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Research Article

Fixed Point Theorems for ψ-Contractive Mappings in Ordered Metric Spaces

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We obtain some new fixed point theorems for ψ-contractive mappings in ordered metric spaces. Our results generalize or improve many recent fixed point theorems in the literature (e.g., Harjani et al., 2011 and 2010).

1. Introduction and Preliminaries

Throughout this paper, by $\mathbb{R}^+$, we denote the set of all real nonnegative numbers, while $\mathbb{N}$ is the set of all natural numbers. Let (X, d) be a metric space, D a subset of X; and $f : D \to X$ a map. We say f is contractive if there exists $\alpha \in [0, 1)$ such that for all $x, y \in D$,

$$d(fx, fy) \leq \alpha \cdot d(x, y). \tag{1.1}$$

The well-known Banach fixed point theorem asserts that if $D = X$, f is contractive and (X, d) is complete, then f has a unique fixed point in X. It is well known that the Banach contraction principle [1] is a very useful and classical tool in nonlinear analysis. Also, this principle has many generalizations. For instance, a mapping $f : X \to X$ is called a quasicontraction if there exists $k < 1$ such that

$$d(fx, fy) \leq k \cdot \max\{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\}, \tag{1.2}$$

for any $x, y \in X$. In 1974, Ćirić [2] introduced these maps and proved an existence and uniqueness fixed point theorem.

In 1972, Chatterjea [3] introduced the following definition.

Definition 1.1. Let (X, d) be a metric space. A mapping $f : X \to X$ is said to be a $\mathcal{C}$-contraction if there exists $\alpha \in (0, 1/2)$ such that for all $x, y \in X$, the following inequality holds:

$$d(fx, fy) \leq \alpha \cdot (d(x, fy) + d(y, fx)). \tag{1.3}$$

Choudhury [4] introduced a generalization of $\mathcal{C}$-contraction as follows.

Definition 1.2. Let (X, d) be a metric space. A mapping $f : X \to X$ is said to be a weakly $\mathcal{C}$-contraction if for all $x, y \in X$,

$$d(fx, fy) \leq \frac{1}{2}(d(x, fy) + d(y, fx) - \phi(d(x, fy), d(y, fx))), \tag{1.4}$$

where $\phi : \mathbb{R}^{+2} \to \mathbb{R}^+$ is a continuous function such that $\phi(x, y) = 0$ if and only if $x = y = 0$.

In [3, 4], the authors proved some fixed point results for the $\mathcal{C}$-contractions. In [5], Harjani et al. proved some fixed point results for weakly $\mathcal{C}$-contractive mappings in a complete metric space endowed with a partial order.

In the following, we assume that the function $\psi : \mathbb{R}^{+5} \to \mathbb{R}^+$ satisfies the following conditions:

(C1) ψ is a strictly increasing and continuous function in each coordinate, and

(C2) for all $t \in \mathbb{R}^+ \setminus \{0\}$, $\psi(t, t, t, 0, 2t) < t$, $\psi(t, t, t, 2t, 0) < t$, $\psi(0, 0, t, t, 0) < t$, and $\psi(t, 0, 0, t, t) < t$.

Example 1.3. Let $\psi : \mathbb{R}^{+5} \to \mathbb{R}^+$ denote

$$\psi(t_1, t_2, t_3, t_4, t_5) = k \cdot \max\left\{t_1, t_2, t_3, \frac{t_4}{2}, \frac{t_5}{2}\right\}, \quad \text{for } k \in (0, 1). \tag{1.5}$$

Then, ψ satisfies the above conditions (C1) and (C2).

Now, we define the following notion of a ψ-contractive mapping in metric spaces.

Definition 1.4. Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a metric space. The mapping $f : X \to X$ is said to be a ψ-contractive mapping, if

$$d(fx, fy) \leq \psi(d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)), \tag{$*$}$$

for $x \geq y$.

Using Example 1.3, it is easy to get the following examples of ψ-contractive mappings.

