}, \quad i = 1, \ldots, M_0 \end{aligned} \tag{4.2}$$

hold for some real numbers $c_0 > 0$ and $d_0 > 0$ which are independent of the number $L_0^{(i)}$ of the sampled particles at time $k = 0$, $i = 1, \ldots, M_0$.

Also, the convergence of the mean-square errors for the resampling steps of the two filters can easily be established by Assumption 5 and Lemma 5 in [19].

The main difficulty and greatest challenge is to prove the mean-square convergence for the prediction steps and update steps of the two filters. They are, respectively, established by Propositions 4.1 and 4.2.

**Proposition 4.1.** *Suppose that, for all $\varphi \in B(\mathbb{R}^d)$,*

$$E\left[\left(r_{k-1}^{(i),L_{k-1}^{(i)}} - r_{k-1}^{(i)}\right)^2\right] \leq \frac{c_{k-1}}{L_{k-1}^{(i)}}, \quad i = 1, \ldots, M_{k-1}, \tag{4.3}$$

$$E\left[\left(\left\langle p_{k-1}^{(i),L_{k-1}^{(i)}}, \varphi\right\rangle - \left\langle p_{k-1}^{(i)}, \varphi\right\rangle\right)^2\right] \leq \|\varphi\|^2 \frac{d_{k-1}}{L_{k-1}^{(i)}}, \quad i = 1, \ldots, M_{k-1}, \tag{4.4}$$

*hold for some real numbers $c_{k-1} > 0$ and $d_{k-1} > 0$ which are independent of the number $L_{k-1}^{(i)}$ of the resampled particles at time $k - 1$, $i = 1, \ldots, M_{k-1}$.*

Then, after the prediction steps of the SMC-MeMBer and SMC-CBMeMBer filters at time $k$:

$$E\left[\left(r_{P,k|k-1}^{(i),L_{k-1}^{(i)}} - r_{P,k|k-1}^{(i)}\right)^2\right] \le \frac{c_{P,k|k-1}}{L_{k-1}^{(i)}}, \quad i = 1,\ldots,M_{k-1}, \tag{4.5}$$

$$E\left[\left(\left\langle p_{P,k|k-1}^{(i),L_{k-1}^{(i)}},\varphi\right\rangle - \left\langle p_{P,k|k-1}^{(i)},\varphi\right\rangle\right)^2\right] \le \|\varphi\|^2 \frac{d_{P,k|k-1}}{L_{k-1}^{(i)}}, \quad i = 1,\ldots,M_{k-1}, \tag{4.6}$$

$$E\left[\left(\left\langle p_{\Gamma,k|k-1}^{(i),L_{\Gamma,k}^{(i)}},\varphi\right\rangle - \left\langle p_{\Gamma,k|k-1}^{(i)},\varphi\right\rangle\right)^2\right] \le \|\varphi\|^2 \frac{d_{\Gamma,k}}{L_{\Gamma,k}^{(i)}}, \quad i = 1,\ldots,M_{\Gamma,k}, \tag{4.7}$$

hold for a constant $d_{\Gamma,k} > 0$ and some real numbers $c_{P,k|k-1} > 0$ and $d_{P,k|k-1} > 0$ which are independent of $L_{k-1}^{(i)}$, $i = 1,\ldots,M_{k-1}$. $c_{P,k|k-1}$ and $d_{P,k|k-1}$ are defined by (A.8) and (A.18), respectively. The proof of Proposition 4.1 can be found in Appendix A.1.

**Proposition 4.2.** *Suppose that, for all $\varphi \in B(\mathbb{R}^d)$,*

$$E\left[\left(r_{k|k-1}^{(i),L_{k|k-1}^{(i)}} - r_{k|k-1}^{(i)}\right)^2\right] \le \frac{c_{k|k-1}}{L_{k|k-1}^{(i)}}, \quad i = 1,\ldots,M_{k|k-1}, \tag{4.8}$$

$$E\left[\left(\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}},\varphi\right\rangle - \left\langle p_{k|k-1}^{(i)},\varphi\right\rangle\right)^2\right] \le \|\varphi\|^2 \frac{d_{k|k-1}}{L_{k|k-1}^{(i)}}, \quad i = 1,\ldots,M_{k|k-1} \tag{4.9}$$

*hold for some real numbers $c_{k|k-1} > 0$ and $d_{k|k-1} > 0$ which are independent of the number $L_{k|k-1}^{(i)}$ of the predicted particles, $i = 1,\ldots,M_{k|k-1}$. Then, after the update steps of the SMC-MeMBer and SMC-CBMeMBer filters at time $k$:*

$$E\left[\left(r_{L,k}^{(i),L_{k|k-1}^{(i)}} - r_{L,k}^{(i)}\right)^2\right] \le \frac{c_{L,k}}{L_{k|k-1}^{(i)}}, \quad i = 1,\ldots,M_{k|k-1}, \tag{4.10}$$

$$E\left[\left(\left\langle p_{L,k}^{(i),L_{k|k-1}^{(i)}},\varphi\right\rangle - \left\langle p_{L,k}^{(i)},\varphi\right\rangle\right)^2\right] \le \|\varphi\|^2 \frac{d_{L,k}}{L_{k|k-1}^{(i)}}, \quad i = 1,\ldots,M_{k|k-1}, \tag{4.11}$$

$$E\left[\left(r_{U,k}^{L_{k|k-1}^{(i)}}(\mathbf{z}_k) - r_{U,k}(\mathbf{z}_k)\right)^2\right] \le \frac{c_{U,k}}{L_{k|k-1}^{\min}}, \quad \mathbf{z}_k \in Z_k, \tag{4.12}$$

$$E\left[\left(r_{U,k}^{*,L_{k|k-1}^{(i)}}(\mathbf{z}_k) - r_{U,k}^{*}(\mathbf{z}_k)\right)^2\right] \le \frac{c_{U,k}^{*}}{L_{k|k-1}^{\min}}, \quad \mathbf{z}_k \in Z_k, \tag{4.13}$$

$$E\left[\left(\left\langle p_{U,k}^{L_{k|k-1}^{(i)}}(\cdot;\mathbf{z}_k),\varphi\right\rangle - \langle p_{U,k}(\cdot;\mathbf{z}_k),\varphi\rangle\right)^2\right] \le \|\varphi\|^2 \frac{d_{U,k}}{L_{k|k-1}^{\min}}, \quad \mathbf{z}_k \in Z_k \tag{4.14}$$

*hold for some real numbers* $c_{L,k} > 0$, $d_{L,k} > 0$, $c_{U,k} > 0$, $c^*_{U,k} > 0$, *and* $d_{U,k} > 0$, *which are independent of* $L^{(i)}_{k|k-1}$. $c_{L,k}$, $d_{L,k}$, $c_{U,k}$, $c^*_{U,k}$ *and* $d_{U,k}$ *are defined by* (A.29), (A.35), (A.47), (A.55), *and* (A.61), *respectively.* $L^{\min}_{k|k-1} = \min(L^{(1)}_{k|k-1}, \ldots, L^{(M_{k|k-1})}_{k|k-1})$, $\min(\cdot)$ *denotes the minimum. In addition,* $c_{U,k}$, $c^*_{U,k}$, *and* $d_{U,k}$ *depend on the number of targets and decrease with the increase of the target number. From* (A.47) *and* (A.55), *it can also be seen that* $c^*_{U,k} \geq c_{U,k}$. *It indicates that* $r^{*,L^{(i)}_{k|k-1}}_{U,k}(\mathbf{z}_k)$ *may need more particles than* $r^{L^{(i)}_{k|k-1}}_{U,k}(\mathbf{z}_k)$ *to achieve the same mean-square error bound. The proof of the Proposition 4.2 can be found in Appendix A.2.*

Propositions 4.1 and 4.2 show that the bounds for the mean-square error of the SMC-MeMBer and SMC-CBMeMBer prediction steps and update steps at each stage depend on the number of particles. The mean-square errors tend to zero as the number of particles tends to infinity. The bounds for the mean-square errors of these quantities are inversely proportional to the corresponding particle number.

