Research Article

Conservation Laws for Some Systems of Nonlinear Partial Differential Equations via Multiplier Approach

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The conservation laws for the integrable coupled KDV type system, complexly coupled kdv system, coupled system arising from complex-valued KDV in magnetized plasma, Ito integrable system, and Navier stokes equations of gas dynamics are computed by multipliers approach. First of all, we calculate the multipliers depending on dependent variables, independent variables, and derivatives of dependent variables up to some fixed order. The conservation laws fluxes are computed corresponding to each conserved vector. For all understudying systems, the local conservation laws are established by utilizing the multiplier approach.

1. Introduction

The partial differential equations, which arise in the sciences, dynamics, fluid mechanics, electromagnetism, economics and so forth, express conservation of mass, momentum, energy, electric charge, or value of firm. All the conservation laws of partial differential equations may not have physical interpretation but are essential in studying the integrability of the PDE. The high number of conservation laws for a partial differential equation grantees that the partial differential equation is strongly integrable and can be linearized or explicitly solved [1]. Moreover, the conservation laws are used for analysis, particularly, development of numerical schemes, soliton solutions, study of properties such as bi-Hamiltonian structures and recursion operators, and reduction of partial differential equations.

There are different methods for the construction of conservation laws as described by Naz [2], Naz et al. [3], Bluman et al. [4], Hereman et al. [5], and references therein. Rocha Filho and Figueiredo [6] developed computer packages based on Noether's method for the variational problems. Wolf [7] and Wolf et al. [8] introduced computer programmes in REDUCE to calculate conservation laws.

In this work, the multiplier approach is used to derive the conservation laws for some systems of partial differential equations important due to physical point of view. Stuedel [9] introduced the multiplier approach and the conserved vectors were written in a characteristic form as $D_i T^i = \Lambda^\alpha E_\alpha$. The determining equations for the multipliers (characteristics) were obtained by taking the variational derivative of $D_i T^i = Q^\alpha E_\alpha$ for the arbitrary functions not only for solutions of system of partial differential equations [10]. A conserved vector is associated with each multiplier. The conservation laws for the integrable coupled kdv-type system, complexly coupled KDA system, coupled system arising from complex-valued KDV in magnetized plasma, Ito integrable system, and Navier stokes equations of gas dynamics are computed by utilizing the multiplier approach. The conserved vectors derived here can be used in constructing the solutions of underlying systems in the following different ways. The corresponding potential system can be written for the conservation laws, and symmetry reductions [11] can be carried out. Another approach to deduce exact solutions is via the double reduction theory [12–14]. The exact solution can be derived if the conservation laws give physical conserved quantities like Naz et al. [15]. The exact solutions of systems under consideration are subject of future work.

The outline of paper is as follows. In Section 2, some definitions related with multiplier approach are presented. The conservation laws for integrable coupled kdv-type system, complexly coupled kdv system, coupled system arising from complex-valued KDV in magnetized plasma, Ito integrable system, and Navier stokes equations of gas dynamics are constructed in Section 3. Conclusions are summarized in Section 4.

2. Preliminaries

Consider a kth-order system of partial differential equations (PDEs) with n independent variables $x \equiv (x^1, x^2, \ldots, x^n)$ and m dependent variables $u \equiv (u^1, u^2, \ldots, u^m)$ defined as,

$$E_\alpha\big(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(k)}\big) = 0, \quad \alpha = 1, 2, \ldots, m, \tag{2.1}$$

where $u_{(i)}$ is the collection of ith-order partial derivatives of u.

(1) The Euler operator is defined by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} - D_i \frac{\partial}{\partial u_i^\alpha} + D_i D_j \frac{\partial}{\partial u_{ij}^\alpha} - \cdots, \tag{2.2}$$

where

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \quad i = 1, 2, \tag{2.3}$$

is the total derivative operator with respect to x^i.

(2) A conserved vector of (2.1) is an n-tuple $T = (T^1, T^2, \ldots, T^n)$, $T^i \in A$, $i = 1, 2 \ldots n$, such that

$$D_i T^i = 0 \tag{2.4}$$

holds for all solutions of (2.1). Equation (2.4) is called a local conservation law.

(3) The multipliers Λ^α of system (2.1) has the property

$$D_i T^i = \Lambda^\alpha E_\alpha, \tag{2.5}$$

for the arbitrary function u^α [9, 10].

(4) The determining equations for the multipliers are obtained by taking variational derivative of (2.5) (see [10]):

$$\frac{\delta}{\delta u^\alpha}[\Lambda^\alpha E_\alpha] = 0. \tag{2.6}$$

Equation (2.6) holds for the arbitrary functions u^α not only for the solutions of system (2.1).