Example 1.5. Let $X = \mathbb{R}^+$ endowed with usual ordering and with the metric $d : X \times X \to \mathbb{R}^+$ given by

$$d(x, y) = |x - y|, \quad \text{for } x, y \in X. \tag{1.6}$$

Let $\psi : \mathbb{R}^{+5} \to \mathbb{R}^+$ denote

$$\psi(t_1, t_2, t_3, t_4, t_5) = \frac{3}{4} \cdot \max\left\{t_1, t_2, t_3, \frac{t_4}{2}, \frac{t_5}{2}\right\}, \tag{1.7}$$

where $t_1 = d(x, y)$, $t_2 = d(x, fx)$, $t_3 = d(y, fy)$, $t_4 = d(x, fy)$, and $t_5 = d(y, fx)$, for all $x, y \in X$. Let $f : X \to X$ denote

$$f(x) = \frac{1}{3}x. \tag{1.8}$$

Then, f is a ψ-contractive mapping.

Example 1.6. Let $X = \mathbb{R}^+ \times \mathbb{R}^+$ endowed with the coordinate ordering (i.e., $(x, y) \le (z, w) \Leftrightarrow x \le z$ and $y \le w$) and with the metric $d : X \times X \to \mathbb{R}^+$ given by

$$d(x, y) = |x_1 - y_1| + |x_2 - y_2|, \quad \text{for } x = (x_1, x_2),\ y = (y_1, y_2) \in X. \tag{1.9}$$

Let $\psi : \mathbb{R}^{+5} \to \mathbb{R}^+$ denote

$$\psi(t_1, t_2, t_3, t_4, t_5) = \frac{3}{4} \cdot \max\left\{t_1, t_2, t_3, \frac{t_4}{2}, \frac{t_5}{2}\right\}, \tag{1.10}$$

where $t_1 = d(x, y)$, $t_2 = d(x, fx)$, $t_3 = d(y, fy)$, $t_4 = d(x, fy)$, and $t_5 = d(y, fx)$, for all $x, y \in X$. Let $f : X \to X$ denote

$$f(x) = \frac{1}{3}x. \tag{1.11}$$

Then, f is a ψ-contractive mapping.

In this paper, we obtain some new fixed point theorems for ψ-contractive mappings in ordered metric spaces. Our results generalize or improve many recent fixed point theorems in the literature (e.g., [5, 6]).

2. Main Results

We start with the following definition.

Definition 2.1. Let $(X, \le)$ be a partially ordered set and $f : X \to X$. Then one says that f is monotone nondecreasing if, for $x, y \in X$,

$$x \le y \Longrightarrow fx \le fy. \tag{2.1}$$

We now state the main fixed point theorem for ψ-contractive mappings in ordered metric spaces when the operator is nondecreasing, as follows.

Theorem 2.2. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space, and let $f : X \to X$ be a continuous and nondecreasing ψ-contractive mapping. If there exists $x_0 \in X$ with $x_0 \leq f x_0$, then f has a fixed point in X.*

Proof. If $f(x_0) = x_0$, then the proof is finished. Suppose that $x_0 < f(x_0)$. Since f is nondecreasing mapping, by induction, we obtain that

$$x_0 < f x_0 \leq f^2 x_0 \leq f^3 x_0 \leq \cdots \leq f^n x_0 \leq f^{n+1} x_0 \leq \cdots . \tag{2.2}$$

Put $x_{n+1} = f x_n = f^{n+1} x_0$ for $n \in \mathbb{N} \cup \{0\}$. Then, for each $n \in \mathbb{N}$, from $(*)$, and, as the elements x_n and x_{n-1} are comparable, we get