Moreover, from the proofs of Propositions 4.1 and 4.2, it can be seen that

(1) Assumptions 1, 3, and 4 ensure that (4.6) holds;

(2) Assumption 4 ensures that (4.7) holds;

(3) Assumption 2 ensures that (4.12), (4.13), and (4.14) hold;

(4) Assumption 5 ensures the convergence of the mean-square errors for the resampling steps of the two filters.

Assumptions 3, 4, and 5 are concerned with the SMC method. They can be satisfied as long as the appropriate sampling strategies are chosen. Assumptions 1 and 2 are concerned with the likelihood and target transition kernel. They may be too restrictive or unrealistic for some practical applications. However, these convergence results give justification to the SMC implementations of the MeMBer and CBMeMBer filters and show how the order of the mean-square errors are reduced as the number of particles increases.

## 5. Simulations

Here, we briefly describe the application of the convergence results for the SMC-CBMeMBer filter to the nonlinear MTT example presented in Example 1 of [12]. The experiment settings are the same as those of Example 1 except that the number of the particles $L^{(i)}_k$ used for each hypothesized track at time $k$. For convenience, we assume $L^{(i)}_k = L$. Assumptions 1–5 are satisfied in this example. So, the SMC-CBMeMBer filter converges to the ground truth in the mean-square sense.

For the SMC-CBMeMBer filter, the estimates of the multitarget number and states, which are derived from the particle multi-Bernoulli parameter set, are unbiased. Therefore, via comparing the tracking performance of the algorithm in the various particle number $L$, the convergence results for the SMC-CBMeMBer filter can be verified to a great extent.

The standard deviation of the estimated cardinality distribution and the optimal subpattern assignment (OSPA) multitarget miss-distance [20] of order $p = 2$ with cut-off $c = 100$ m, which jointly captures differences in cardinality and individual elements between two finite sets, are used to evaluate the performance of the method. Table 1 shows the time-averaged standard deviation of the estimated cardinality distribution and the time-averaged OSPA in various $L$ via 200 MC simulation experiments.

**Table 1:** Time-averaged standard deviation of the estimated cardinality distribution and time-averaged OSPA (m) in various $L$.

| Particle number $L$ | 100 | 500 | 1000 | 1500 | 2000 |
|---|---|---|---|---|---|
| Time-averaged standard deviation of the estimated cardinality distribution from the SMC-CBMeMBer filter | 2.69 | 2.03 | 1.48 | 1.23 | 1.04 |
| OSPA (m) from the SMC-CBMeMBer filter | 65.2 | 51.9 | 42.3 | 34.8 | 27.7 |

Table 1 shows that both the standard deviation of the estimated cardinality distribution and OSPA decrease with the increase of the particle number $L$. This phenomenon can be reasonably explained by the convergence results derived in this paper: first, the mean-square error of the particle multi-Bernoulli parameter set decreases as the number of the particles increases; then, the more precise estimates of the cardinality distribution and multitarget states can be derived from the more precise particle multi-Bernoulli parameter set, which eventually leads to the results presented in Table 1.

## 6. Conclusions and Future Work

This paper presents the mathematical proofs of the convergence for the SMC-MeMBer and SMC-CBMeMBer filters and gives the bounds for the mean-square errors. In the linear-Gaussian condition, Vo et al. presented the analytic solutions to the MeMBer and CBMeMBer recursions: GM-MeMBer and GM-CBMeMBer filters [12]. The future work is focused on studying the convergence results and error bounds for the two filters.

## Appendix

### A.

In deriving the proofs, we use the Minkowski inequality, which states that, for any two random variables $X$ and $Y$ in $L^2$,

$$E\left[(X+Y)^2\right]^{1/2} \leq E\left[X^2\right]^{1/2} + E\left[X^2\right]^{1/2}. \tag{A.1}$$

Using Minkowski's inequality, we obtain that, for all $\varphi \in B(\mathbb{R}^d)$,

$$\begin{aligned} &E\left[\left(r^{(i),L^{(i)}}\left\langle p^{(i),L^{(i)}},\varphi\right\rangle - r^{(i)}\left\langle p^{(i)},\varphi\right\rangle\right)^2\right]^{1/2} \\ &\quad = E\left[\left(r^{(i),L^{(i)}}\left\langle p^{(i),L^{(i)}},\varphi\right\rangle - r^{(i)}\left\langle p^{(i),L^{(i)}},\varphi\right\rangle + r^{(i)}\left\langle p^{(i),L^{(i)}},\varphi\right\rangle - r^{(i)}\left\langle p^{(i)},\varphi\right\rangle\right)^2\right]^{1/2} \end{aligned} \tag{A.2}$$

$$\leq E\left[\left\langle p^{(i),L^{(i)}},\varphi\right\rangle^2\left(r^{(i),L^{(i)}} - r^{(i)}\right)^2\right]^{1/2} + r^{(i)}E\left[\left(\left\langle p^{(i),L^{(i)}},\varphi\right\rangle - \left\langle p^{(i)},\varphi\right\rangle\right)^2\right]^{1/2} \tag{A.3}$$

$$\leq \|\varphi\| E\left[\left(r^{(i),L^{(i)}} - r^{(i)}\right)^2\right]^{1/2} + r^{(i)}E\left[\left(\left\langle p^{(i),L^{(i)}},\varphi\right\rangle - \left\langle p^{(i)},\varphi\right\rangle\right)^2\right]^{1/2} \tag{A.4}$$

holds, $i = 1,\ldots,M$, for the multi-Bernoulli density $\pi = \{(r^{(i)},p^{(i)})\}_{i=1}^M$ and its particle approximation $\pi^{L^{(i)}} = \{(r^{(i),L^{(i)}},p^{(i),L^{(i)}})\}_{i=1}^M$.

### *A.1. Proof of Proposition 4.1*

We first prove (4.5). From (2.4) and (3.2), we have

$$E\left[\left(r_{P,k|k-1}^{(i),L_{k-1}^{(i)}}-r_{P,k|k-1}^{(i)}\right)^2\right]^{1/2}=E\left[\left(r_{k-1}^{(i),L_{k-1}^{(i)}}\left\langle p_{k-1}^{(i),L_{k-1}^{(i)}},p_{S,k}\right\rangle-r_{k-1}^{(i)}\left\langle p_{k-1}^{(i)},p_{S,k}\right\rangle\right)^2\right]^{1/2} \tag{A.5}$$

(by (A.4))

$$\le\|p_{S,k}\|E\left[\left(r_{k-1}^{(i),L_{k-1}^{(i)}}-r_{k-1}^{(i)}\right)^2\right]^{1/2}+r_{k-1}^{(i)}E\left[\left(\left\langle p_{k-1}^{(i),L_{k-1}^{(i)}},p_{S,k}\right\rangle-\left\langle p_{k-1}^{(i)},p_{S,k}\right\rangle\right)^2\right]^{1/2} \tag{A.6}$$

(by (4.3), (4.4), and $0\le r_{k-1}^{(i)}\le 1$)

$$\le\|p_{S,k}\|\frac{\sqrt{c_{k-1}}+\sqrt{d_{k-1}}}{\sqrt{L_{k-1}^{(i)}}}. \tag{A.7}$$

So that (4.5) is proved with

$$c_{P,k|k-1}=\|p_{S,k}\|^2\left(\sqrt{c_{k-1}}+\sqrt{d_{k-1}}\right)^2. \tag{A.8}$$

Now turn to (4.6). From (2.5), we have

$$\begin{aligned}&E\left[\left(\left\langle p_{P,k|k-1}^{(i),L_{k-1}^{(i)}},\varphi\right\rangle-\left\langle p_{P,k|k-1}^{(i)},\varphi\right\rangle\right)^2\right]^{1/2}\\&\quad=E\left[\left(\left\langle p_{P,k|k-1}^{(i),L_{k-1}^{(i)}},\varphi\right\rangle-\left\langle\frac{\left\langle f_{k|k-1},p_{k-1}^{(i)}p_{S,k}\right\rangle}{\left\langle p_{k-1}^{(i)},p_{S,k}\right\rangle},\varphi\right\rangle\right)^2\right]^{1/2}\end{aligned} \tag{A.9}$$

