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Research Article

Robust Control for Uncertain Switched Systems with Interval Nondifferentiable Time-Varying Delays

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This paper addresses the conditions for robust stabilization of a class of uncertain switched systems with delay. The system to be considered is autonomous and the state delay is time-varying. Using Lyapunov functional approach, restriction on the derivative of time-delay function is not required to design switching rule for the robust stabilization of switched systems with time-varying delays. The delay-dependent stability conditions are presented in terms of the solution of LMIs which can be solved by various available algorithms. A numerical example is given to illustrate the effectiveness of theoretical results.

1. Introduction

The switched system is a type of hybrid systems that consist of a family of a differential or difference equations and a switching rule to indicate which subsystem will be activated at a specific interval of time. For applications, switched systems can be used to describe several physical or chemical processes which are concerned by more than one dynamics: some systems work at some interval time then stop and other systems take over such as the automatic system in airplane, car energy system, traffic system, and machine industrial system, see [1, 2]. Thus, a switching strategy must be designed in the study of stability of switched systems, also see [3–5].

The delay system has been considered in many research, especially the real processes in our world often involve time-delay; that is, the present state depends on the past states which brings more difficulty to investigate the stability of the system, especially time varying delay system, see [6–10]. In general, the following assumption on the derivative of the delay is made, namely $\dot{h}(t) < 1$, see [11, 12] and references cited therein. This assumption may leads to conservativeness; for example, it might not be used when the delay is a fast or a nondifferential time varying function.

Moreover, in study of real world applications, the systems are in general influenced by disturbances which might cause inaccuracy of the data. The system can become unstable or less capable because of disturbance. Consequently, the study of robust stability of switched systems with time varying delay becomes important and has been studied by many researchers, see [7, 12–17].

In this paper, we study the problem of robust stability for a class of switched systems with time-varying delay. Compared with existing results in the literature, the novelty of our results is twofold. Firstly, the state delay is time-varying in which the restriction on the derivative of the time-delay function is not required to design switching rule in term of a dwell time for the robust stability of the system. Secondly, the obtained conditions for the robust stability are delay-dependent and formulated in terms of the solution of standard LMIs which can be solved by various available algorithms [18]. The paper is organized as follows. Section 2 presents notations, definitions, and auxiliary propositions required for the proof of the main results. Switching design for the robust stability of the system with illustrative examples is presented in Sections 3 and 4, respectively. The paper ends with a conclusion followed by cited references.

2. Problem Formulation and Preliminaries

Throughout this paper, the following notations will be used:

$\mathbb{R}^n$—the n dimensional Euclidean space;

$\mathbb{R}^{n\times n}$—the set of all $n \times n$ real matrices;

$\mathbb{N}$—the set of all positive integers;

$\|x\|$—the Euclidean norm of vector $x \in \mathbb{R}^n$;

$\mathrm{diag}\{\cdot\}$—the block diagonal matrix;

I—the identity matrix;

A^T—the transpose of matrix A;

A^{-1}—the inverse of matrix A;

$\begin{bmatrix} A & B \\ * & C \end{bmatrix}$—$*$ the symmetric form of matrix, namely, $* = B^T$;

$\|A\| = \sup\{\|Ax\| : x \in \mathbb{R}^n, \|x\| = 1\}$ for any $A \in \mathbb{R}^{n\times n}$;

C_{h_2}—space of continuous vector-valued function defined on $[-h_2, 0]$;

$x_t(\theta) = x(t+\theta), -h_2 \le \theta \le 0,$ where $x_t \in C_{h_2}$;

$\|x_t\|_{c_{h_2}} = \sup_{-h_2\le\theta\le 0}\|x(t+\theta)\|$;

$\lambda_{\max}(A) = \max\{\mathrm{Re}(\lambda) : \lambda \text{ is eigenvalue of } A\}$;

$\lambda_{\min}(A) = \min\{\mathrm{Re}(\lambda) : \lambda \text{ is eigenvalue of } A\}$;

$M = \{1, 2, \ldots, k\}$.

Consider the following uncertain switched system with time varying delay:

$$\begin{aligned}\dot{x}(t) &= \left(A_{\sigma(t)} + \Delta A_{\sigma(t)}(t)\right)x(t) + \left(D_{\sigma(t)} + \Delta D_{\sigma(t)}(t)\right)x\left(t - h_{\sigma(t)}(t)\right)\\ &\quad + \left(B_{\sigma(t)} + \Delta B_{\sigma(t)}(t)\right)u_{\sigma(t)}(t), \quad t > 0,\\ x(t) &= \phi(t), \quad t \in [-h_2, 0],\end{aligned} \tag{2.1}$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in \mathbb{R}^n$ is the state vector. A_i, D_i, B_i, $i \in M$ are known constant matrices, $\Delta A_i, \Delta D_i, \Delta B_i$ are uncertainty matrices which are of the form

$$\begin{aligned}\Delta A_i &= E_{1,i}F_{1,i}(t)H_{1,i}, \qquad \Delta D_i = E_{2,i}F_{2,i}(t)H_{2,i},\\ \Delta B_i &= E_{3,i}F_{3,i}(t)H_{3,i}, \qquad F_{j,i}^T(t)F_{j,i}(t) \le I, \quad j = 1, 2, 3,\end{aligned} \tag{2.2}$$

where $F_{1,i}(t), F_{2,i}(t), F_{3,i}(t)$ are unknown matrices, I is the identity matrix of appropriate dimension. $h_i(t)$ is the delay function for an ith subsystem which satisfies the following condition:

$$0 \le h_1 \le h_{1,i} \le h_i(t) \le h_{2,i} \le h_2. \tag{2.3}$$

Let $\phi(t)$ be an initial condition. $\sigma(t) : R^+ \cup \{0\} \to M$, $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$, $\sigma(t)$ is called the switching signal; we have the switching sequence $\{x_{t_0}; (i_0, t_0), \ldots \mid i_k \in M, k = 0, 1, 2, \ldots\}$, which means that when $t \in [t_k, t_{k+1})$, the i_kth subsystem is activated.

Definition 2.1. $T^* = \inf\{t_l - t_{l-1}\}$ is called the dwell time of switched system.

Definition 2.2. The system (2.1) is said to be stabilizabled if there exists a feedback controller $u(t) \in \mathbb{R}^m$ such that the closed loop switched systems (without uncertainties) is asymptotically stable.

Definition 2.3. The system (2.1) is said to be robustly stabilizable if there exists a feedback controller $u(t) \in \mathbb{R}^m$ such that the closed loop uncertain switched systems are robustly stable.

The following lemmas will be used throughout this paper.

Lemma 2.4 (Schur complement lemma). *Given constant symmetric matrices Q, S and $R \in \mathbb{R}^{n\times n}$, where $R > 0$, $Q = Q^T$, and $R = R^T$, one has*

$$\begin{bmatrix} Q & S \\ S^T & -R \end{bmatrix} < 0 \Longleftrightarrow Q + SR^{-1}S^T < 0. \tag{2.4}$$

Lemma 2.5 (see [13]). *Given $\varepsilon > 0$ and matrices D, E, and F with $F^TF \le I$, one has*

$$DEF + E^TF^TD^T \le \varepsilon DD^T + \varepsilon^{-1}E^TE. \tag{2.5}$$

Lemma 2.6 (see [13]). *Given a positive definite matrix $P \in \mathbb{R}^{n\times n}$, any symmetric matrix $Q \in \mathbb{R}^{n\times n}$ and $x \in \mathbb{R}^n$, then*

$$\lambda_{\min}\left(P^{-1}Q\right)x^T Px \le x^T Qx \le \lambda_{\max}\left(P^{-1}Q\right)x^T Px. \tag{2.6}$$

Lemma 2.7 (see [13]). *For any positive semidefinite matrix $M \in \mathbb{R}^{n\times n}$, a scalar $\gamma > 0$, and a vector function $\omega : [0,\gamma] \to \mathbb{R}^{n\times n}$ such that the integrals concerned are well-defined, one has*

$$\left(\int_0^\gamma \omega(s)ds\right)^T M\left(\int_0^\gamma \omega(s)ds\right) \le \gamma\int_0^\gamma \omega^T(s)M\omega(s)ds. \tag{2.7}$$

Lemma 2.8 (Cauchy inequality). *For any symmetric positive definite matrix $N \in M^{m\times m}$ and $x, y \in \mathbb{R}^n$, one has*

$$\pm 2x^T y \le x^T Nx + y^T N^{-1}y. \tag{2.8}$$

For simplicity of later presentation, one gives the following notations.