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \psi\big(d(x_n, x_{n-1}), d(x_n, f x_n), d(x_{n-1}, f x_{n-1}), d(x_n, f x_{n-1}), d(x_{n-1}, f x_n)\big) \\ &\leq \psi(d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n), d(x_n, x_n), d(x_{n-1}, x_{n+1})) \\ &\leq \psi(d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n), 0, d(x_{n-1}, x_n) + d(x_n, x_{n+1})), \end{aligned} \tag{2.3}$$

and so we can deduce that, for each $n \in \mathbb{N}$,

$$d(x_{n+1}, x_n) \leq d(x_n, x_{n-1}). \tag{2.4}$$

Let we denote $c_m = d(x_{m+1}, x_m)$. Then, c_m is a nonincreasing sequence and bounded below. Thus, it must converge to some $c \geq 0$. If $c > 0$, then by the above inequalities, we have

$$c \leq c_{n+1} \leq \psi(c_n, c_n, c_n, 0, 2c_n). \tag{2.5}$$

Passing to the limit, as $n \to \infty$, we have

$$c \leq c \leq \psi(c, c, c, 0, 2c) < c, \tag{2.6}$$

which is a contradiction. So $c = 0$.

We next claim that that the following result holds.

For each $\gamma > 0$, there is $n_0(\gamma) \in \mathbb{N}$ such that for all $m > n > n_0(\gamma)$,

$$d(x_m, x_n) < \gamma. \tag{*}$$

We will prove (*) by contradiction. Suppose that (*) is false. Then, there exists some $\gamma > 0$ such that for all $k \in \mathbb{N}$, there exist m_k and n_k with $m_k > n_k > k$ such that

$$d(x_{m_k}, x_{n_k}) \geq \gamma, \qquad d(x_{m_k - 1}, x_{n_k}) < \gamma. \tag{2.7}$$

Using the triangular inequality:

$$\begin{aligned}\gamma &\leq d(x_{m_k}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{n_k}) \\ &< \gamma + d(x_{m_k}, x_{m_k-1}),\end{aligned} \tag{2.8}$$

and letting $k \to \infty$, we have

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \gamma. \tag{2.9}$$

Since f is a ψ-contractive mapping, we also have

$$\begin{aligned}\gamma &\leq d(x_{m_k}, x_{n_k}) = d\left(fx_{m_k-1}, fx_{n_k-1}\right) \\ &\leq \psi(d(x_{m_k-1}, x_{n_k-1}), d(x_{m_k-1}, x_{m_k}), d(x_{n_k-1}, x_{n_k}), d(x_{m_k-1}, x_{n_k}), d(x_{n_k-1}, x_{m_k})) \\ &\leq \psi(c_{m_k-1} + d(x_{m_k}, x_{n_k}) + c_{n_k-1}, c_{m_k-1}, c_{n_k-1}, c_{m_k-1} + d(x_{m_k}, x_{n_k}), d(x_{m_k}, x_{n_k}) + c_{n_k-1}).\end{aligned} \tag{2.10}$$

Letting $k \to \infty$. Then, we get

$$\gamma \leq \psi(\gamma, 0, 0, \gamma, \gamma) < \gamma, \tag{2.11}$$

a contradiction. It follows from (*) that the sequence $\{x_n\}$ must be a Cauchy sequence.

Similary, we also conclude that for each $n \in \mathbb{N}$,

$$\begin{aligned}d(x_n, x_{n+1}) &\leq \psi\left(d(x_{n-1}, x_n), d\left(x_{n-1}, fx_{n-1}\right), d\left(x_n, fx_n\right), d\left(x_{n-1}, fx_n\right), d\left(x_n, fx_{n-1}\right)\right) \\ &\leq \psi(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), d(x_n, x_n)) \\ &\leq \psi(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_n) + d(x_n, x_{n+1}), 0),\end{aligned} \tag{2.12}$$

and so we have that for each $n \in \mathbb{N}$,

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n). \tag{2.13}$$

Let us denote $b_m = d(x_m, x_{m+1})$. Then, b_m is a nonincreasing sequence and bounded below. Thus, it must converge to some $b \geq 0$. If $b > 0$, then by the above inequalities, we have

$$b \leq b_{n+1} \leq \psi(b_n, b_n, b_n, 2b_n, 0). \tag{2.14}$$

Passing to the limit, as $n \to \infty$, we have

$$b \leq b \leq \psi(b, b, b, 2b, 0) < b, \tag{2.15}$$

which is a contradiction. So $b = 0$. By the above argument, we also conclude that $\{x_n\}$ is a Cauchy sequence.