(adding and subtracting a new term)

$$\begin{aligned}&=E\left[\left(\left\langle p_{P,k|k-1}^{(i),L_{k-1}^{(i)}},\varphi\right\rangle-\left\langle\frac{\left\langle f_{k|k-1},p_{k-1}^{(i),L_{k-1}^{(i)}}p_{S,k}\right\rangle}{\left\langle p_{k-1}^{(i),L_{k-1}^{(i)}},p_{S,k}\right\rangle},\varphi\right\rangle\right.\right.\\&\qquad\left.\left.+\left\langle\frac{\left\langle f_{k|k-1},p_{k-1}^{(i),L_{k-1}^{(i)}}p_{S,k}\right\rangle}{\left\langle p_{k-1}^{(i),L_{k-1}^{(i)}},p_{S,k}\right\rangle},\varphi\right\rangle-\left\langle\frac{\left\langle f_{k|k-1},p_{k-1}^{(i)}p_{S,k}\right\rangle}{\left\langle p_{k-1}^{(i)},p_{S,k}\right\rangle},\varphi\right\rangle\right)^2\right]^{1/2}\end{aligned} \tag{A.10}$$

(using Minkowski's inequality)

$$\leq E\left[\left(\left\langle p_{P,k|k-1}^{(i),L_{k-1}^{(i)}},\varphi\right\rangle-\left\langle\frac{\left\langle f_{k|k-1},p_{k-1}^{(i),L_{k-1}^{(i)}}p_{S,k}\right\rangle}{\left\langle p_{k-1}^{(i),L_{k-1}^{(i)}},p_{S,k}\right\rangle},\varphi\right\rangle\right)^2\right]^{1/2}$$
$$+E\left[\left(\frac{\left\langle p_{k-1}^{(i),L_{k-1}^{(i)}},p_{S,k}\langle f_{k|k-1},\varphi\rangle\right\rangle}{\left\langle p_{k-1}^{(i),L_{k-1}^{(i)}},p_{S,k}\right\rangle}-\frac{\left\langle p_{k-1}^{(i)},p_{S,k}\langle f_{k|k-1},\varphi\rangle\right\rangle}{\left\langle p_{k-1}^{(i)},p_{S,k}\right\rangle}\right)^2\right]^{1/2}. \tag{A.11}$$

By Assumption 3 and Lemma 1 in [13], we easily obtain that the first term in (A.11) becomes

$$E\left[\left(\left\langle p_{P,k|k-1}^{(i),L_{k-1}^{(i)}},\varphi\right\rangle-\left\langle\frac{\left\langle f_{k|k-1},p_{k-1}^{(i),L_{k-1}^{(i)}}p_{S,k}\right\rangle}{\left\langle p_{k-1}^{(i),L_{k-1}^{(i)}},p_{S,k}\right\rangle},\varphi\right\rangle\right)^2\right]^{1/2}$$
$$\leq\frac{\|\varphi\|}{\sqrt{L_{k-1}^{(i)}}}\left(\left\|\frac{f_{k|k-1}p_{S,k}}{q_k^{(i)}}\right\|^2+\|f_{k|k-1}p_{S,k}\|^2\right)^{1/2} \tag{A.12}$$

(since $\|f_{k|k-1}/q_k^{(i)}\|\leq B_2$ by Assumption 4 and $f_{k|k-1}p_{S,k}\in C_b(\mathbb{R}^d)$ by Assumption 1)

$$\leq\frac{\|\varphi\|}{\sqrt{L_{k-1}^{(i)}}}\left(\|p_{S,k}\|^2B_2^2+\|f_{k|k-1}p_{S,k}\|^2\right)^{1/2}. \tag{A.13}$$

Adding and subtracting a new term in the second term of (A.11), we have

$$E\left[\left(\frac{\left\langle p_{k-1}^{(i),L_{k-1}^{(i)}},p_{S,k}\langle f_{k|k-1},\varphi\rangle\right\rangle}{\left\langle p_{k-1}^{(i),L_{k-1}^{(i)}},p_{S,k}\right\rangle}-\frac{\left\langle p_{k-1}^{(i)},p_{S,k}\langle f_{k|k-1},\varphi\rangle\right\rangle}{\left\langle p_{k-1}^{(i)},p_{S,k}\right\rangle}\right)^2\right]^{1/2}$$

$$= E\left[\left(\frac{\left\langle p_{k-1}^{(i),L_{k-1}^{(i)}}, p_{S,k}\left\langle f_{k|k-1},\varphi\right\rangle\right\rangle}{\left\langle p_{k-1}^{(i),L_{k-1}^{(i)}}, p_{S,k}\right\rangle} - \frac{\left\langle p_{k-1}^{(i),L_{k-1}^{(i)}}, p_{S,k}\left\langle f_{k|k-1},\varphi\right\rangle\right\rangle}{\left\langle p_{k-1}^{(i)}, p_{S,k}\right\rangle}\right.\right.$$
$$\left.\left. + \frac{\left\langle p_{k-1}^{(i),L_{k-1}^{(i)}}, p_{S,k}\left\langle f_{k|k-1},\varphi\right\rangle\right\rangle}{\left\langle p_{k-1}^{(i)}, p_{S,k}\right\rangle} - \frac{\left\langle p_{k-1}^{(i)}, p_{S,k}\left\langle f_{k|k-1},\varphi\right\rangle\right\rangle}{\left\langle p_{k-1}^{(i)}, p_{S,k}\right\rangle}\right)^2\right]^{1/2} \tag{A.14}$$

(using Minkowski's inequality)

$$\leq \frac{1}{\left\langle p_{k-1}^{(i)}, p_{S,k}\right\rangle} E\left[\left(\frac{\left\langle p_{k-1}^{(i),L_{k-1}^{(i)}}, p_{S,k}\left\langle f_{k|k-1},\varphi\right\rangle\right\rangle}{\left\langle p_{k-1}^{(i),L_{k-1}^{(i)}}, p_{S,k}\right\rangle}\left(\left\langle p_{k-1}^{(i)}, p_{S,k}\right\rangle - \left\langle p_{k-1}^{(i),L_{k-1}^{(i)}}, p_{S,k}\right\rangle\right)\right)^2\right]^{1/2} \tag{A.15}$$
$$+ \frac{1}{\left\langle p_{k-1}^{(i)}, p_{S,k}\right\rangle} E\left[\left(\left\langle p_{k-1}^{(i),L_{k-1}^{(i)}}, p_{S,k}\left\langle f_{k|k-1},\varphi\right\rangle\right\rangle - \left\langle p_{k-1}^{(i)}, p_{S,k}\left\langle f_{k|k-1},\varphi\right\rangle\right\rangle\right)^2\right]^{1/2}$$

$$\leq 2\frac{\left\|\left\langle f_{k|k-1},\varphi\right\rangle\right\|}{\left\langle p_{k-1}^{(i)}, p_{S,k}\right\rangle} E\left[\left(\left(\left\langle p_{k-1}^{(i)}, p_{S,k}\right\rangle - \left\langle p_{k-1}^{(i),L_{k-1}^{(i)}}, p_{S,k}\right\rangle\right)\right)^2\right]^{1/2} \tag{A.16}$$

(by (4.4))

$$\leq 2\frac{\|\varphi\|\cdot\|p_{S,k}\|}{\inf(p_{S,k})}\sqrt{\frac{d_{k-1}}{L_{k-1}^{(i)}}}, \tag{A.17}$$

where $\inf(\cdot)$ denotes the infimum.

Finally, substituting (A.12) and (A.17) into (A.11), (4.6) is proved with

$$d_{P,k|k-1} = \left(\sqrt{\left\|p_{S,k}\right\|^2 B_2^2 + \left\|f_{k|k-1}p_{S,k}\right\|^2} + \frac{2\|p_{S,k}\|\sqrt{d_{k-1}}}{\inf(p_{S,k})}\right)^2. \tag{A.18}$$

Now, turn to (4.7). By Lemma 0 in [13] and the boundedness of $\|p_{\Gamma,k}^{(i)}/b_k^{(i)}\| \leq B_1$ ($i = 1,\ldots,M_{k-1}$) in Assumption 4, we get that (4.7) holds for a constant $d_{\Gamma,k}$. This completes the proof.