$$\begin{gathered}
\Sigma_{11,i} = P_iA_i^T + A_iP_i - B_iB_i^T + \left(2 + h_1^2 + h_2^2\right)Q_i - R_i; \quad \Sigma_{12,i} = P_iA_i^T - 0.5B_iB_i^T; \\
\Sigma_{13,i} = 0.5R_i; \quad \Sigma_{14,i} = 0.5R_i; \quad \Sigma_{15,i} = D_iP_i; \quad \Sigma_{22,i} = \left(h_{1,i}^2 + h_{2,i}^2\right)R_i + \delta_i^2Q_i - 2P_i; \\
\Sigma_{25,i} = D_iP_i; \quad \Sigma_{33,i} = -Q_i - 0.5R_i - 0.5U_i; \quad \Sigma_{35,i} = 0.5U_i; \quad \Sigma_{44,i} = -Q_i - 0.5R_i - 0.5U_i; \\
\Sigma_{45,i} = 0.5U_i; \quad \Sigma_{55,i} = -U_i; \quad \lambda_1 = \max_{i\in M}\left\{\lambda_{\max}\left(P_i^{-1}\right)\right\}; \quad \lambda_2 = \min_{i\in M}\left\{\lambda_{\min}\left(P_i^{-1}\right)\right\}; \\
\lambda_3 = \max_{i\in M}\{\lambda_{\max}(P_i)\}; \quad \lambda_4 = \min_{i\in I_k}\{\lambda_{\min}(-\Omega_{1,i})\}; \quad \lambda_5 = \min_{i\in M}\{\lambda_{\min}(-\Omega_{2,i})\}; \\
\lambda^* = \min\{\lambda_4, \lambda_5\}; \quad h_{a,i} = \frac{1}{2}(h_{1,i} + h_{2,i}); \quad \delta_i = (h_{2,i} - h_{1,i}); \quad h^* = \min_{i\in M}\{h_{2,i} - h_{a,i}\}; \quad \delta = h_2 - h_1.
\end{gathered} \tag{2.9}$$

3. Main Results

3.1. Asymptotical Stabilization for Nominal Switched Systems with Interval Time-Varying Delay

The nominal switched systems are given by

$$\begin{gathered}
\dot{x}(t) = A_{\sigma(t)}x(t) + D_{\sigma(t)}x\left(t - h_{\sigma(t)}(t)\right) + B_{\sigma(t)}u_{\sigma(t)}(t), \quad t > 0 \\
x(t) = \phi(t), \quad t \in [-h_2, 0].
\end{gathered} \tag{3.1}$$

We now state the main result on sufficient condition for stabilization of the switched systems (3.1).

Theorem 3.1. *Given $\alpha \in (0,1)$. If there exists symmetric positive definite matrices P_i, Q_i, R_i, U_i such that the following conditions hold:*

$$\Omega_{1,i} = \Omega_i - \begin{bmatrix}000 & -II\end{bmatrix}^T U_i \begin{bmatrix}000 & -II\end{bmatrix} < 0,$$
$$\Omega_{2,i} = \Omega_i - \begin{bmatrix}00I0 & -I\end{bmatrix}^T U_i \begin{bmatrix}00I0 & -I\end{bmatrix} < 0,$$
$$\Omega_i = \begin{bmatrix} \Sigma_{11,i} & \Sigma_{12,i} & \Sigma_{13,i} & \Sigma_{14,i} & \Sigma_{15,i} \\ * & \Sigma_{22,i} & 0 & 0 & \Sigma_{25,i} \\ * & * & \Sigma_{33,i} & 0 & \Sigma_{35,i} \\ * & * & * & \Sigma_{44,i} & \Sigma_{45,i} \\ * & * & * & * & \Sigma_{55,i} \end{bmatrix}, \tag{3.2}$$

and if

$$T^* \geq \frac{1}{\rho} \ln\left(\frac{\lambda_1}{\alpha\lambda_2}\right), \tag{3.3}$$

where $\rho = \min\{h^*\lambda^*/\delta\lambda_3, 1/2h_2, 1/2\}$, T^* *is the dwell time, then for any switching rule satisfying* (3.3) *the switched system* (3.1) *is stabilizable under the feedback controller*

$$u_i(t) = -\frac{1}{2}B_i^T P_i^{-1} x(t), \quad t \geq 0. \tag{3.4}$$

Proof. Let $Y_i = P_i^{-1}, y(t) = Y_i x(t)$. Using the feedback controller (3.4), we choose a Lyapunov-Krasovskii functional candidates as

$$V_i(x_t) = V_{1,i}(x_t) + \cdots + V_{8,i}(x_t), \tag{3.5}$$

where

$$V_{1,i}(x_t) = x^T(t) Y_i x(t),$$
$$V_{2,i}(x_t) = \int_{t-h_{1,i}}^{t} x^T(s) Y_i Q_i Y_i x(s) ds,$$
$$V_{3,i}(x_t) = \int_{t-h_{2,i}}^{t} x^T(s) Y_i Q_i Y_i x(s) ds,$$
$$V_{4,i}(x_t) = h_{1,i} \int_{-h_{1,i}}^{0} \int_{t+s}^{t} \dot{x}^T(\theta) Y_i R_i Y_i \dot{x}(\theta) d\theta\, ds,$$

$$\begin{aligned}
V_{5,i}(x_t) &= h_{2,i}\int_{-h_{2,i}}^{0}\int_{t+s}^{t}\dot{x}^T(\theta)Y_iR_iY_i\dot{x}(\theta)d\theta\, ds,\\
V_{6,i}(x_t) &= \delta_i\int_{-h_{2,i}}^{-h_{1,i}}\int_{t+s}^{t}\dot{x}^T(\theta)Y_iU_iY_i\dot{x}(\theta)d\theta\, ds,\\
V_{7,i}(x_t) &= h_{1,i}\int_{-h_{1,i}}^{0}\int_{t+s}^{t}x^T(\theta)Y_iQ_iY_ix(\theta)d\theta\, ds,\\
V_{8,i}(x_t) &= h_{2,i}\int_{-h_{2,i}}^{0}\int_{t+s}^{t}x^T(\theta)Y_iQ_iY_ix(\theta)d\theta\, ds.
\end{aligned} \tag{3.6}$$

It is easy to see that

$$V_i(x_t) \geq c_1\|x(t)\|^2, \tag{3.7}$$

for some $c_1 > 0$. Taking the derivative of $V_i(x_t)$ with respect to t along any trajectory of solution of (3.1) yields

$$\begin{aligned}
\dot{V}_{1,i}(x_t) &= 2x^T(t)Y_i\dot{x}(t),\\
&= y^T(t)\left[P_iA_i^T + A_iP_i\right]y(t) - y^T(t)B_i^TB_iy(t) + 2y^T(t)D_iP_iy(t-h_i(t)),
\end{aligned} \tag{3.8}$$

$$\dot{V}_{2,i}(x_t) = y^T(t)Q_iy(t) - y^T(t-h_{1,i})Q_iy(t-h_{1,i}), \tag{3.9}$$

$$\dot{V}_{3,i}(x_t) = y^T(t)Q_iy(t) - y^T(t-h_{2,i})Q_iy(t-h_{2,i}), \tag{3.10}$$

$$\dot{V}_{4,i}(x_t) = h_{1,i}^2\dot{y}^T(t)R_i\dot{y}(t) - \frac{h_{1,i}}{2}\int_{t-h_{1,i}}^{t}\dot{y}^T(s)R_i\dot{y}(s)ds - \frac{h_{1,i}}{2}\int_{t-h_{1,i}}^{t}\dot{y}^T(s)R_i\dot{y}(s)ds, \tag{3.11}$$

$$\dot{V}_{5,i}(x_t) = h_{2,i}^2\dot{y}^T(t)R_i\dot{y}(t) - \frac{h_{2,i}}{2}\int_{t-h_{2,i}}^{t}\dot{y}^T(s)R_i\dot{y}(s)ds - \frac{h_{2,i}}{2}\int_{t-h_{2,i}}^{t}\dot{y}^T(s)R_i\dot{y}(s)ds, \tag{3.12}$$

$$\dot{V}_{6,i}(x_t) = \delta_i^2\dot{y}^T(t)U_i\dot{y}(t) - \frac{\delta_i}{2}\int_{t-h_{2,i}}^{t-h_{1,i}}\dot{y}^T(s)U_i\dot{y}(s)ds - \frac{\delta_i}{2}\int_{t-h_{2,i}}^{t-h_{1,i}}\dot{y}^T(s)R_i\dot{y}(s)ds, \tag{3.13}$$

$$\dot{V}_{7,i}(x_t) = h_{1,i}^2y^T(t)Q_iy(t) - h_{1,i}\int_{t-h_{1,i}}^{t}x^T(s)Y_iQ_iY_ix(s)ds, \tag{3.14}$$

$$\dot{V}_{8,i}(x_t) = h_{2,i}^2y^T(t)Q_iy(t) - h_{2,i}\int_{t-h_{2,i}}^{t}x^T(s)Y_iQ_iY_ix(s)ds. \tag{3.15}$$

Then by applying Lemma 2.7 and Leibniz-Newton formular, we have

$$
\begin{aligned}
&-\frac{h_{1,i}}{2}\int_{t-h_{1,i}}^{t}\dot{y}^T(s)R_i\dot{y}(s)ds \\
&\quad\le y^T(t)\left(-\frac{R_i}{2}\right)y(t)+y^T(t)R_iy(t-h_{1,i})+y^T(t-h_{1,i})\left(-\frac{R_i}{2}\right)y(t-h_{1,i}),\\
&-\frac{h_{2,i}}{2}\int_{t-h_{2,i}}^{t}\dot{y}^T(s)R_i\dot{y}(s)ds \\
&\quad\le y^T(t)\left(-\frac{R_i}{2}\right)y(t)+y^T(t)R_iy(t-h_{2,i})+y^T(t-h_{2,i})\left(-\frac{R_i}{2}\right)y(t-h_{2,i}).
\end{aligned}
\tag{3.16}
$$