Since X is complete, there exists $\mu \in X$ such that $\lim_{n\to\infty} x_n = \mu$. Moreover, the continuity of f implies that

$$\mu = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f(\mu). \tag{2.16}$$

So we complete the proof. □

In what follows, we prove that Theorem 2.2 is still valid for f not necessarily continuous, assuming the following hypothesis in X (which appears in Theorem 1 of [7]).

If $\{x_n\}$ is a nondecreasing sequence in X, such that

$$x_n \longrightarrow x, \text{ then } x_n \le x \quad \forall n \in \mathbb{N}. \tag{**}$$

Theorem 2.3. *Let $(X, \le)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Assume that X satisfies (**), and let $f : X \to X$ be a nondecreasing ψ-contractive mapping. If there exists $x_0 \in X$ with $x_0 \le f(x_0)$, then f has a fixed point in X.*

Proof. Following the proof of Theorem 2.2, we only have to check that $f(\mu) = \mu$. As $\{x_n\}$ is a nondecreasing sequence in X and $x_n \to \mu$, then the condition (**) gives us that $x_n \le \mu$ for every $n \in \mathbb{N}$. Since $f : X \to X$ is a nondecreasing ψ-contractive mapping, we have

$$\begin{aligned} d(x_{n+1}, f\mu) &= d(fx_n, f\mu) \\ &\le \psi(d(x_n, \mu), d(x_n, fx_n), d(\mu, f\mu), d(x_n, f\mu), d(\mu, fx_n)) \\ &\le \psi(d(x_n, \mu), d(x_n, x_{n+1}), d(\mu, f\mu), d(x_n, f\mu), d(\mu, x_{n+1})). \end{aligned} \tag{2.17}$$

Letting $n \to \infty$ and using the continuity of ψ, we have

$$\begin{aligned} d(\mu, f\mu) &\le \psi(0, 0, d(\mu, f\mu), d(\mu, f\mu), 0) \\ &< d(\mu, f\mu), \end{aligned} \tag{2.18}$$

and this is a contraction unless $d(\mu, f\mu) = 0$, or equivalently, $\mu = f\mu$. □

In what follows, we give a sufficient condition for the uniqueness of the fixed point in Theorems 2.2 and 2.3. This condition is the following and it appears in [8]:

$$\text{for } x, y \in X, \text{ there exists a lower bound or an upper bound.} \tag{2.19}$$

In [7], it is proved that the above-mentioned condition is equivalent to the following:

$$\text{for } x, y \in X, \text{ there exists } z \in X \text{ which is comparable to } x \text{ and } y. \tag{***}$$

Theorem 2.4. *Adding condition (***) to the hypothesis of Theorem 2.2 (or Theorem 2.3) and the condition for all* $t \in \mathbb{R}^+$, $\varphi(t,0,2t,t,t) < t$ *(or,* $\varphi(t,2t,0,0,t) < t$*) to the function* φ*, one obtains the uniqueness of the fixed point of* f.

Proof. Suppose that there exist $\mu, \nu \in X$ which are fixed points of f. We distinguish two cases.

Case 1. If μ and ν are comparable and $\mu \neq \nu$, then $f^n\mu = \mu$ is comparable to $f^n\nu = \nu$ for all $n \in \mathbb{N}$, and

$$\begin{aligned}
d(\mu,\nu)& \\
&= d(f^n\mu, f^n\nu) \\
&\leq \varphi\left(d\left(f^{n-1}\mu, f^{n-1}\nu\right), d\left(f^{n-1}\mu, f^n\mu\right), d\left(f^{n-1}\nu, f^n\nu\right), d\left(f^{n-1}\mu, f^n\nu\right), d\left(f^{n-1}\nu, f^n\mu\right)\right) \\
&\leq \varphi(d(\mu,\nu), d(\mu,\mu), d(\nu,\nu), d(\mu,\nu), d(\nu,\mu)) \\
&= \varphi(d(\mu,\nu), 0, 0, d(\mu,\nu), d(\nu,\mu)) \\
&< d(\mu,\nu),
\end{aligned} \tag{2.20}$$

and this is a contradiction unless $d(\mu,\nu) = 0$, that is, $\mu = \nu$.