Equation (2.6) yields multipliers for all local conservation laws. Then conserved vectors can be derived systematically using (2.5) as the determining equation. But in some problems it is not difficult to construct the conserved vectors by elementary manipulations once the multiplier has been determined.

3. Conservation Laws for Nonlinear Systems of Partial Differential Equations

3.1. Integrable Coupled System

Consider the integrable coupled system [16, 17]

$$
\begin{aligned}
E_1 &= u_t + \left[u_{xx} - (k+3)(k+6)u^2 - k^2 v^2\right]_x + 2k[(k+6)vu_x + (k+3)uv_x = 0, \\
E_2 &= v_t + \left[v_{xx} - k(k-3)v^2 - (k+3)^2 u^2\right]_x + 2(k+3)[kvu_x + (k-3)uv_x = 0.
\end{aligned} \tag{3.1}
$$

The group invariant solution of (3.1) was derived in [17]. Here we will construct conservation laws for coupled system (3.1). Consider simple multipliers of the form $\Lambda_1(t, x, u, v)$ and $\Lambda_2(t, x, u, v)$. Multipliers Λ_1 and Λ_2 for the system (3.1) have the property that

$$\Lambda_1 E_1 + \Lambda_2 E_2 = D_t T^1 + D_x T^2, \tag{3.2}$$

for all functions $u(t, x)$ and $v(t, x)$ where the total derivative operators D_t and D_x from (2.3) are

$$
\begin{aligned}
D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + v_t \frac{\partial}{\partial v} + u_{tt} \frac{\partial}{\partial u_t} + v_{tt} \frac{\partial}{\partial v_t} + u_{tx} \frac{\partial}{\partial u_x} + v_{tx} \frac{\partial}{\partial v_x} + \cdots, \\
D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + v_x \frac{\partial}{\partial v} + u_{xx} \frac{\partial}{\partial u_x} + v_{xx} \frac{\partial}{\partial v_x} + u_{xt} \frac{\partial}{\partial u_t} + v_{xt} \frac{\partial}{\partial v_t} + \cdots.
\end{aligned} \tag{3.3}
$$

The right-hand side of (3.2) is a divergence expression and T^1 and T^2 are the components of the conserved vector $T = (T^1, T^2)$. The determining equations for the multipliers

Λ_1 and Λ_2 are

$$\frac{\delta}{\delta u}[\Lambda_1 E_1 + \Lambda_2 E_2] = 0, \tag{3.4}$$

$$\frac{\delta}{\delta v}[\Lambda_1 E_1 + \Lambda_2 E_2] = 0, \tag{3.5}$$

where $\delta/\delta u$ and $\delta/\delta v$ are the standard Euler operators defined in (2.2), which annihilate divergence expressions:

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t \frac{\partial}{\partial u_t} - D_x \frac{\partial}{\partial u_x} + D_t^2 \frac{\partial}{\partial u_{tt}} + D_t D_x \frac{\partial}{\partial u_{tx}} + D_x^2 \frac{\partial}{\partial u_{xx}} - \cdots, \tag{3.6}$$

$$\frac{\delta}{\delta v} = \frac{\partial}{\partial v} - D_t \frac{\partial}{\partial v_t} - D_x \frac{\partial}{\partial v_x} + D_t^2 \frac{\partial}{\partial v_{tt}} + D_t D_x \frac{\partial}{\partial v_{tx}} + D_x^2 \frac{\partial}{\partial v_{xx}} - \cdots. \tag{3.7}$$

Separating (3.4) and (3.5), after expansion, with respect to different combinations of derivatives of u and v, yields the following overdetermined system:

$$\Lambda_{2xx} = 0, \qquad \Lambda_{2vx} = 0, \qquad \Lambda_{2vv} = 0,$$

$$\Lambda_{1t} = -\frac{12}{k}(k+3)[(u-v)k+3u]\Lambda_{2x}, \qquad \Lambda_{1x} = -\frac{(k+3)}{k}\Lambda_{2x},$$

$$\Lambda_{2t} = 12[(k+3)u - kv]\Lambda_{2x},$$

$$\Lambda_{1u} = \frac{(k+3)\left[(2k^2+6k)\Lambda_2 + 2\Lambda_1 k^2 + (k+3)((u-v)k+3u)\right]\Lambda_{2v}}{k^2[(u-v)k+3u]}, \tag{3.8}$$

$$\Lambda_{2u} = \frac{(-3k-k^2)\Lambda_2 - \Lambda_1 k^2 - (k+3)((u-v)k+3u)\Lambda_{2v}}{[(u-v)k+3u]k},$$

$$\Lambda_{1v} = \frac{(-3k-k^2)\Lambda_2 - \Lambda_1 k^2 - (k+3)((u-v)k+3u)\Lambda_{2v}}{[(u-v)k+3u]k}.$$