### *A.2. Proof of Proposition 4.2*

Now turn to (4.10). From (2.7) and (3.8), we have

$$
\begin{aligned}
&E\left[\left(r_{L,k}^{(i)}-r_{L,k}^{(i),L_{k|k-1}^{(i)}}\right)^2\right]^{1/2}\\
&\quad=E\left[\left(r_{k|k-1}^{(i)}\frac{1-\left\langle p_{k|k-1}^{(i)},p_{D,k}\right\rangle}{1-r_{k|k-1}^{(i)}\left\langle p_{k|k-1}^{(i)},p_{D,k}\right\rangle}-r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\frac{1-\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}},p_{D,k}\right\rangle}{1-r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}},p_{D,k}\right\rangle}\right)^2\right]^{1/2}
\end{aligned}
\tag{A.19}
$$

(adding and subtracting a new term)

$$
=E\left[\left(\begin{array}{l}
r_{k|k-1}^{(i)}\dfrac{1-\left\langle p_{k|k-1}^{(i)},p_{D,k}\right\rangle}{1-r_{k|k-1}^{(i)}\left\langle p_{k|k-1}^{(i)},p_{D,k}\right\rangle}-r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\dfrac{1-\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}},p_{D,k}\right\rangle}{1-r_{k|k-1}^{(i)}\left\langle p_{k|k-1}^{(i)},p_{D,k}\right\rangle}\\
+r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\dfrac{1-\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}},p_{D,k}\right\rangle}{1-r_{k|k-1}^{(i)}\left\langle p_{k|k-1}^{(i)},p_{D,k}\right\rangle}-r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\dfrac{1-\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}},p_{D,k}\right\rangle}{1-r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}},p_{D,k}\right\rangle}
\end{array}\right)^2\right]^{1/2}
\tag{A.20}
$$

(using Minkowski's inequality)

$$
\begin{aligned}
&\leq\frac{E\left[\left(r_{k|k-1}^{(i)}\left(1-\left\langle p_{k|k-1}^{(i)},p_{D,k}\right\rangle\right)-r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\left(1-\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}},p_{D,k}\right\rangle\right)\right)^2\right]^{1/2}}{1-r_{k|k-1}^{(i)}\left\langle p_{k|k-1}^{(i)},p_{D,k}\right\rangle}\\
&\quad+E\left[\left(r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\frac{1-\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}},p_{D,k}\right\rangle}{1-r_{k|k-1}^{(i)}\left\langle p_{k|k-1}^{(i)},p_{D,k}\right\rangle}-r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\frac{1-\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}},p_{D,k}\right\rangle}{1-r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}},p_{D,k}\right\rangle}\right)^2\right]^{1/2}.
\end{aligned}
\tag{A.21}
$$

The numerator of the first term in (A.21) is

$$
\begin{aligned}
&E\left[\left(r_{k|k-1}^{(i)}\left(1-\left\langle p_{k|k-1}^{(i)},p_{D,k}\right\rangle\right)-r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\left(1-\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}},p_{D,k}\right\rangle\right)\right)^2\right]^{1/2}\\
&\quad=E\left[\left(\left(r_{k|k-1}^{(i)}-r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\right)+\left(r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}},p_{D,k}\right\rangle-r_{k|k-1}^{(i)}\left\langle p_{k|k-1}^{(i)},p_{D,k}\right\rangle\right)\right)^2\right]^{1/2}
\end{aligned}
\tag{A.22}
$$

(using Minkowski's inequality and then (A.4))

$$\begin{aligned}&\leq E\left[\left(r_{k|k-1}^{(i)}-r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\right)^{2}\right]^{1/2}+\|p_{D,k}\|E\left[\left(r_{k|k-1}^{(i)}-r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\right)^{2}\right]^{1/2}\\&\quad+r_{k|k-1}^{(i)}E\left[\left(\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}},p_{D,k}\right\rangle-\left\langle p_{k|k-1}^{(i)},p_{D,k}\right\rangle\right)^{2}\right]^{1/2}\end{aligned}\tag{A.23}$$

(by (4.8) and (4.9))

$$\leq\frac{(1+\|p_{D,k}\|)\sqrt{c_{k|k-1}}+r_{k|k-1}^{(i)}\|p_{D,k}\|\sqrt{d_{k|k-1}}}{\sqrt{L_{k|k-1}^{(i)}}}.\tag{A.24}$$

The second term in (A.21) is

$$\begin{aligned}&E\left[\left(r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\frac{1-\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}},p_{D,k}\right\rangle}{1-r_{k|k-1}^{(i)}\left\langle p_{k|k-1}^{(i)},p_{D,k}\right\rangle}-r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\frac{1-\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}},p_{D,k}\right\rangle}{1-r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}},p_{D,k}\right\rangle}\right)^{2}\right]^{1/2}\\&\quad=E\left[\left(\frac{r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}-r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}},p_{D,k}\right\rangle}{1-r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}},p_{D,k}\right\rangle}\right)^{2}\right.\\&\qquad\left.\times\left(\frac{r_{k|k-1}^{(i)}\left\langle p_{k|k-1}^{(i)},p_{D,k}\right\rangle-r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}},p_{D,k}\right\rangle}{1-r_{k|k-1}^{(i)}\left\langle p_{k|k-1}^{(i)},p_{D,k}\right\rangle}\right)^{2}\right]^{1/2}\end{aligned}\tag{A.25}$$

(by $0\leq r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\leq 1$)

$$\leq\frac{E\left[\left(r_{k|k-1}^{(i)}\left\langle p_{k|k-1}^{(i)},p_{D,k}\right\rangle-r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}},p_{D,k}\right\rangle\right)^{2}\right]^{1/2}}{1-r_{k|k-1}^{(i)}\left\langle p_{k|k-1}^{(i)},p_{D,k}\right\rangle}\tag{A.26}$$

(by (A.4), (4.8) and (4.9))

$$\leq\frac{\|p_{D,k}\|\sqrt{c_{k|k-1}}+r_{k|k-1}^{(i)}\|p_{D,k}\|\sqrt{d_{k|k-1}}}{\left(1-r_{k|k-1}^{(i)}\left\langle p_{k|k-1}^{(i)},p_{D,k}\right\rangle\right)\sqrt{L_{k|k-1}^{(i)}}}.\tag{A.27}$$

Substituting (A.24) and (A.27) into (A.21), and then using $0 \le r^{(i)}_{k|k-1} \le 1$, we get

$$\begin{aligned}
&E\left[\left(r^{(i)}_{L,k} - r^{(i),L^{(i)}_{k|k-1}}_{L,k}\right)^2\right]^{1/2} \\
&\quad\le \frac{\left(1+2\|p_{D,k}\|\right)\sqrt{c_{k|k-1}} + 2r^{(i)}_{k|k-1}\|p_{D,k}\|\sqrt{d_{k|k-1}}}{\left(1 - r^{(i)}_{k|k-1}\left\langle p^{(i)}_{k|k-1}, p_{D,k}\right\rangle\right)\sqrt{L^{(i)}_{k|k-1}}} \\
&\quad\le \frac{\left(1+2\|p_{D,k}\|\right)\sqrt{c_{k|k-1}} + 2\|p_{D,k}\|\sqrt{d_{k|k-1}}}{\left(1 - \|p_{D,k}\|\right)\sqrt{L^{(i)}_{k|k-1}}}.
\end{aligned} \tag{A.28}$$

Finally, (4.10) is proved with

$$c_{L,k} = \left(\frac{\left(1+2\|p_{D,k}\|\right)\sqrt{c_{k|k-1}} + 2\|p_{D,k}\|\sqrt{d_{k|k-1}}}{1 - \|p_{D,k}\|}\right)^2. \tag{A.29}$$