Case 2. If μ and ν are not comparable, then there exists $x \in X$ comparable to μ and ν. Monotonicity of f implies that $f^n x$ is comparable to $f^n\mu$ and $f^n\nu$ for all $n \in \mathbb{N}$. We also distinguish two cases.

Subcase 2.1. If there exists $n_0 \in \mathbb{N}$ with $f^{n_0}x = \mu$, then we have

$$\begin{aligned}
d(\mu,\nu)& \\
&= d(f\mu, f\nu) \\
&= d\left(f^{n_0+1}x, f^{n_0+1}\nu\right) \\
&\leq \varphi\left(d(f^{n_0}x, f^{n_0}\nu), d\left(f^{n_0}x, f^{n_0+1}x\right), d\left(f^{n_0}\nu, f^{n_0+1}\nu\right), d\left(f^{n_0}x, f^{n_0+1}\nu\right), d\left(f^{n_0}\nu, f^{n_0+1}x\right)\right) \\
&= \varphi(d(\mu,\nu), d(\mu, f\mu), d(\nu,\nu), d(\mu,\nu), d(\nu, f\mu)) \\
&= \varphi(d(\mu,\nu), 0, 0, d(\mu,\nu), d(\nu,\mu)) \\
&< d(\mu,\nu),
\end{aligned} \tag{2.21}$$

and this is a contradiction unless $d(\mu,\nu) = 0$, that is, $\mu = \nu$.

Subcase 2.2. For all $n \in \mathbb{N}$ with $f^n x \neq \mu$, since f is a nondecreasing ψ-contractive mapping, we have

$$\begin{aligned}
&d(\mu, f^n x)\\
&\quad = d(f^n \mu, f^n x)\\
&\quad \leq \psi\left(d\left(f^{n-1}\mu, f^{n-1}x\right), d\left(f^{n-1}\mu, f^n \mu\right), d\left(f^{n-1}x, f^n x\right), d\left(f^{n-1}\mu, f^n x\right), d\left(f^{n-1}x, f^n \mu\right)\right)\\
&\quad \leq \psi\left(d\left(\mu, f^{n-1}x\right), d(\mu, \mu), d\left(f^{n-1}x, f^n x\right), d(\mu, f^n x), d\left(f^{n-1}x, \mu\right)\right)\\
&\quad \leq \psi\left(d\left(\mu, f^{n-1}x\right), 0, d\left(f^{n-1}x, \mu\right) + d(\mu, f^n x), d(\mu, f^n x), d\left(f^{n-1}x, \mu\right)\right).
\end{aligned} \tag{2.22}$$

Using the above inequality, we claim that for each $n \in \mathbb{N}$,

$$d(\mu, f^n x) < d\left(\mu, f^{n-1}x\right). \tag{2.23}$$

If not, we assume that $d(\mu, f^{n-1}x) \leq d(\mu, f^n x)$, then by the definition of ψ and $\psi(t, 0, 2t, t, t) < t$, we have

$$\begin{aligned}
d(\mu, f^n x) &\leq \psi\left(d\left(\mu, f^{n-1}x\right), 0, d\left(f^{n-1}x, \mu\right) + d(\mu, f^n x), d(\mu, f^n x), d\left(f^{n-1}x, \mu\right)\right)\\
&\leq \psi(d(\mu, f^n x), 0, 2d(f^n x, \mu), d(\mu, f^n x), d(f^n x, \mu))\\
&< d(\mu, f^n x),
\end{aligned} \tag{2.24}$$

which implies a contradiction. Therefore, our claim is proved.

This proves that the nonnegative decreasing sequence $\{d(\mu, f^n x)\}$ is convergent. Put $\lim_{n\to\infty} d(\mu, f^n x) = \eta$, $\eta \geq 0$. We now claim that $\eta = 0$. If $\eta > 0$, then making $n \to \infty$, we get

$$\eta = \lim_{n\to\infty} d(\mu, f^n x) \leq \psi(\eta, 0, 2\eta, \eta, \eta) < \eta, \tag{2.25}$$

this is a contradiction. So $\eta = 0$, that is, $\lim_{n\to\infty} d(\mu, f^n x) = 0$.