Note that

$$
\begin{aligned}
-\frac{\delta_i}{2}\int_{t-h_{2,i}}^{t-h_{1,i}}\dot{y}^T(s)U_i\dot{y}(s)ds &= -\frac{\delta_i}{2}\int_{t-h_{2,i}}^{t-h_i(t)}\dot{y}^T(s)U_i\dot{y}(s)ds-\frac{\delta_i}{2}\int_{t-h_i(t)}^{t-h_{1,i}}\dot{y}^T(s)U_i\dot{y}(s)ds\\
&= -\frac{h_{2,i}-h_i(t)}{2}\int_{t-h_{2,i}}^{t-h(t)}\dot{y}^T(s)U_i\dot{y}(s)ds-\frac{h(t)-h_{1,i}}{2}\\
&\quad\times\int_{t-h_{2,i}}^{t-h_i(t)}\dot{y}^T(s)U_i\dot{y}(s)ds-\frac{h_i(t)-h_{1,i}}{2}\int_{t-h_i(t)}^{t-h_{1,i}}\dot{y}^T(s)U_i\dot{y}(s)ds\\
&\quad-\frac{h_{2,i}-h_i(t)}{2}\int_{t-h(t)}^{t-h_{1,i}}\dot{y}^T(s)U_i\dot{y}(s)ds.
\end{aligned}
\tag{3.17}
$$

Using Lemma 2.7 yields

$$
\begin{aligned}
&-\frac{h_{2,i}-h_i(t)}{2}\int_{t-h_{2,i}}^{t-h_i(t)}\dot{y}^T(s)U_i\dot{y}(s)ds\\
&\quad\le \left[y(t-h_i(t))-y(t-h_{2,i})\right]^T\left(-\frac{U_i}{2}\right)\left[y(t-h_i(t))-y(t-h_{2,i})\right],\\
&-\frac{h_i(t)-h_{1,i}}{2}\int_{t-h_i(t)}^{t-h_{1,i}}\dot{y}^T(s)U_i\dot{y}(s)ds\\
&\quad\le \left[y(t-h_{1,i})-y(t-h_i(t))\right]^T\left(-\frac{U_i}{2}\right)\left[y(t-h_{1,i})-y(t-h_i(t))\right].
\end{aligned}
\tag{3.18}
$$

Let $\beta = (h_{2,i} - h_i(t))/(h_{2,i} - h_{1,i}) \le 1$. Then

$$\begin{aligned}
-\frac{h_{2,i} - h_i(t)}{2}\int_{t-h_i(t)}^{t-h_{1,i}} \dot{y}^T(s)U_i\dot{y}(s)ds &= -\beta\int_{t-h_i(t)}^{t-h_{1,i}} (h_{2,i} - h_{1,i})\dot{y}^T(s)\left(\frac{U_i}{2}\right)\dot{y}(s)ds \\
&\le -\beta\int_{t-h_i(t)}^{t-h_{1,i}} (h_i(t) - h_{1,i})\dot{y}^T(s)\left(\frac{U_i}{2}\right)\dot{y}(s)ds \\
&\le -\beta\left[y(t-h_{1,i}) - y(t-h_i(t))\right]^T\left(\frac{U_i}{2}\right) \\
&\quad \times \left[y(t-h_{1,i}) - y(t-h_i(t))\right]. \\
-\frac{h_i(t) - h_{1,i}}{2}\int_{t-h_{2,i}}^{t-h_i(t)} \dot{y}^T(s)U_i\dot{y}(s)ds &= -(1-\beta)\int_{t-h_{2,i}}^{t-h_i(t)} (h_{2,i} - h_{1,i})\dot{y}^T(s)\left(\frac{U_i}{2}\right)\dot{y}(s)ds \\
&\le -(1-\beta)\int_{t-h_{2,i}}^{t-h_i(t)} (h_{2,i} - h_i(t))\dot{y}^T(s)\left(\frac{U_i}{2}\right)\dot{y}(s)ds \\
&\le -(1-\beta)\left[y(t-h_i(t)) - y(t-h_{2,i})\right]^T\left(\frac{U_i}{2}\right) \\
&\quad \times \left[y(t-h_i(t)) - y(t-h_{2,i})\right].
\end{aligned} \tag{3.19}$$

Therefore from (3.18)-(3.19), we have

$$\begin{aligned}
-\frac{\delta_i}{2}\int_{t-h_{2,i}}^{t-h_{1,i}} \dot{y}^T(s)U_i\dot{y}(s)ds \le{}& \left[y(t-h_i(t)) - y(t-h_{2,i})\right]^T\left(-\frac{U_i}{2}\right)\left[y(t-h_i(t)) - y(t-h_{2,i})\right] \\
&+ \left[y(t-h_i(t)) - y(t-h_{2,i})\right]^T\left(-\frac{U_i}{2}\right)\left[y(t-h_i(t)) - y(t-h_{2,i})\right] \\
&- \beta\left[y(t-h_{1,i}) - y(t-h_i(t))\right]^T\left(\frac{U_i}{2}\right)\left[y(t-h_{1,i}) - y(t-h_i(t))\right] \\
&- (1-\beta)\left[y(t-h_i(t)) - y(t-h_{2,i})\right]^T\left(\frac{U_i}{2}\right) \\
&\times \left[y(t-h_i(t)) - y(t-h_{2,i})\right].
\end{aligned} \tag{3.20}$$

Furthermore, from the following zero equation

$$-P_i\dot{y}(t) + A_iP_iy(t) + D_iP_iy(t-h_i(t)) - 0.5B_iB_i^Ty(t) = 0, \tag{3.21}$$

we obtain

$$-2\dot{y}^TP_i\dot{y}(t) + 2\dot{y}^TA_iP_iy(t) + 2\dot{y}^TD_iP_iy(t-h_i(t)) - 2\dot{y}^T0.5B_iB_i^Ty(t) = 0. \tag{3.22}$$

Hence, from (3.5),(3.8)–(3.16),(3.20), and (3.22), we can get

$$\begin{aligned}
\dot{V}_i(x_t) \leq{}& \xi^T(t)\Omega_i\xi(t) - \beta\left[y(t-h_{1,i}) - y(t-h(t))\right]^T\left(\frac{U_i}{2}\right)\left[y(t-h_{1,i}) - y(t-h_i(t))\right] \\
&- (1-\beta)\left[y(t-h_i(t)) - y(t-h_{2,i})\right]^T\left(\frac{U_i}{2}\right)\left[y(t-h_i(t)) - y(t-h_{2,i})\right] \\
&- \frac{h_{1,i}}{2}\int_{t-h_{1,i}}^{t}\dot{y}^T(s)R_i\dot{y}(s)ds - \frac{h_{2,i}}{2}\int_{t-h_{2,i}}^{t}\dot{y}^T(s)R_i\dot{y}(s)ds \\
&- \frac{\delta_i}{2}\int_{t-h_{2,i}}^{t-h_{1,i}}\dot{y}^T(s)R_i\dot{y}(s)ds - h_{1,i}\int_{t-h_{1,i}}^{t}x^T(s)Y_iQ_iY_ix(s)ds \\
&- h_{2,i}\int_{t-h_{2,i}}^{t}x^T(s)Y_iQ_iY_ix(s)ds \\
={}& \xi^T(t)\left[(1-\beta)\Omega_{1,i} + \beta\Omega_{2,i}\right]\xi(t) - \frac{h_{1,i}}{2}\int_{t-h_{1,i}}^{t}\dot{y}^T(s)R_i\dot{y}(s)ds \\
&- \frac{h_{2,i}}{2}\int_{t-h_{2,i}}^{t}\dot{y}^T(s)R_i\dot{y}(s)ds - \frac{\delta_i}{2}\int_{t-h_{2,i}}^{t-h_{1,i}}\dot{y}^T(s)R_i\dot{y}(s)ds, \\
&- h_{1,i}\int_{t-h_{1,i}}^{t}x^T(s)Y_iQ_iY_ix(s)ds - h_{2,i}\int_{t-h_{2,i}}^{t}x^T(s)Y_iQ_iY_ix(s)ds,
\end{aligned} \tag{3.23}$$

where $\xi^T(t) = \left[\, y^T(t)\ \dot{y}^T(t)\ y^T(t-h_{1,i})\ y^T(t-h_{2,i})\ y^T(t-h_i(t)) \,\right]$.

Suppose τ is the time when the system switches from state j to state i, that is $I(\tau^+) = i$ and $I(\tau^-) = j$, where τ^+ and τ^- are the right and the left limit of the time τ, respectively. According to Lemma 2.6, we obtain

$$\begin{aligned}
V_{1,i}(x_{\tau^+}) &= x^T(\tau)Y_ix(\tau) \\
&\leq \frac{\lambda_1}{\lambda_2}\lambda_2x^T(\tau)x(\tau) \\
&\leq \frac{\lambda_1}{\lambda_2}x^T(\tau)P_jx(\tau) \\
&= \frac{\lambda_1}{\lambda_2}V_{1,j}(x_{\tau^-}).
\end{aligned} \tag{3.24}$$

We can apply this argument to integral terms in the Lypunov-Krasovskii function, so we get

$$V_i(x_{\tau^+}) \leq \frac{\lambda_1}{\lambda_2}V_j(x_{\tau^-}). \tag{3.25}$$

Now let ν be the time when the system switches from state k to state j, that is, $I(\nu^+) = j$ and $I(\nu^-) = k$, where ν^+ and ν^- are a right and a left limit of the time ν, respectively. In order to show that the switched system is stable, we need to compare $V_i(x_{\tau^+})$ with $V_j(x_{\nu^+})$ and estimate the upper bound for the term $\xi^T(t)[(1-\beta)\Omega_{1,i}+\beta\Omega_{2,i}]\xi(t)$ in the inequality (3.23). Hence we consider two following possible cases.