Now turn to (4.11). From (2.8) and (3.9), we have

$$\begin{aligned}
&E\left[\left(\left\langle p^{(i)}_{L,k}, \varphi\right\rangle - \left\langle p^{(i),L^{(i)}_{k|k-1}}_{L,k}, \varphi\right\rangle\right)^2\right]^{1/2} \\
&\quad= E\left[\left(\frac{\left\langle p^{(i)}_{k|k-1}, \varphi(1-p_{D,k})\right\rangle}{1 - \left\langle p^{(i)}_{k|k-1}, p_{D,k}\right\rangle} - \frac{\left\langle p^{(i),L^{(i)}_{k|k-1}}_{k|k-1}, \varphi(1-p_{D,k})\right\rangle}{1 - \left\langle p^{(i),L^{(i)}_{k|k-1}}_{k|k-1}, p_{D,k}\right\rangle}\right)^2\right]^{1/2}
\end{aligned} \tag{A.30}$$

(adding and subtracting a new term)

$$\begin{aligned}
&= E\Bigg[\Bigg(\frac{\left\langle p^{(i)}_{k|k-1}, \varphi(1-p_{D,k})\right\rangle}{1 - \left\langle p^{(i)}_{k|k-1}, p_{D,k}\right\rangle} - \frac{\left\langle p^{(i),L^{(i)}_{k|k-1}}_{k|k-1}, \varphi(1-p_{D,k})\right\rangle}{1 - \left\langle p^{(i)}_{k|k-1}, p_{D,k}\right\rangle} \\
&\qquad + \frac{\left\langle p^{(i),L^{(i)}_{k|k-1}}_{k|k-1}, \varphi(1-p_{D,k})\right\rangle}{1 - \left\langle p^{(i)}_{k|k-1}, p_{D,k}\right\rangle} - \frac{\left\langle p^{(i),L^{(i)}_{k|k-1}}_{k|k-1}, \varphi(1-p_{D,k})\right\rangle}{1 - \left\langle p^{(i),L^{(i)}_{k|k-1}}_{k|k-1}, p_{D,k}\right\rangle}\Bigg)^2\Bigg]^{1/2}
\end{aligned} \tag{A.31}$$

(using Minkowski's inequality)

$$\leq \frac{E\left[\left(\left\langle p_{k|k-1}^{(i)}, \varphi(1-p_{D,k})\right\rangle - \left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}}, \varphi(1-p_{D,k})\right\rangle\right)^2\right]^{1/2}}{1-\left\langle p_{k|k-1}^{(i)}, p_{D,k}\right\rangle} + E\left[\left(\frac{\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}}, \varphi(1-p_{D,k})\right\rangle}{1-\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}}, p_{D,k}\right\rangle}\right)^2 \cdot \left(\frac{\left\langle p_{k|k-1}^{(i)}, p_{D,k}\right\rangle - \left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}}, p_{D,k}\right\rangle}{1-\left\langle p_{k|k-1}^{(i)}, p_{D,k}\right\rangle}\right)^2\right]^{1/2} \tag{A.32}$$

$$\leq \frac{E\left[\left(\left\langle p_{k|k-1}^{(i)}, \varphi(1-p_{D,k})\right\rangle - \left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}}, \varphi(1-p_{D,k})\right\rangle\right)^2\right]^{1/2}}{1-\left\langle p_{k|k-1}^{(i)}, p_{D,k}\right\rangle} + \frac{\|\varphi\| E\left[\left(\left\langle p_{k|k-1}^{(i)}, p_{D,k}\right\rangle - \left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}}, p_{D,k}\right\rangle\right)^2\right]^{1/2}}{1-\left\langle p_{k|k-1}^{(i)}, p_{D,k}\right\rangle} \tag{A.33}$$

(by (4.9))

$$\leq \frac{\|\varphi\|\sqrt{d_{k|k-1}}}{(1-\|p_{D,k}\|)\sqrt{L_{k|k-1}^{(i)}}}. \tag{A.34}$$

Finally, (4.11) is proved with

$$d_{L,k} = \frac{d_{k|k-1}}{(1-\|p_{D,k}\|)^2}. \tag{A.35}$$

Now, turn to (4.12). From (2.9) and (3.10), we have

$$E\left[\left(r_{U,k}^{L_{k|k-1}^{(i)}}(\mathbf{z}_k) - r_{U,k}(\mathbf{z}_k)\right)^2\right]^{1/2} = E\left[\left(\frac{\sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i),L_{k|k-1}^{(i)}}(\mathbf{z}_k)}{\kappa_k(\mathbf{z}_k) + \sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i),L_{k|k-1}^{(i)}}(\mathbf{z}_k)} - \frac{\sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i)}(\mathbf{z}_k)}{\kappa_k(\mathbf{z}_k) + \sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i)}(\mathbf{z}_k)}\right)^2\right]^{1/2} \tag{A.36}$$

(adding and subtracting a new term)

$$= E\left[\left(\frac{\sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i),L_{k|k-1}^{(i)}}(\mathbf{z}_k)}{\kappa_k(\mathbf{z}_k)+\sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i),L_{k|k-1}^{(i)}}(\mathbf{z}_k)} - \frac{\sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i),L_{k|k-1}^{(i)}}(\mathbf{z}_k)}{\kappa_k(\mathbf{z}_k)+\sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i)}(\mathbf{z}_k)} + \frac{\sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i),L_{k|k-1}^{(i)}}(\mathbf{z}_k)}{\kappa_k(\mathbf{z}_k)+\sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i)}(\mathbf{z}_k)} - \frac{\sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i)}(\mathbf{z}_k)}{\kappa_k(\mathbf{z}_k)+\sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i)}(\mathbf{z}_k)}\right)^2\right]^{1/2} \tag{A.37}$$

(using Minkowski's inequality)

$$\leq E\left[\left(r_{U,k}^{L_{k|k-1}^{(i)}}(\mathbf{z}_k)\frac{\sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i)}(\mathbf{z}_k) - \sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i),L_{k|k-1}^{(i)}}(\mathbf{z}_k)}{\kappa_k(\mathbf{z}_k)+\sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i)}(\mathbf{z}_k)}\right)^2\right]^{1/2} + E\left[\left(\frac{\sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i),L_{k|k-1}^{(i)}}(\mathbf{z}_k) - \sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i)}(\mathbf{z}_k)}{\kappa_k(\mathbf{z}_k)+\sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i)}(\mathbf{z}_k)}\right)^2\right]^{1/2} \tag{A.38}$$

(using $\kappa_k(\mathbf{z}_k) \geq 0$, $0 \leq r_{U,k}^{L_{k|k-1}^{(i)}}(\mathbf{z}_k) \leq 1$)

$$\leq \frac{2\sum_{i=1}^{M_{k|k-1}} E\left[\left(r_{\alpha,k}^{(i),L_{k|k-1}^{(i)}}(\mathbf{z}_k) - r_{\alpha,k}^{(i)}(\mathbf{z}_k)\right)^2\right]^{1/2}}{\sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i)}(\mathbf{z}_k)}. \tag{A.39}$$

From (2.12) and (3.14), the expectation in the summation of (A.39) is

$$E\left[\left(r_{\alpha,k}^{(i),L_{k|k-1}^{(i)}}(\mathbf{z}_k) - r_{\alpha,k}^{(i)}(\mathbf{z}_k)\right)^2\right]^{1/2} = E\left[\left(\frac{r_{k|k-1}^{(i)}\left\langle p_{k|k-1}^{(i)}, \psi_{k,\mathbf{z}_k}\right\rangle}{1-r_{k|k-1}^{(i)}\left\langle p_{k|k-1}^{(i)}, p_{D,k}\right\rangle} - \frac{r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}}, \psi_{k,\mathbf{z}_k}\right\rangle}{1-r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}}, p_{D,k}\right\rangle}\right)^2\right]^{1/2} \tag{A.40}$$

(adding and subtracting a new term)

$$= E\left[\left(\frac{r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}},\psi_{k,\mathbf{z}_k}\right\rangle}{1-r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}},p_{D,k}\right\rangle}-\frac{r_{k|k-1}^{(i)}\left\langle p_{k|k-1}^{(i)},\psi_{k,\mathbf{z}_k}\right\rangle}{1-r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}},p_{D,k}\right\rangle}\right.\right.$$
$$\left.\left.+\frac{r_{k|k-1}^{(i)}\left\langle p_{k|k-1}^{(i)},\psi_{k,\mathbf{z}_k}\right\rangle}{1-r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}},p_{D,k}\right\rangle}-\frac{r_{k|k-1}^{(i)}\left\langle p_{k|k-1}^{(i)},\psi_{k,\mathbf{z}_k}\right\rangle}{1-r_{k|k-1}^{(i)}\left\langle p_{k|k-1}^{(i)},p_{D,k}\right\rangle}\right)^2\right]^{1/2} \tag{A.41}$$