Analogously, it can be proved that $\lim_{n\to\infty} d(\nu, f^n x) = 0$.

Finally, the uniqueness of the limit gives us $\mu = \nu$.

This finishes the proof. □

In the following, we present a fixed point theorem for a ψ-contractive mapping when the operator f is nonincreasing. We start with the following definition.

Definition 2.5. Let $(X, \leq)$ be a partially ordered set and $f : X \to X$. Then one says that f is monotone nonincreasing if, for $x, y \in X$,

$$x \leq y \Longrightarrow fx \geq fy. \tag{2.26}$$

Using a similar argument to that in the proof of Theorem 3.1 of [5], we get the following point results.

Theorem 2.6. *Let $(X, \leq)$ be a partially ordered set satisfying condition (***) and suppose that there exists a metric d in X such that (X, d) is a complete metric space, and let f be a nonincreasing ψ-contractive mapping. If there exists $x_0 \in X$ with $x_0 \leq fx_0$ or $x_0 \geq fx_0$, then* $\inf\{d(x, fx) : x \in X\} = 0$*. Moreover, if in addition, X is compact and f is continuous, then f has a unique fixed point in X.*

Proof. If $fx_0 = x_0$, then it is obvious that $\inf\{d(x, fx) : x \in X\} = 0$. Suppose that $x_0 < fx_0$ (the same argument serves for $x_0 < fx_0$). Since f is nonincreasing the consecutive terms of the sequence $\{f^n x_0\}$ are comparable, we have

$$\begin{aligned}
&d\left(f^{n+1}x_0, f^n x_0\right) \\
&\quad \leq \psi\left(d\left(f^n x_0, f^{n-1}x_0\right), d\left(f^n x_0, f^{n+1}x_0\right), d\left(f^{n-1}x_0, f^n x_0\right), d\left(f^{n-1}x_0, f^{n+1}x_0\right), d(f^n x_0, f^n x_0)\right) \\
&\quad \leq \psi\left(d\left(f^n x_0, f^{n-1}x_0\right), d\left(f^n x_0, f^{n+1}x_0\right), d\left(f^{n-1}x_0, f^n x_0\right), d\left(f^{n-1}x_0, f^{n+1}x_0\right), 0\right) \\
&\quad \leq \psi\left(d\left(f^n x_0, f^{n-1}x_0\right), d\left(f^n x_0, f^{n+1}x_0\right), d\left(f^{n-1}x_0, f^n x_0\right), d\left(f^{n-1}x_0, f^n x_0\right) + d\left(f^n x_0, f^{n+1}x_0\right)\right),
\end{aligned} \tag{2.27}$$

and so we conclude that for each $n \in \mathbb{N}$,

$$d\left(f^{n+1}x_0, f^n x_0\right) < d\left(f^n x_0, f^{n-1}x_0\right). \tag{2.28}$$

Thus, $\{d(f^{n+1}x_0, f^n x_0)\}$ is a decreasing sequence and bounded below, and it must converge to $\eta \geq 0$. We claim that $\eta = 0$. If $\eta > 0$, then by the above inequalities and the continuity of ψ, letting $n \to \infty$, we have

$$\begin{aligned}
\eta &= \lim_{n\to\infty} d\left(f^{n+1}x_0, f^n x_0\right) \\
&\leq \psi(\eta, \eta, \eta, 2\eta, 0) \\
&< \eta,
\end{aligned} \tag{2.29}$$

which is a contradiction. So $\eta = 0$, that is, $\lim_{n\to\infty} d(f^{n+1}x_0, f^n x_0) = 0$. Consequently, $\inf\{d(x, fx) : x \in X\} = 0$.

Further, since f is continuous and X is compact, we can find $\mu \in X$ such that

$$d(\mu, f\mu) = \inf\{d(x, fx) : x \in X\} = 0, \tag{2.30}$$

and, therefore, μ is a fixed point of f.

The uniqueness of the fixed point is proved as in Theorem 2.4. □

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