Case 1 $(0 \le h_1 \le h_{1,i} \le h(t) \le h_{a,i} \le h_{2,i} \le h_2)$. Since $0 \le \beta \le 1, \Omega_{1,i} < 0$ and $\Omega_{2,i} < 0$, we have

$$\begin{aligned}
&\xi^T(t)\left[(1-\beta)\Omega_{1,i}+\beta\Omega_{2,i}\right]\xi(t)\\
&\quad \le \xi^T(t)\left[\beta\Omega_{2,i}\right]\xi(t)\\
&\quad \le -\left(\frac{h_{2,i}-h_{a,i}}{\delta_i}\right)\lambda_5\Big(\|y(t)\|^2+\|\dot{y}(t)\|^2+\|y(t-h_{1,i})\|^2\\
&\qquad\qquad +\|y(t-h_{2,i})\|^2+\|y(t-h_i(t))\|^2\Big)\\
&\quad \le -\left(\frac{h_{2,i}-h_{a,i}}{\delta_i}\right)\lambda_5\left(\|y(t)\|^2\right)\\
&\quad = -\left(\frac{h_{2,i}-h_{a,i}}{\delta_i}\right)\frac{\lambda_5\lambda_3}{\lambda_3}\left(\|y(t)\|^2\right)\\
&\quad \le -\left(\frac{h_{2,i}-h_{a,i}}{\delta}\right)\frac{\lambda_5}{\lambda_3}x^T(t)P_i^{-1}P_iP_i^{-1}x(t)\\
&\quad \le -\left(\frac{h_{2,i}-h_{a,i}}{\delta}\right)\frac{\lambda_5}{\lambda_3}V_{1,i}\\
&\quad \le -\left(\frac{h_{2,i}-h_{a,i}}{\delta}\right)\frac{\lambda^*}{\lambda_3}V_{1,i}\\
&\quad \le -\left(\frac{h^*}{\delta}\right)\frac{\lambda^*}{\lambda_3}V_{1,i}.
\end{aligned} \tag{3.26}$$

Case 2 $(0 \le h_1 \le h_{1,i} \le h_{a,i} \le h(t) \le h_{2,i} \le h_2)$. Since $0 \le \beta \le 1, \Omega_{1,i} < 0$ and $\Omega_{2,i} < 0$, we get

$$\begin{aligned}
&\xi^T(t)\left[(1-\beta)\Omega_{1,i}+\beta\Omega_{2,i}\right]\xi(t)\\
&\quad \le \xi^T(t)\left[(1-\beta)\Omega_{1,i}\right]\xi(t)\\
&\quad \le -\left(\frac{h_{a,i}-h_{1,i}}{\delta_i}\right)\lambda_4\Big(\|y(t)\|^2+\|\dot{y}(t)\|^2+\|y(t-h_{1,i})\|^2\\
&\qquad\qquad +\|y(t-h_{2,i})\|^2+\|y(t-h_i(t))\|^2\Big)\\
&\quad \le -\left(\frac{h_{a,i}-h_{1,i}}{\delta_i}\right)\lambda_4\left(\|y(t)\|^2\right)
\end{aligned}$$

$$
\begin{aligned}
&= -\left(\frac{h_{a,i} - h_{1,i}}{\delta_i}\right)\frac{\lambda_4\lambda_3}{\lambda_3}\left(\|y(t)\|^2\right) \\
&\leq -\left(\frac{h_{a,i} - h_{1,i}}{\delta}\right)\frac{\lambda_4}{\lambda_3}x^T(t)P_i^{-1}P_iP_i^{-1}x(t) \\
&\leq -\left(\frac{h_{a,i} - h_{1,i}}{\delta}\right)\frac{\lambda_4}{\lambda_3}V_{1,i}.
\end{aligned}
\tag{3.27}
$$

Note that $h_{2,i} - h_{a,i} = h_{a,i} - h_{1,i}$, so we obtain

$$
\begin{aligned}
\xi^T(t)\left[(1-\beta)\Omega_{1,i} + \beta\Omega_{2,i}\right]\xi(t) &\leq -\left(\frac{h_{2,i} - h_{a,i}}{\delta}\right)\frac{\lambda^*}{\lambda_3}V_{1,i} \\
&\leq -\left(\frac{h^*}{\delta}\right)\frac{\lambda^*}{\lambda_3}V_{1,i}.
\end{aligned}
\tag{3.28}
$$

Moreover, from the definition of $V_{2,i}, \ldots, V_{8,i}$, we can get

$$
\begin{aligned}
-\frac{h_{1,i}}{2}\int_{t-h_{1,i}}^{t}\dot{y}^T(s)R_i\dot{y}(s)ds &\leq -\frac{1}{2}\int_{-h_{1,i}}^{0}\int_{t+s}^{t}\dot{x}^T(\theta)Y_iR_iY_i\dot{x}(\theta)d\theta\, ds \\
&\leq -\frac{1}{2h_{2,i}}V_{4,i}, \\
-\frac{h_{2,i}}{2}\int_{t-h_{2,i}}^{t}\dot{y}^T(s)R_i\dot{y}(s)ds &\leq -\frac{1}{2}\int_{-h_{2,i}}^{0}\int_{t+s}^{t}\dot{x}^T(\theta)Y_iR_iY_i\dot{x}(\theta)d\theta\, ds \\
&\leq -\frac{1}{2h_{1,i}}V_{5,i}, \\
-\frac{\delta_i}{2}\int_{t-h_{2,i}}^{t-h_{1,i}}\dot{y}^T(s)U_i\dot{y}(s)ds &\leq -\frac{1}{2}\int_{-h_{2,i}}^{-h_{1,i}}\int_{t+s}^{t}\dot{x}^T(\theta)Y_iU_iY_i\dot{x}(\theta)d\theta\, ds \\
&\leq -\frac{1}{2\delta_i}V_{6,i}.
\end{aligned}
\tag{3.29}
$$

Since

$$
\begin{aligned}
-h_{1,i}\int_{t-h_{1,i}}^{t}x^T(s)Y_iQ_iY_ix(s)ds &\leq -\int_{-h_{1,i}}^{0}\int_{t+s}^{t}x^T(\theta)Y_iQ_iY_ix(\theta)d\theta\, ds \\
&\leq -\frac{1}{h_{1,i}}V_{7,i},
\end{aligned}
\tag{3.30}
$$

we can get

$$\begin{aligned} &- h_{1,i}\int_{t-h_{1,i}}^{t} x^T(s)Y_iQ_iY_ix(s)ds - h_{1,i}\int_{t-h_{1,i}}^{t} x^T(s)Y_iQ_iY_ix(s)ds \\ &\quad\le -\frac{1}{h_{1,i}}V_{7,i} - h_{1,i}\int_{t-h_{1,i}}^{t} x^T(s)Y_iQ_iY_ix(s)ds. \end{aligned} \tag{3.31}$$

So we obtain

$$-h_{1,i}\int_{t-h_{1,i}}^{t} x^T(s)Y_iQ_iY_ix(s)ds \le -\frac{1}{2h_{1,i}}V_{7,i} - \frac{1}{2}V_{2,i}. \tag{3.32}$$

Similar to (3.32), we have

$$-h_{2,i}\int_{t-h_{2,i}}^{t} x^T(s)Y_iQ_iY_ix(s)ds \le -\frac{1}{2h_{2,i}}V_{3,i} - \frac{1}{2}V_{8,i}. \tag{3.33}$$

According to (3.23), (3.28)–(3.33), we have

$$\begin{aligned} \dot{V}_i(x_t) &\le -\min\left\{\frac{h^*\lambda^*}{\delta\lambda_3}, \frac{1}{2h_{1,i}}, \frac{1}{2h_{2,i}}, \frac{1}{2\delta_i}, \frac{1}{2}\right\}V_i(x_t) \\ &\le -\min\left\{\frac{h^*\lambda^*}{\delta\lambda_3}, \frac{1}{2h_2}, \frac{1}{2}\right\}V_i(x_t) \\ &= -\rho V_i(x_t). \end{aligned} \tag{3.34}$$

As a result, we get

$$\dot{V}_j(x_t) \le -\rho V_j(x_t), \tag{3.35}$$

which yields

$$\frac{1}{V_j(x_t)}dV_j(x_t) \le -\rho dt. \tag{3.36}$$

Integrating (3.36) on (ν, τ), we obtain

$$V_j(x_{\tau^-}) \le V_j(x_{\nu^+})e^{-\rho(\tau-\nu)}. \tag{3.37}$$

From (3.25) and (3.37), we have

$$V_i(x_{\tau^+}) \le \frac{\lambda_1}{\lambda_2}V_j(x_{\nu^+})e^{-\rho(\tau-\nu)}. \tag{3.38}$$