(using Minkowski's inequality)

$$\leq E\left[\left(\frac{r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}},\psi_{k,\mathbf{z}_k}\right\rangle-r_{k|k-1}^{(i)}\left\langle p_{k|k-1}^{(i)},\psi_{k,\mathbf{z}_k}\right\rangle}{1-r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}},p_{D,k}\right\rangle}\right)^2\right]^{1/2}$$
$$+\frac{r_{k|k-1}^{(i)}\left\langle p_{k|k-1}^{(i)},\psi_{k,\mathbf{z}_k}\right\rangle}{1-r_{k|k-1}^{(i)}\left\langle p_{k|k-1}^{(i)},p_{D,k}\right\rangle}E\left[\left(\frac{r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}},p_{D,k}\right\rangle-r_{k|k-1}^{(i)}\left\langle p_{k|k-1}^{(i)},p_{D,k}\right\rangle}{1-r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}},p_{D,k}\right\rangle}\right)^2\right]^{1/2} \tag{A.42}$$

(by $0\leq r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\leq 1$, Assumption 2 and (A.4))

$$\leq\frac{\left\|\psi_{k,\mathbf{z}_k}\right\|E\left[\left(r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}-r_{k|k-1}^{(i)}\right)^2\right]^{1/2}+r_{k|k-1}^{(i)}E\left[\left(\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}},\psi_{k,\mathbf{z}_k}\right\rangle-\left\langle p_{k|k-1}^{(i)},\psi_{k,\mathbf{z}_k}\right\rangle\right)^2\right]^{1/2}}{1-\left\|p_{D,k}\right\|}$$
$$+\frac{r_{k|k-1}^{(i)}\left\langle p_{k|k-1}^{(i)},\psi_{k,\mathbf{z}_k}\right\rangle}{1-r_{k|k-1}^{(i)}\left\langle p_{k|k-1}^{(i)},p_{D,k}\right\rangle}$$
$$\times\frac{\left\|p_{D,k}\right\|E\left[\left(r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}-r_{k|k-1}^{(i)}\right)^2\right]^{1/2}+r_{k|k-1}^{(i)}E\left[\left(\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}},p_{D,k}\right\rangle-\left\langle p_{k|k-1}^{(i)},p_{D,k}\right\rangle\right)^2\right]^{1/2}}{1-\left\|p_{D,k}\right\|} \tag{A.43}$$

(by (4.8), (4.9), and $0 \leq r_{k|k-1}^{(i)} \leq 1$)

$$\leq \frac{\left\|\psi_{k,\mathbf{z}_k}\right\|\left(\sqrt{c_{k|k-1}}+\sqrt{d_{k|k-1}}\right)}{\left(1-\left\|p_{D,k}\right\|\right)^2\sqrt{L_{k|k-1}^{(i)}}}. \tag{A.44}$$

From (2.12), $0 \leq r_{k|k-1}^{(i)} \leq 1$ and Assumption 2, the denominator of (A.39) is

$$\begin{aligned}\sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i)}(\mathbf{z}_k) &= \sum_{i=1}^{M_{k|k-1}} \frac{r_{k|k-1}^{(i)}\left\langle p_{k|k-1}^{(i)}, \psi_{k,\mathbf{z}_k}\right\rangle}{1-r_{k|k-1}^{(i)}\left\langle p_{k|k-1}^{(i)}, p_{D,k}\right\rangle} \geq \sum_{i=1}^{M_{k|k-1}} r_{k|k-1}^{(i)}\left\langle p_{k|k-1}^{(i)}, \psi_{k,\mathbf{z}_k}\right\rangle \\ &\geq \inf\left(\psi_{k,\mathbf{z}_k}\right)\sum_{i=1}^{M_{k|k-1}} r_{k|k-1}^{(i)} = \inf\left(\psi_{k,\mathbf{z}_k}\right) n_{k|k-1},\end{aligned} \tag{A.45}$$

where $n_{k|k-1} = \sum_{i=1}^{M_{k|k-1}} r_{k|k-1}^{(i)}$ is the number of the predicted targets at time $k$.

Substituting (A.44) and (A.43) into (A.39), we get

$$\begin{aligned}E\left[\left(r_{U,k}^{L_{k|k-1}^{(i)}}(\mathbf{z}_k)-r_{U,k}(\mathbf{z}_k)\right)^2\right]^{1/2} &\leq \frac{2\left\|\psi_{k,\mathbf{z}_k}\right\|\left(\sqrt{c_{k|k-1}}+\sqrt{d_{k|k-1}}\right)}{\left(1-\left\|p_{D,k}\right\|\right)^2\inf\left(\psi_{k,\mathbf{z}_k}\right)n_{k|k-1}}\sum_{i=1}^{M_{k|k-1}}\left(\frac{1}{\sqrt{L_{k|k-1}^{(i)}}}\right) \\ &\leq \frac{2\left\|\psi_{k,\mathbf{z}_k}\right\|\left(\sqrt{c_{k|k-1}}+\sqrt{d_{k|k-1}}\right)M_{k|k-1}}{\inf\left(\psi_{k,\mathbf{z}_k}\right)n_{k|k-1}\left(1-\left\|p_{D,k}\right\|\right)^2\sqrt{L_{k|k-1}^{\min}}},\end{aligned} \tag{A.46}$$

where $L_{k|k-1}^{\min} = \min(L_{k|k-1}^{(1)}, \ldots, L_{k|k-1}^{(M_{k|k-1})})$, $\min(\cdot)$ denotes the minimum.

Finally, (4.12) is proved with

$$c_{U,k} = \left(\frac{2\left\|\psi_{k,\mathbf{z}_k}\right\|\left(\sqrt{c_{k|k-1}}+\sqrt{d_{k|k-1}}\right)M_{k|k-1}}{\inf\left(\psi_{k,\mathbf{z}_k}\right)n_{k|k-1}\left(1-\left\|p_{D,k}\right\|\right)^2}\right)^2. \tag{A.47}$$

Now, turn to (4.13). First, from (2.12), $0 \leq r_{k|k-1}^{(i)} \leq 1$ and Assumption 2, we have

$$r_{\alpha,k}^{(i)}(\mathbf{z}_k) = \frac{r_{k|k-1}^{(i)}\left\langle p_{k|k-1}^{(i)}, \psi_{k,\mathbf{z}_k}\right\rangle}{1-r_{k|k-1}^{(i)}\left\langle p_{k|k-1}^{(i)}, p_{D,k}\right\rangle} \leq \frac{\left\|\psi_{k,\mathbf{z}_k}\right\|}{1-\left\|p_{D,k}\right\|}. \tag{A.48}$$

Then, from (2.13), (3.11), and (A.39), we get

$$E\left[\left(r_{U,k}^{*,L_{k|k-1}^{(i)}}(\mathbf{z}_k)-r_{U,k}^{*}(\mathbf{z}_k)\right)^2\right]^{1/2}$$

$$\leq \frac{\sum_{i=1}^{M_{k|k-1}}\left(E\left[\left(\begin{array}{c} r_{\alpha,k}^{(i),L_{k|k-1}^{(i)}}(\mathbf{z}_k) \\ -r_{\alpha,k}^{(i)}(\mathbf{z}_k)\end{array}\right)^2\right]^{1/2}+E\left[\left(\begin{array}{c}\dfrac{r_{\alpha,k}^{(i),L_{k|k-1}^{(i)}}(\mathbf{z}_k)\left(1-r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\right)}{1-r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}},p_{D,k}\right\rangle} \\ -\dfrac{r_{\alpha,k}^{(i)}(\mathbf{z}_k)\left(1-r_{k|k-1}^{(i)}\right)}{1-r_{k|k-1}^{(i)}\left\langle p_{k|k-1}^{(i)},p_{D,k}\right\rangle}\end{array}\right)^2\right]^{1/2}\right)}{\sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i)}(\mathbf{z}_k)} \tag{A.49}$$

(adding and subtracting a new term in the second expectation in the summation)

$$\leq \frac{\sum_{i=1}^{M_{k|k-1}}\left(\begin{array}{c} E\left[\left(r_{\alpha,k}^{(i),L_{k|k-1}^{(i)}}(\mathbf{z}_k)-r_{\alpha,k}^{(i)}(\mathbf{z}_k)\right)^2\right]^{1/2} \\ +E\left[\left(\begin{array}{c}\dfrac{r_{\alpha,k}^{(i),L_{k|k-1}^{(i)}}(\mathbf{z}_k)\left(1-r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\right)}{1-r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}},p_{D,k}\right\rangle}-\dfrac{r_{\alpha,k}^{(i)}(\mathbf{z}_k)\left(1-r_{k|k-1}^{(i)}\right)}{1-r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}},p_{D,k}\right\rangle} \\ +\dfrac{r_{\alpha,k}^{(i)}(\mathbf{z}_k)\left(1-r_{k|k-1}^{(i)}\right)}{1-r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}},p_{D,k}\right\rangle}-\dfrac{r_{\alpha,k}^{(i)}(\mathbf{z}_k)\left(1-r_{k|k-1}^{(i)}\right)}{1-r_{k|k-1}^{(i)}\left\langle p_{k|k-1}^{(i)},p_{D,k}\right\rangle}\end{array}\right)^2\right]^{1/2}\end{array}\right)}{\sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i)}(\mathbf{z}_k)} \tag{A.50}$$

It holds that (using Minkowski's inequality for the second term in the summation)

$$\leq \frac{\sum_{i=1}^{M_{k|k-1}}\left(\begin{array}{c} E\left[\left(r_{\alpha,k}^{(i),L_{k|k-1}^{(i)}}(\mathbf{z}_k)-r_{\alpha,k}^{(i)}(\mathbf{z}_k)\right)^2\right]^{1/2} \\ +E\left[\left(\dfrac{r_{\alpha,k}^{(i),L_{k|k-1}^{(i)}}(\mathbf{z}_k)-r_{\alpha,k}^{(i)}(\mathbf{z}_k)+r_{\alpha,k}^{(i)}(\mathbf{z}_k)r_{k|k-1}^{(i)}-r_{\alpha,k}^{(i),L_{k|k-1}^{(i)}}(\mathbf{z}_k)r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}}{1-r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}},p_{D,k}\right\rangle}\right)^2\right]^{1/2} \\ +\dfrac{r_{\alpha,k}^{(i)}(\mathbf{z}_k)\left(1-r_{k|k-1}^{(i)}\right)}{1-r_{k|k-1}^{(i)}\left\langle p_{k|k-1}^{(i)},p_{D,k}\right\rangle}E\left[\left(\dfrac{r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}},p_{D,k}\right\rangle-r_{k|k-1}^{(i)}\left\langle p_{k|k-1}^{(i)},p_{D,k}\right\rangle}{1-r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}},p_{D,k}\right\rangle}\right)^2\right]^{1/2}\end{array}\right)}{\sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i)}(\mathbf{z}_k)} \tag{A.51}$$

(by $0 \leq r_{k|k-1}^{(i),L_{k|k-1}^{(i)}} \leq 1$, (A.4), and Minkowski's inequality)

$$\leq \frac{\sum_{i=1}^{M_{k|k-1}} \left( \begin{array}{l} (2-\|p_{D,k}\|)E\left[\left(r_{\alpha,k}^{(i),L_{k|k-1}^{(i)}}(\mathbf{z}_k)-r_{\alpha,k}^{(i)}(\mathbf{z}_k)\right)^2\right]^{1/2} \\ +E\left[\left(\begin{array}{l} r_{\alpha,k}^{(i)}(\mathbf{z}_k)r_{k|k-1}^{(i)}-r_{\alpha,k}^{(i)}(\mathbf{z}_k)r_{k|k-1}^{(i),L_{k|k-1}^{(i)}} \\ +r_{\alpha,k}^{(i)}(\mathbf{z}_k)r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}-r_{\alpha,k}^{(i),L_{k|k-1}^{(i)}}(\mathbf{z}_k)r_{k|k-1}^{(i),L_{k|k-1}^{(i)}} \end{array}\right)^2\right]^{1/2} \\ +\dfrac{r_{\alpha,k}^{(i)}(\mathbf{z}_k)\left(1-r_{k|k-1}^{(i)}\right)}{1-r_{k|k-1}^{(i)}\left\langle p_{k|k-1}^{(i)},p_{D,k}\right\rangle}\left(\begin{array}{l} \|p_{D,k}\|E\left[\left(r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}-r_{k|k-1}^{(i)}\right)^2\right]^{1/2} \\ +r_{k|k-1}^{(i)}E\left[\left(\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}},p_{D,k}\right\rangle-\left\langle p_{k|k-1}^{(i)},p_{D,k}\right\rangle\right)^2\right]^{1/2} \end{array}\right) \end{array} \right)}{(1-\|p_{D,k}\|)\sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i)}(\mathbf{z}_k)} \tag{A.52}$$

(using $0 \leq r_{k|k-1}^{(i),L_{k|k-1}^{(i)}} \leq 1$ and Minkowski's inequality again for the second term in the summation)

$$\leq \frac{\sum_{i=1}^{M_{k|k-1}} \left( \begin{array}{l} (3-\|p_{D,k}\|)E\left[\left(r_{\alpha,k}^{(i),L_{k|k-1}^{(i)}}(\mathbf{z}_k)-r_{\alpha,k}^{(i)}(\mathbf{z}_k)\right)^2\right]^{1/2}+r_{\alpha,k}^{(i)}(\mathbf{z}_k)E\left[\left(r_{k|k-1}^{(i)}-r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}\right)^2\right]^{1/2} \\ +\dfrac{r_{\alpha,k}^{(i)}(\mathbf{z}_k)\left(1-r_{k|k-1}^{(i)}\right)}{1-r_{k|k-1}^{(i)}\left\langle p_{k|k-1}^{(i)},p_{D,k}\right\rangle}\left(\begin{array}{l} \|p_{D,k}\|E\left[\left(r_{k|k-1}^{(i),L_{k|k-1}^{(i)}}-r_{k|k-1}^{(i)}\right)^2\right]^{1/2} \\ +r_{k|k-1}^{(i)}E\left[\left(\left\langle p_{k|k-1}^{(i),L_{k|k-1}^{(i)}},p_{D,k}\right\rangle-\left\langle p_{k|k-1}^{(i)},p_{D,k}\right\rangle\right)^2\right]^{1/2} \end{array}\right) \end{array} \right)}{(1-\|p_{D,k}\|)\sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i)}(\mathbf{z}_k)} \tag{A.53}$$

(by $0 \leq r_{k|k-1}^{(i)} \leq 1$, (4.8), (4.9) (A.44), (A.45), and Assumption 2)

$$\begin{aligned} &\leq \frac{\|\varphi_{k,\mathbf{z}_k}\|\left(\sqrt{c_{k|k-1}}+\sqrt{d_{k|k-1}}\right)}{(1-\|p_{D,k}\|)^3\inf(\varphi_{k,\mathbf{z}_k})n_{k|k-1}}\sum_{i=1}^{M_{k|k-1}}\left(\frac{4-\left(r_{k|k-1}^{(i)}+1\right)\|p_{D,k}\|}{\sqrt{L_{k|k-1}^{(i)}}}\right) \\ &\leq \frac{\|\varphi_{k,\mathbf{z}_k}\|\left(\sqrt{c_{k|k-1}}+\sqrt{d_{k|k-1}}\right)\left(4M_{k|k-1}-\left(n_{k|k-1}+M_{k|k-1}\right)\|p_{D,k}\|\right)}{(1-\|p_{D,k}\|)^3\inf(\varphi_{k,\mathbf{z}_k})n_{k|k-1}\sqrt{L_{k|k-1}^{\min}}}. \end{aligned} \tag{A.54}$$