Since

$$\tau - \nu \geq T_0 \geq \frac{1}{\rho} \ln\left(\frac{\lambda_1}{\alpha\lambda_2}\right), \tag{3.39}$$

we obtain from (3.38) that

$$V_i(x_{\tau^+}) \leq \alpha V_j(x_{\nu^+}). \tag{3.40}$$

Let $N(t)$ be the number of time the system is switched on $[0, t)$. Since the switched system (3.1) undergoes infinite times of switches on $[0, \infty)$, we obtain $\lim_{t\to\infty} N(t) = \infty$. Assume that $I(t) = i$ and $I(0) = i_0$. Then, according to (3.7) and (3.40), we have

$$c_1 \|x(t)\|^2 \leq V_i(x_t) \leq \alpha^{N(t)} V_{i_0}(x_0). \tag{3.41}$$

Since $0 < \alpha < 1$ and $\lim_{t\to\infty} N(t) = \infty$, it follows from (3.41) that $\|x(t)\| \to 0$ as $t \to \infty$. We conclude that the zero solution of (3.1) is stabilizable. □

3.2. Robust Stabilization for Switched Systems with Interval Time-Varying Delay

Theorem 3.2. *Given $\alpha \in (0, 1)$. If there exists symmetric positive definite matrices P_i, Q_i, R_i, U_i such that the following conditions hold:*

$$\Omega_{1,i} = \Omega_i - \begin{bmatrix} 000 & -II \end{bmatrix}^T U_i \begin{bmatrix} 000 & -II \end{bmatrix} < 0, \tag{3.42}$$

$$\Omega_{2,i} = \Omega_i - \begin{bmatrix} 00I0 & -I \end{bmatrix}^T U_i \begin{bmatrix} 00I0 & -I \end{bmatrix} < 0, \tag{3.43}$$

$$\Omega_{3,i} = \begin{bmatrix} \Omega_{311,i} & P_i H_{1,i}^T & P_i H_{1,i}^T & \dfrac{B_i H_{3,i}^T}{2} & \dfrac{B_i H_{3,i}^T}{2} \\ * & -\epsilon_1 I & 0 & 0 & 0 \\ * & * & -\epsilon_2 I & 0 & 0 \\ * & * & * & \dfrac{-\epsilon_5 I}{2} & 0 \\ * & * & * & * & \dfrac{-\epsilon_6 I}{2} \end{bmatrix} < 0, \tag{3.44}$$

$$\Omega_{4,i} = \begin{bmatrix} \Omega_{411,i} & P_i H_{2,i}^T & P_i H_{2,i}^T \\ * & -\epsilon_3 I & 0 \\ * & * & -\epsilon_4 I \end{bmatrix} < 0, \tag{3.45}$$

$$\Omega_i = \begin{bmatrix} \Lambda_{11,i} & \Lambda_{12,i} & \Lambda_{13,i} & \Lambda_{14,i} & \Lambda_{15,i} \\ * & \Lambda_{22,i} & 0 & 0 & \Lambda_{25,i} \\ * & * & \Lambda_{33,i} & 0 & \Lambda_{35,i} \\ * & * & * & \Lambda_{44,i} & \Lambda_{45,i} \\ * & * & * & * & \Lambda_{55,i} \end{bmatrix}, \tag{3.46}$$

where

$$\begin{aligned}
\Lambda_{11,i} &= P_iA_i^T + A_iP_i - 0.5B_iB_i^T + \left(2 + h_{1,i}^2 + h_{2,i}^2\right)Q_i - 0.5R_i + \epsilon_1E_1^TE_1 + \epsilon_3E_2^TE_2 + \frac{\epsilon_5}{2}E_3^TE_3;\\
\Lambda_{12,i} &= P_iA_i^T - 0.5B_iB_i^T;\\
\Lambda_{13,i} &= 0.5R_i;\\
\Lambda_{14,i} &= 0.5R_i;\\
\Lambda_{15,i} &= D_iP_i;\\
\Lambda_{22,i} &= \left(h_{1,i}^2 + h_{2,i}^2 + \delta_i^2\right)Q_i - 2R_i + \epsilon_2E_1^TE_1 + \epsilon_4E_2^TE_2 + \frac{\epsilon_6}{2}E_3^TE_3;\\
\Lambda_{25,i} &= D_iP_i;\\
\Lambda_{33,i} &= -Q_i - 0.5R_i - 0.5U_i;\\
\Lambda_{35,i} &= 0.5U_i;\\
\Lambda_{44,i} &= -Q_i - 0.5R_i - 0.5U_i;\\
\Lambda_{45,i} &= 0.5U_i;\\
\Lambda_{55,i} &= -0.5U_i;\\
\Omega_{311,i} &= -0.5B_iB_i^T - 0.5R_i;\\
\Omega_{411,i} &= -0.5U_i.
\end{aligned} \tag{3.47}$$

Then for any switching rule satisfying (3.3) *the switched system* (2.1) *is robustly stabilizable under the feedback controller given by* (3.4)*.*

Proof. Construct Lyapunov-Krasovskii functional as in (3.5), we can proof this theorem by using a similar argument as in the proof of Theorem 3.1. By replacing A_i, D_i, B_i in (3.8) with $A_i + E_{1,i}F_{1,i}(t)H_{1,i}$, $D_i + E_{2,i}F_{2,i}(t)H_{2,i}$, $B_i + E_{3,i}F_{3,i}(t)H_{3,i}$, respectively, we obtain

$$\begin{aligned}
\dot{V}_i(x_t) \le{}& y^T(t)\left[P_i(A_i + E_{1,i}F_{1,i}(t)H_{1,i})^T + (A_i + E_{1,i}F_{1,i}(t)H_{1,i})P_i\right]y(t)\\
&+ 2y^T(t)(D_i + E_{2,i}F_{2,i}(t)H_{2,i})P_iy(t - h_i(t)) - y^T(t)B_i^TB_iy(t)\\
&- y^T(t)E_{3,i}F_{3,i}(t)H_{3,i}B_i^Ty(t) + y^T(t)\left(2 + h_{1,i}^2 + h_{2,i}^2\right)Q_iy(t)\\
&- y^T(t - h_1)Q_iy(t - h_1) - y^T(t - h_2)Q_iy(t - h_2)\\
&+ \dot{y}^T(t)\left[\left(h_{1,i}^2 + h_{1,i}^2\right)R_i + \delta_i^2U_i\right]\dot{y}(t) + y^T(t)\left(-\frac{R_i}{2}\right)y(t)\\
&+ y^T(t)R_iy(t - h_{1,i}) + y^T(t - h_{1,i})\left(-\frac{R_i}{2}\right)y(t - h_{1,i})
\end{aligned}$$

$$
\begin{aligned}
&+ y^T(t)\left(-\frac{R_i}{2}\right)y(t) + y^T(t)R_i y(t-h_{2,i}) + y^T(t-h_{2,i})\left(-\frac{R_i}{2}\right)y(t-h_{2,i}) \\
&+ \left[y(t-h_i(t)) - y(t-h_{2,i})\right]^T\left(-\frac{U_i}{2}\right)\left[y(t-h_i(t)) - y(t-h_{2,i})\right] \\
&+ \left[y(t-h_i(t)) - y(t-h_{2,i})\right]^T\left(-\frac{U_i}{2}\right)\left[y(t-h_i(t)) - y(t-h_{2,i})\right] \\
&- \beta\left[y(t-h_{1,i}) - y(t-h_i(t))\right]^T\left(-\frac{U_i}{2}\right)\left[y(t-h_{1,i}) - y(t-h_i(t))\right] \\
&- (1-\beta)\left[y(t-h_i(t)) - y(t-h_{2,i})\right]^T\left(-\frac{U_i}{2}\right)\left[y(t-h_i(t)) - y(t-h_{2,i})\right] \\
&- 2\dot{y}^T P_i \dot{y}(t) + 2\dot{y}^T(A_i + E_{1,i}F_{1,i}(t)H_{1,i})P_i y(t) + 2\dot{y}^T(D_i + E_{2,i}F_{2,i}(t)H_{2,i})P_i \\
&\times y(t-h_i(t)) - 2\dot{y}^T 0.5 B_i B_i^T y(t) - \dot{y}^T E_{3,i}F_{3,i}(t)H_{3,i}B_i^T y(t) \\
&- \frac{h_{1,i}}{2}\int_{t-h_{1,i}}^{t}\dot{y}^T(s)R_i\dot{y}(s)ds - \frac{h_{2,i}}{2}\int_{t-h_{2,i}}^{t}\dot{y}^T(s)R_i\dot{y}(s)ds \\
&- \frac{\delta_i}{2}\int_{t-h_{2,i}}^{t-h_{1,i}}\dot{y}^T(s)R_i\dot{y}(s)ds - h_{1,i}\int_{t-h_{1,i}}^{t}x^T(s)Y_iQ_iY_ix(s)ds \\
&- h_{2,i}\int_{t-h_{2,i}}^{t}x^T(s)Y_iQ_iY_ix(s)ds.
\end{aligned}
\tag{3.48}
$$

And using Lemma 2.5, we can get the following upper bounds for the uncertain terms in (3.48):

$$
\begin{aligned}
2y^T(t)E_{1,i}F_{1,i}(t)H_{1,i}P_i y(t) &\le \epsilon_{1,i}y^T(t)E_{1,i}^TE_{1,i}y(t) + \epsilon_{1,i}^{-1}y^T(t)P_iH_{1,i}^TH_{1,i}P_iy(t), \\
2\dot{y}^T(t)E_{1,i}F_{1,i}(t)H_{1,i}P_i y(t) &\le \epsilon_{2,i}\dot{y}^T(t)E_{1,i}^TE_{1,i}\dot{y}(t) + \epsilon_{2,i}^{-1}y^T(t)P_iH_{1,i}^TH_{1,i}P_iy(t), \\
2y^T(t)E_{2,i}F_{2,i}(t)H_{2,i}P_i y(t-h_i(t)) &\le \epsilon_{3,i}y^T(t)E_{2,i}^TE_{2,i}y(t) \\
&\quad + \epsilon_{3,i}^{-1}y^T(t-h_i(t))P_iH_{2,i}^TH_{2,i}P_iy(t-h_i(t)), \\
2\dot{y}^T(t)E_{2,i}F_{2,i}(t)H_{2,i}P_i y(t-h_i(t)) &\le \epsilon_{4,i}\dot{y}^T(t)E_{2,i}^TE_{2,i}\dot{y}(t) \\
&\quad + \epsilon_{4,i}^{-1}y^T(t-h_i(t))P_iH_{2,i}^TH_{2,i}P_iy(t-h_i(t)), \\
-y^T(t)E_{3,i}F_{3,i}(t)H_{3,i}B_i^T y(t) &\le \frac{\epsilon_{5,i}}{2}y^T(t)E_{3,i}^TE_{3,i}y(t) + \frac{\epsilon_{5,i}^{-1}}{2}y^T(t)B_iH_{3,i}^TH_{3,i}B_iy(t), \\
-\dot{y}^T(t)E_{3,i}F_{3,i}(t)H_{3,i}B_i^T y(t) &\le \frac{\epsilon_{6,i}}{2}\dot{y}^T(t)E_{3,i}^TE_{3,i}\dot{y}(t) + \frac{\epsilon_{6,i}^{-1}}{2}y^T(t)B_iH_{3,i}^TH_{3,i}B_iy(t).
\end{aligned}
\tag{3.49}
$$

According to (3.48)-(3.49), it follows that

$$\begin{aligned}
\dot{V}_i(x_t) \leq{} & \xi^T(t)\left[(1-\beta)\Omega_{1,i} + \beta\Omega_{2,i}\right]\xi(t) + y^T(t)\Psi_{3,i}y(t) + y^T(t-h_i(t))\Psi_{4,i}y(t-h_i(t)) \\
& - \frac{h_{1,i}}{2}\int_{t-h_{1,i}}^{t} \dot{y}^T(s)R_i\dot{y}(s)ds - \frac{h_{2,i}}{2}\int_{t-h_{2,i}}^{t} \dot{y}^T(s)R_i\dot{y}(s)ds \\
& - \frac{\delta_i}{2}\int_{t-h_{2,i}}^{t-h_{1,i}} \dot{y}^T(s)R_i\dot{y}(s)ds, -h_{1,i}\int_{t-h_{1,i}}^{t} x^T(s)Y_iQ_iY_ix(s)ds \\
& - h_{2,i}\int_{t-h_{2,i}}^{t} x^T(s)Y_iQ_iY_ix(s)ds,
\end{aligned} \tag{3.50}$$

where

$$\begin{aligned}
\Psi_{3,i} &= \Omega_{311,i} + \varepsilon_{1,i}^{-1}P_iH_{1,i}^TH_{1,i}P_i + \epsilon_{2,i}^{-1}P_iH_{1,i}^TH_{1,i}P_i + \frac{\epsilon_{5,i}^{-1}}{2}B_iH_{3,i}^TH_{3,i}B_i^T + \frac{\epsilon_{6,i}^{-1}}{2}B_iH_{3,i}^TH_{3,i}B_i^T, \\
\Psi_{4,i} &= \Omega_{311,i} + \epsilon_{3,i}^{-1}P_iH_{2,i}^TH_{2,i}P_i + \epsilon_{4,i}^{-1}P_iH_{2,i}^TH_{2,i}P_i.
\end{aligned} \tag{3.51}$$

Applying Lemma 2.4, the LMIs $\Psi_{3,i} < 0$ and $\Psi_{4,i} < 0$ are equivalent to $\Omega_{3,i} < 0$ and $\Omega_{4,i} < 0$, respectively. Similarly to the previous theorem we can conclude that the switched system (2.1) is robustly stabilizable. □

Remark 3.3. Our proposed method can remove the conservative restrictions on the derivatives of the time-varying delays, meanwhile traditional design methods require the condition $\dot{h}(t) \leq h_D$ (see [14, 19, 20]). So our method can deal with fast time-varying delays. Moreover, the improved method need not to introduce any free-weighting matrix variables which turn out to be less conservative than results in [14, 20, 21].

4. Numerical Examples

In this section, we provide some examples to illustrate the effectiveness of our results in Theorems 3.1 and 3.2.

Example 4.1. Consider the nominal switched systems with interval time-varying delay

$$\dot{x}(t) = A_ix(t) + D_ix(t-h_i(t)) + B_iu_i(t), \quad i = 1,2, \tag{4.1}$$

with

$$\begin{aligned}
A_1 &= \begin{bmatrix} -2 & 0 \\ 0 & 0.5 \end{bmatrix}, & D_1 &= \begin{bmatrix} -0.2 & 0.1 \\ 0.1 & -0.1 \end{bmatrix}, & B_1 &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\
A_2 &= \begin{bmatrix} -3 & 1 \\ 0 & 0.1 \end{bmatrix}, & D_2 &= \begin{bmatrix} -0.1 & 0.1 \\ 0.1 & -0.1 \end{bmatrix}, & B_2 &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\end{aligned} \tag{4.2}$$

Table 1: Maximum allowable upper bounds h_2 of the time-varying delay for different values of the lower bounds h_1 in example 1 of [8].

Result	h_1	h_2		
		[8]	[4]	Our results No restriction on $\dot{h}(t)$
$h_d = 0.4$	0	1.2644	—	1.2843
$h_d \geq 1$ or unknown	0	0.9339	—	1.2843
$h_d \geq 1$ or unknown	0.3	—	0.9514	1.3120
$h_d \geq 1$ or unknown	0.6	—	1.04	1.3913

And we choose $\alpha = 0.9$, $h_1(t) = 0.1 + |\sin t|$, $h_2(t) = 0.1 + |\cos t|$, so we get $h_1 = 0.1, h_2 = 1.1$. Then, by using the LMI control toolbox in Matlab, solutions of LMIs (3.2) are given by

$$P_1 = \begin{bmatrix} 0.8273 & -0.0052 \\ -0.0052 & 0.8167 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 0.2445 & 0.0040 \\ 0.0040 & 0.0665 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 0.1238 & 0.0030 \\ 0.0030 & 0.5581 \end{bmatrix},$$
$$U_1 = \begin{bmatrix} 0.2249 & -0.0084 \\ -0.0084 & 0.3873 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 3.4735 & 1.2444 \\ 1.2444 & 4.2337 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0.7132 & 0.1563 \\ 0.1563 & 0.1691 \end{bmatrix}, \tag{4.3}$$
$$R_2 = \begin{bmatrix} 0.4429 & 0.9777 \\ 0.9777 & 2.9654 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 0.5684 & 0.4259 \\ 0.4259 & 1.8023 \end{bmatrix},$$

with stabilizing controllers

$$u_1(t) = \begin{bmatrix} -0.0039 & -0.6122 \end{bmatrix} x(t), \qquad u_2(t) = \begin{bmatrix} 0.0473 & -0.1320 \end{bmatrix} x(t). \tag{4.4}$$

By computation, we obtain $\lambda^* = 0.0461$, $\lambda_1 = 1.2276$, $\lambda_2 = 0.1940$, $\lambda_3 = 5.1547$, $h^* = 0.5$. Thus, $\rho = 0.0054$ and from (3.3) we obtain

$$T_0 \geq 361.1694. \tag{4.5}$$

Hence, from Theorem 3.1, the switched systems (4.1) with arbitrary switching law subject to (4.5) are stabilizable under the feedback controllers which are shown in (4.4).

By choosing initial condition as $\phi(t) = \begin{bmatrix} \sin(t) \\ \sin(t) \end{bmatrix}$, $t \in [-1.1, 0]$, the trajectories of solutions of the switched system and the trajectories of solutions of subsystems 1 and 2 for this example are shown in Figures 1, 2, 3, and 4.

In Tables 1 and 2 we give comparison of maximum allowable value of h_2 of asymptotic stability of nominal switched systems obtained in [14, 22] and Theorem 3.1. As we can see that for some of h_1, the maximum allowable bounds for h_2 obtained from Theorem 3.1 are greater than that obtained in [14, 22]. More important, the differentiability of the time delay $h(t)$ is not required in our theorem.