Finally, (4.13) is proved with

$$c^*_{U,k} = \left( \frac{\|\varphi_{k,\mathbf{z}_k}\| \left(\sqrt{c_{k|k-1}} + \sqrt{d_{k|k-1}}\right)}{\left(1 - \|p_{D,k}\|\right)^3 \inf(\varphi_{k,\mathbf{z}_k})} \left( \frac{\left(4 - \|p_{D,k}\|\right) M_{k|k-1}}{n_{k|k-1}} - \|p_{D,k}\| \right) \right)^2. \tag{A.55}$$

Now turn to (4.14). From (2.10) and (3.12), we get

$$\begin{aligned} &E\left[\left(\langle p_{U,k}(\cdot;\mathbf{z}_k),\varphi\rangle - \left\langle p_{U,k}^{L^{(i)}_{k|k-1}}(\cdot;\mathbf{z}_k),\varphi\right\rangle\right)^2\right]^{1/2} \\ &\quad = E\left[\left(\frac{\left\langle \sum_{i=1}^{M_{k|k-1}} p_{\alpha,k}^{(i),L^{(i)}_{k|k-1}}(\mathbf{x}_k;\mathbf{z}_k),\varphi\right\rangle}{\sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i),L^{(i)}_{k|k-1}}(\mathbf{z}_k)} - \frac{\left\langle \sum_{i=1}^{M_{k|k-1}} p_{\alpha,k}^{(i)}(\mathbf{x}_k;\mathbf{z}_k),\varphi\right\rangle}{\sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i)}(\mathbf{z}_k)}\right)^2\right]^{1/2} \end{aligned} \tag{A.56}$$

(adding and subtracting a new term)

$$\begin{aligned} = E\Bigg[\Bigg( &\frac{\left\langle \sum_{i=1}^{M_{k|k-1}} p_{\alpha,k}^{(i),L^{(i)}_{k|k-1}}(\cdot;\mathbf{z}_k),\varphi\right\rangle}{\sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i),L^{(i)}_{k|k-1}}(\mathbf{z}_k)} - \frac{\left\langle \sum_{i=1}^{M_{k|k-1}} p_{\alpha,k}^{(i),L^{(i)}_{k|k-1}}(\cdot;\mathbf{z}_k),\varphi\right\rangle}{\sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i)}(\mathbf{z}_k)} \\ &+ \frac{\left\langle \sum_{i=1}^{M_{k|k-1}} p_{\alpha,k}^{(i),L^{(i)}_{k|k-1}}(\cdot;\mathbf{z}_k),\varphi\right\rangle}{\sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i)}(\mathbf{z}_k)} - \frac{\left\langle \sum_{i=1}^{M_{k|k-1}} p_{\alpha,k}^{(i)}(\cdot;\mathbf{z}_k),\varphi\right\rangle}{\sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i)}(\mathbf{z}_k)}\Bigg)^2\Bigg]^{1/2} \end{aligned} \tag{A.57}$$

(using Minkowski's inequality)

$$\begin{aligned} \leq\ & E\left[\left(\left\langle \sum_{i=1}^{M_{k|k-1}} p_{\alpha,k}^{(i),L^{(i)}_{k|k-1}}(\cdot;\mathbf{z}_k),\varphi\right\rangle\left(\frac{\sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i),L^{(i)}_{k|k-1}}(\mathbf{z}_k) - \sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i)}(\mathbf{z}_k)}{\sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i)}(\mathbf{z}_k)\sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i),L^{(i)}_{k|k-1}}(\mathbf{z}_k)}\right)\right)^2\right]^{1/2} \\ &+ \frac{E\left[\left(\left\langle \sum_{i=1}^{M_{k|k-1}} p_{\alpha,k}^{(i),L^{(i)}_{k|k-1}}(\cdot;\mathbf{z}_k) - \sum_{i=1}^{M_{k|k-1}} p_{\alpha,k}^{(i)}(\cdot;\mathbf{z}_k),\varphi\right\rangle\right)^2\right]^{1/2}}{\sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i)}(\mathbf{z}_k)} \end{aligned} \tag{A.58}$$

(by (2.12) and (3.14))

$$\leq \frac{2\|\varphi\|}{\sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i)}(\mathbf{z}_k)} E\left[\left(\sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i),L^{(i)}_{k|k-1}}(\mathbf{z}_k) - \sum_{i=1}^{M_{k|k-1}} r_{\alpha,k}^{(i)}(\mathbf{z}_k)\right)^2\right]^{1/2} \tag{A.59}$$

(by (A.39) and (A.47))

$$\leq \frac{\|\varphi\|}{\sqrt{L_{k|k-1}^{\min}}} c_{U,k}. \tag{A.60}$$

Finally, (4.14) is proved with

$$d_{U,k} = c_{U,k} = \left( \frac{2\|\psi_{k,\mathbf{z}_k}\| \left(\sqrt{c_{k|k-1}} + \sqrt{d_{k|k-1}}\right) M_{k|k-1.}}{\inf(\psi_{k,\mathbf{z}_k}) n_{k|k-1} \left(1 - \|p_{D,k}\|\right)^2} \right)^2. \tag{A.61}$$

This completes the proof.

## Nomenclature

| | |
|---|---|
| $\mathbf{x}_k$: | State vector of a single target at time $k$ |
| $\mathbf{z}_k$: | Single measurement vector at time $k$ |
| $n_k$: | Number of existing targets at time $k$ |
| $m_k$: | Number of measurements collected at time $k$ |
| $X_k = \{\mathbf{x}_{i,k}\}_{i=1}^{n_k}$: | Finite set of multitarget state-vectors at time $k$ |
| $Z_k = \{\mathbf{z}_{i,k}\}_{i=1}^{m_k}$: | Finite set of measurements collected at time $k$ |
| $f_{k\|k-1}(\mathbf{x}_k \mid \mathbf{x}_{k-1})$: | Single-target Markov transition density at time $k$ |
| $p_{S,k}(\mathbf{x}_{k-1})$: | Probability of target survival at time $k$ |
| $p_{D,k}(\mathbf{x}_k)$: | Probability of detection at time $k$ |
| $\kappa_k(\mathbf{z}_k)$: | Intensity of Poisson clutter process at time $k$ |
| $\psi_{k,\mathbf{z}_k}(\mathbf{x}_k) = f_k(\mathbf{z}_k \mid \mathbf{x}_k)$: | Single-sensor/target likelihood density at time $k$ |
| $\delta_{\mathbf{x}}(\cdot)$: | Dirac delta function centered at $\mathbf{x}$ |
| $\mathbb{R}^d$: | $d$-dimensional real space |
| $C_b(\mathbb{R}^d)$: | Set of continuous bounded functions on $\mathbb{R}^d$ |
| $B(\mathbb{R}^d)$: | Set of bounded Borel measurable functions on $\mathbb{R}^d$ |
| $\pi(Y^{(i)})$: | Probability density of a Bernoulli random finite set (RFS) $Y^{(\mathbf{i})}$ |
| $\pi(Y)$: | Probability density of multi-Bernoulli RFS $Y = \bigcup_{i=1}^{M} Y^{(i)}$ |
| $\pi = \{(r^{(i)}, p^{(i)})\}_{i=1}^{M}$: | Abbreviation of $\pi(Y)$. $\{(r^{(i)}, p^{(i)})\}_{i=1}^{M}$ is the multi-Bernoulli parameter set |
| $\tilde{\pi} = \{(r^{(i),L^{(i)}}, p^{(i),L^{(i)}})\}_{i=1}^{M}$: | Particle approximation of $\pi = \{(r^{(i)}, p^{(i)})\}_{i=1}^{M}$. $(r^{(i),L^{(i)}}, p^{(i),L^{(i)}})$ denotes that $(r^{(i)}, p^{(i)})$ is comprised of the number $L^{(i)}$ of the particles |
| $\|\cdot\|$: | Supremum norm. $\|\varphi\| \triangleq \sup(\mid \varphi \mid)$, $\sup(\cdot)$ denotes the supremum |
| $\langle \cdot, \cdot \rangle$: | Inner product. If the measure in $\langle \cdot, \cdot \rangle$ is continuous, it defines the integral inner product; if the measure in $\langle \cdot, \cdot \rangle$ is discrete, it defines the summation inner product. |

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