Example 4.2. Consider the following uncertain switched systems with interval time-varying delay

$$\dot{x}(t) = (A_i + \Delta A_i)x(t) + (D_i + \Delta D_i)x(t - h_i(t)) + (B_i + \Delta B_i)u_i(t), \quad i = 1, 2, \tag{4.6}$$

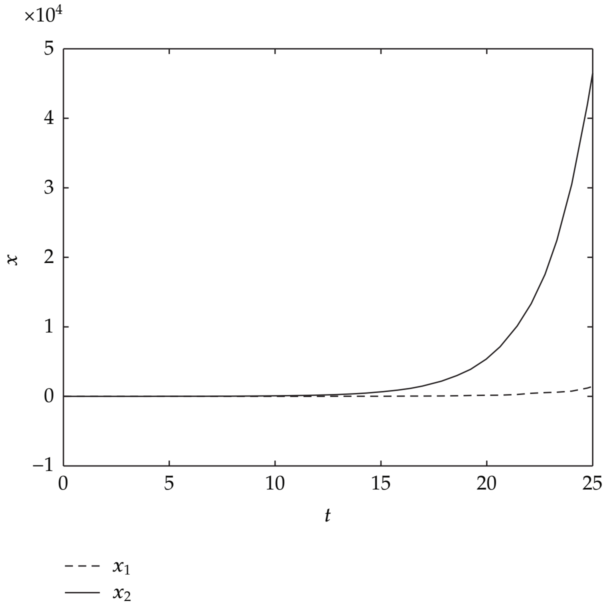

Figure 1: The trajectory of solution of system $i = 1$.

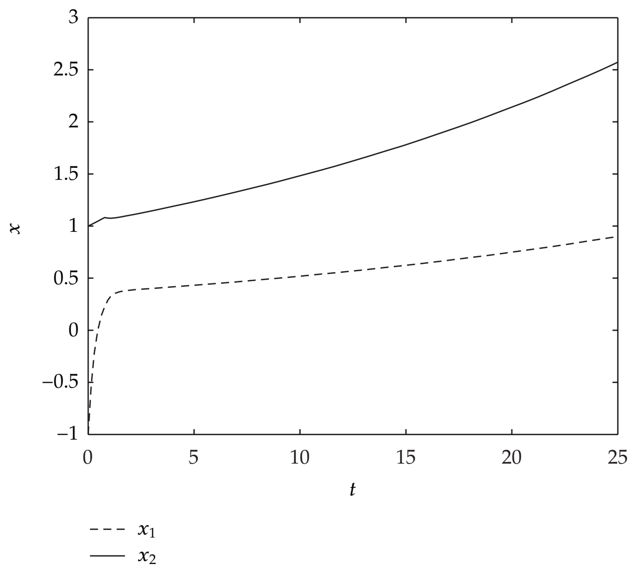

Figure 2: The trajectory of solution of system $i = 2$.

Table 2: Maximum allowable upper bounds h_2 of the time-varying delay for different values of the lower bounds h_1 in example 2 of [4].

Result	h_1	h_2	
		[4]	Our results No restriction on $\dot{h}(t)$
$h_d \geq 1$ or unknown	0	0.687	0.9871
$h_d \geq 1$ or unknown	0.4	0.856	1.0227

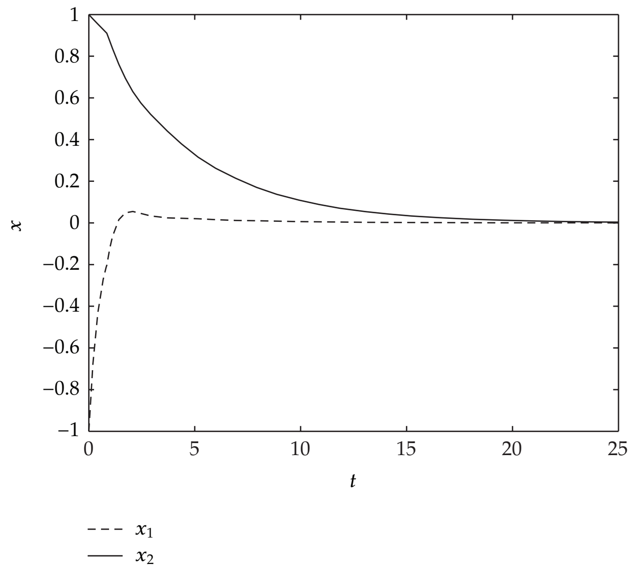

Figure 3: The trajectory of solution of system $i = 1$ under the feedback controller (4.4).

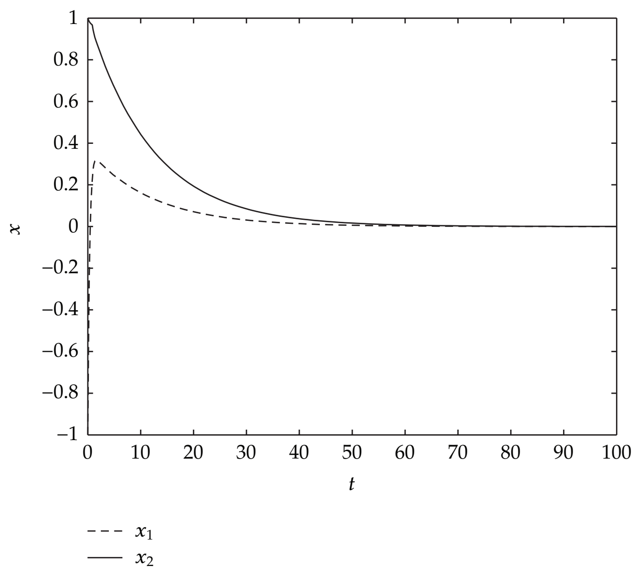

Figure 4: The trajectory of solution of system $i = 2$ under the feedback controller (4.4).

with

$$A_1 = \begin{bmatrix} -2 & 1 \\ 0 & 0.1 \end{bmatrix}, \qquad D_1 = \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & -0.1 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -3 & 0 \\ 0 & 0.1 \end{bmatrix}, \qquad D_2 = \begin{bmatrix} -0.1 & 0.1 \\ 0.1 & -0.1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \tag{4.7}$$

$$E_{1,i} = E_{2,i} = E_{3,i} = \begin{bmatrix} 0.02 & 0.01 \\ -0.01 & 0.01 \end{bmatrix}, \qquad H_{1,i} = H_{2,i} = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.01 \end{bmatrix}, \qquad H_{3,i} = \begin{bmatrix} 0.01 \\ 0 \end{bmatrix}, \quad i = 1, 2. \tag{4.8}$$

As an illustration, we choose $\alpha = 0.9$, $\epsilon_{j,i} = 1$ for $j = 1, 2, \ldots 6, i = 1, 2$,

$$h(t) = 0.1 + |\sin t|, \qquad h_2(t) = 0.1 + |\cos t|, \qquad F_{1,1}(t) = F_{1,2}(t) = F_{2,1}(t) = F_{2,2}(t) = \begin{bmatrix} \sin t & 0 \\ 0 & \sin t \end{bmatrix}. \tag{4.9}$$

In this case, we can take $h_1 = 0.1, h_M = 1.1$. Then, by using the LMI control toolbox in Matlab, solutions of LMIs (3.42) and (3.45) are given by

$$P_1 = \begin{bmatrix} 0.0438 & 0.0046 \\ 0.0046 & 0.0081 \end{bmatrix}, \qquad Q_1 = \begin{bmatrix} 0.0153 & 0.0005 \\ 0.0005 & 0.0001 \end{bmatrix}, \qquad R_1 = \begin{bmatrix} 0.0037 & 0.0005 \\ 0.0005 & 0.0010 \end{bmatrix},$$
$$U_1 = \begin{bmatrix} 0.0122 & -0.0005 \\ -0.0005 & 0.0006 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 0.0272 & -0.0010 \\ -0.0010 & 0.0088 \end{bmatrix}, \qquad Q_2 = \begin{bmatrix} 0.0059 & 0.0001 \\ 0.0001 & 0.0001 \end{bmatrix}, \tag{4.10}$$
$$R_2 = \begin{bmatrix} 0.0006 & -0.0002 \\ -0.0002 & 0.0022 \end{bmatrix}, \qquad U_2 = \begin{bmatrix} 0.0038 & -0.0010 \\ -0.0010 & 0.0012 \end{bmatrix},$$

with stabilizing controllers

$$u_1(t) = \begin{bmatrix} 0.6907 & -6.5520 \end{bmatrix} x(t), \qquad u_2(t) = \begin{bmatrix} -0.1997 & -5.6729 \end{bmatrix} x(t). \tag{4.11}$$

By computation, we obtain $\lambda^* = 0.000007$, $\lambda_1 = 132.7976$, $\lambda_2 = 22.5048$, $\lambda_3 = 0.0444$, $h^* = 0.5$. Thus, $\rho = 7.8829 \times 10^{-5}$, and from (3.3) we obtain

$$T_0 \geq 2.3855 \times 10^4. \tag{4.12}$$

Hence, from Theorem 3.2, the switched systems (4.6) with arbitrary switching law subject to (4.12) are robustly stabilizable under the feedback controllers which are shown in (4.11).

By choosing the same initial condition as in Example 4.1, the trajectories of solutions of the switched system and the trajectories of solutions of subsystems 1 and 2 for this example are shown in Figures 5, 6, 7, and 8.

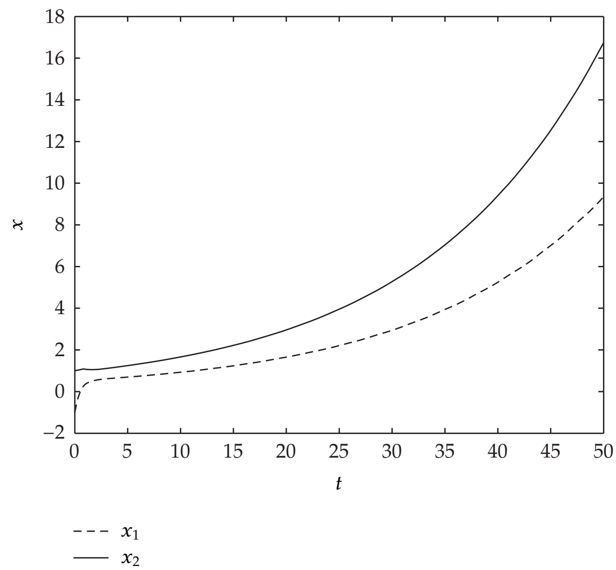

Figure 5: The trajectory of solution of system $i = 1$.

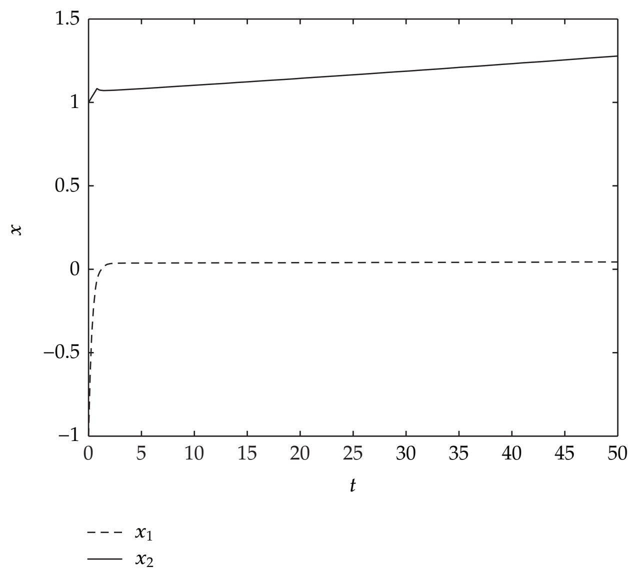

Figure 6: The trajectory of solution of system $i = 2$.

5. Conclusion

In this paper, we study the problem of robust stabilization for a class of switched systems with time-varying delay. Comparing with some existing results in the literature, the novelty of our results is twofold. Firstly, the state delay is time-varying in which the restriction on the derivative of the time-delay function is not required to design switching rule for the robust

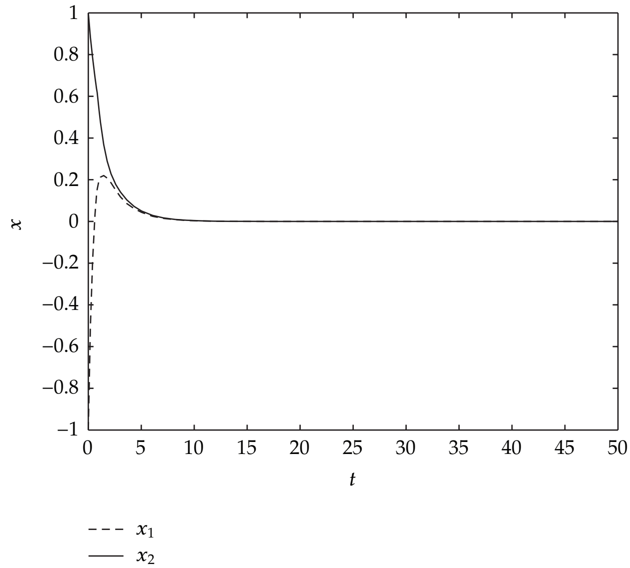

Figure 7: The trajectory of solution of system $i = 1$ under the feedback controller (4.12).

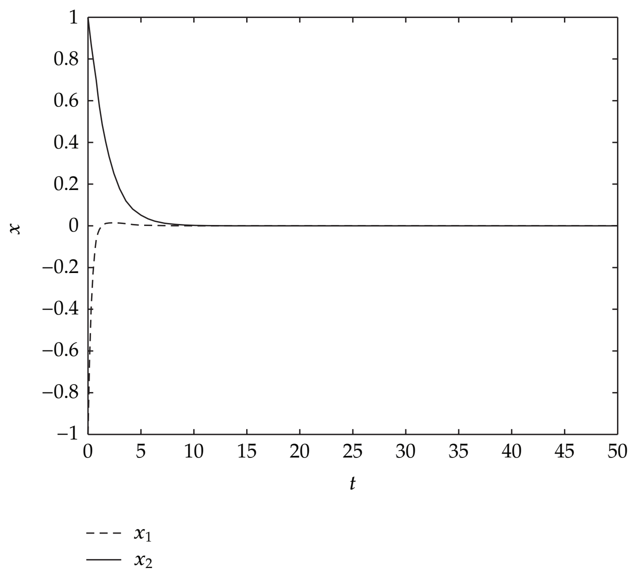

Figure 8: The trajectory of solution of system $i = 2$ under the feedback controller (4.12).

stability of the system. Secondly, the obtained conditions for the robust stability are delay-dependent and formulated in terms of the solution of standard LMIs which can be solved by various available algorithms. Numerical example is given to illustrate the effectiveness of the theoretical result.

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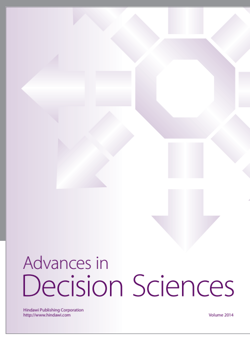
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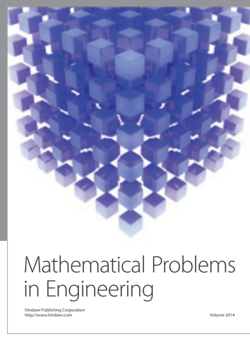
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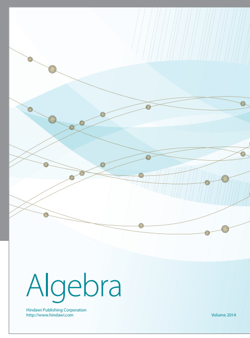
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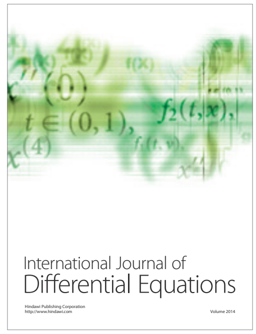
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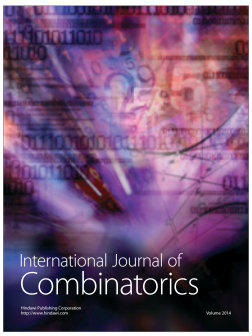
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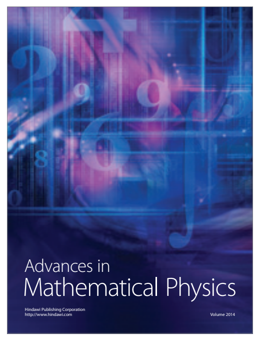
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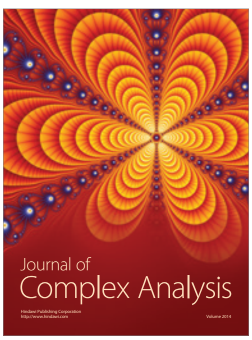
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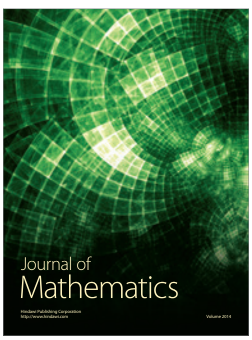
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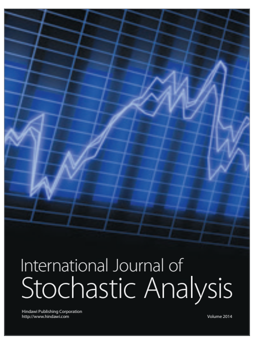
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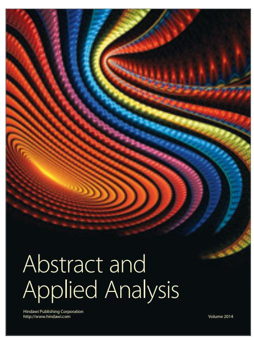
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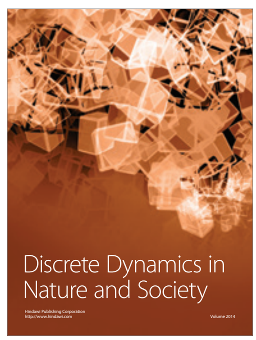
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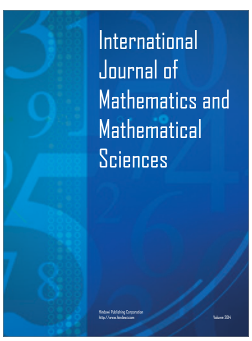
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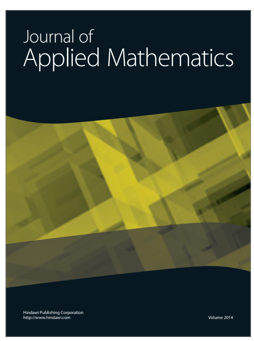
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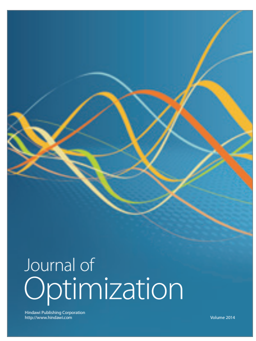
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