The solution of system (3.8) yields following four multipliers:

$$\Lambda_1^{(1)} = 1, \qquad \Lambda_2^{(1)} = -\frac{k+3}{k},$$

$$\Lambda_1^{(2)} = -\frac{1}{k^2}(k+3)^2 u, \qquad \Lambda_2^{(2)} = v,$$

$$\Lambda_1^{(3)} = \frac{1}{k}[-kv + 2(k+3)u], \qquad \Lambda_2^{(3)} = u, \tag{3.9}$$

$$\Lambda_1^{(4)} = -\frac{k+3}{k}[x + 12(k+3)ut - 12kvt],$$

$$\Lambda_2^{(4)} = x + 12(k+3)ut - 12ktv.$$

From (3.2) and (3.9), we obtained following four conserved vectors:

$$T_1^{(1)} = -\frac{1}{k}[(k+3)u - kv],$$

$$T_2^{(1)} = \frac{1}{k}\left[6(k+3)^2u^2 - 12k(k+3)uv + kv_{xx} - ku_{xx} - 3u_{xx}\right],$$

$$T_1^{(2)} = -\frac{1}{2k^2}\left[u^2(k+3)^2 - k^2v^2\right],$$

$$T_2^{(2)} = \frac{1}{6k^2}\left[3(k+3)^2u_x^2 - 3k^2v_x^2 + 4(k+6)(k+3)^3u^3 - 4k^3(k-3)v^3\right.$$

$$\left. -12k(k+3)^3u^2v + 12k^3(k+3)uv^2 - 6(k+3)^2uu_{xx} + 6k^2vv_{xx}\right],$$

$$T_1^{(3)} = -\frac{1}{k}\left[(k+3)u^2 - kuv\right],$$

$$T_2^{(3)} = \frac{1}{3k}\left[3(k+3)u_x^2 + 2(k+3)^2(k+12)u^3\right.$$

$$- 2k^3v^3 - 6k(k+3)(k+9)u^2v + 6k^2(k+6)uv^2$$

$$\left. -6(k+3)uu_{xx} + 3ku_{xx}v + 3kuv_{xx} - 3ku_xv_x\right],$$

$$T_1^{(4)} = -\frac{1}{k}\left[6(k+3)^2tu^2 + 6k^2tv^2\right.$$

$$\left. -12k(k+3)tuv + (k+3)u_x - kv_x\right],$$

$$T_2^{(4)} = \frac{1}{k}\left[-(k+3)(x+12(k+3)tu - 12ktv)u_{xx}\right.$$

$$+ (x + 12(k+3)ut - 12ktv)kv_{xx} + 6(k+3)^2tu_x^2 + 6tk^2v_x^2$$

$$- 12k(k+3)tu_xv_x + (k+3)u_x - kv_x + 48(k+3)^3tu^3$$

$$- 48k^3tv^3 - 144k(k+3)^2tu^2v + 144k^2(k+3)tuv^2$$

$$\left. +6(k+3)^2xu^2 + 6k^2xv^2 - 12k(k+3)xuv\right].$$

$$(3.10)$$

3.2. Higher-Order Conservation Laws for Complexly Coupled KDV System

The conservation laws of complexly coupled KDV were discussed in Naz [18]

$$u_t - 6uu_x - 6vv_x - u_{xxx} = 0,$$

$$v_t - 6uv_x - 6vu_x - v_{xxx} = 0,$$

$$(3.11)$$

and six conserved vectors were derived by multipliers approach with multipliers of form $\Lambda_1(t, x, u, v)$ and $\Lambda_2(t, x, u, v)$. Now we will consider higher-order multipliers and derive the associated conservation laws fluxes. The determining equations for multipliers of the form $\Lambda_1(t, x, u, v, u_x, v_x, u_{xx}, v_{xx})$ and $\Lambda_2(t, x, u, v, u_x, v_x, u_{xx}, v_{xx})$ from (2.6) are

$$\frac{\delta}{\delta u}[\Lambda_1(u_t - 6uu_x - 6vv_x - u_{xxx}) + \Lambda_2(v_t - 6uv_x - 6vu_x - v_{xxx})] = 0, \tag{3.12}$$

$$\frac{\delta}{\delta v}[\Lambda_1(u_t - 6uu_x - 6vv_x - u_{xxx}) + \Lambda_2(v_t - 6uv_x - 6vu_x - v_{xxx})] = 0, \tag{3.13}$$

where the standard Euler operators $\delta/\delta u$ and $\delta/\delta v$ are given by (3.6) and (3.7). Equations (3.12) and (3.13) are separated, after expansion, according to different combinations of derivatives of u and v and after some simplification the following system of equations for Λ_1, Λ_2 is obtained:

$$\Lambda_{1xx} = 0, \qquad \Lambda_{1vx} = 0, \qquad \Lambda_{1xv_{xx}} = 0, \qquad \Lambda_{1vv_{xx}} = 0, \qquad \Lambda_{1u_x} = 0, \qquad \Lambda_{1v_x} = 0,$$

$$\Lambda_{2xx} = 0, \qquad \Lambda_{2vx} = 0, \qquad \Lambda_{2xv_{xx}} = 0, \qquad \Lambda_{2vv_{xx}} = 0, \qquad \Lambda_{2u_x} = 0, \qquad \Lambda_{2v_x} = 0,$$

$$\Lambda_{1v_{xx}v_{xx}} = 0, \qquad \Lambda_{2v_{xx}v_{xx}} = 0, \qquad \Lambda_{2vv} = 6\Lambda_{1v_{xx}}, \qquad \Lambda_{1vv} = 6\Lambda_{2v_{xx}},$$

$$\Lambda_{1t} = 6\Lambda_{2x}v + 6\Lambda_{1x}u, \qquad \Lambda_{2t} = 6\Lambda_{1x}v + 6\Lambda_{2x}u,$$

$$\Lambda_{1u} = \Lambda_{2v}, \qquad \Lambda_{2u} = \Lambda_{1v}, \qquad \Lambda_{1u_{xx}} = \Lambda_{2v_{xx}}, \qquad \Lambda_{2u_{xx}} = \Lambda_{1v_{xx}}. \tag{3.14}$$

The solution of system (3.14) yields

$$\Lambda_1 = c_1 + u(c_3 + c_4 t) + v(c_5 + c_6 t) + \frac{x}{6}c_4 + \left(3u^2 + 3v^2 + u_{xx}\right)c_7 + (6uv + v_{xx})c_8,$$

$$\Lambda_2 = c_2 + v(c_3 + c_4 t) + u(c_5 + c_6 t) + \frac{x}{6}c_6 + (6uv + v_{xx})c_7 + \left(3u^2 + 3v^2 + u_{xx}\right)c_8, \tag{3.15}$$

where $c_1, c_2, \ldots, c_8$ are arbitrary constants. The first six multipliers are same as derived in [18]. The two new multipliers are actually the higher-order multipliers associated with constants c_7 and c_8

$$\Lambda_1^{(7)} = 3u^2 + 3v^2 + u_{xx}, \qquad \Lambda_2^{(7)} = 6uv + v_{xx},$$

$$\Lambda_1^{(8)} = 6uv + v_{xx}, \qquad \Lambda_2^{(8)} = 3u^2 + 3v^2 + u_{xx}. \tag{3.16}$$

Equation (2.5) with multipliers given in (3.16) gives two new conservation laws

$$T_1^{(7)} = 3uv^2 + u^3 + \frac{1}{2}uu_{xx} + \frac{1}{2}vv_{xx},$$

$$T_2^{(7)} = -6uvv_{xx} - 27u^2v^2 - \frac{1}{2}vv_{tx} - \frac{9}{2}u^4 - \frac{1}{2}u_{xx}^2$$

$$- 3u_{xx}v^2 - 3u^2u_{xx} - \frac{9}{2}v^4 - \frac{1}{2}v_{xx}^2 - \frac{1}{2}uu_{tx} + \frac{1}{2}v_xv_t + \frac{1}{2}u_xu_t,$$

$$T_1^{(8)} = 3u^2v + v^3 + \frac{1}{2}uv_{xx} + \frac{1}{2}u_{xx}v,$$

$$T_2^{(8)} = -\frac{1}{2}uv_{tx} - 18u^3v - u_{xx}v_{xx} - 18uv^3 - 3v^2v_{xx}$$

$$- 3u^2v_{xx} - \frac{1}{2}vu_{tx} - 6uvu_{xx} + \frac{1}{2}u_xv_t + \frac{1}{2}v_xu_t.$$

$$(3.17)$$

The two new higher-order conservation laws (3.17) are obtained for the system (3.11).

3.3. Conservation Laws for Complex-Valued KDV in Magnetized Plasma

The complexly coupled KDV

$$w_t - \left(|w|^2w\right)_x - w_{xxx} = 0 \tag{3.18}$$

arises in the study of the asymptotic investigation of electrostatic waves of a magnetized plasma [19]. The variable w is the complex field amplitude $w = u + iv$. The representation of (3.18) in real field variables u and v is

$$u_t - 3u^2u_x - 2uvv_x - v^2u_x - u_{xxx} = 0,$$
$$v_t - 3v^2v_x - 2uvu_x - u^2v_x - v_{xxx} = 0. \tag{3.19}$$

The conservation laws for system (3.19) are derived here by using multiplier approach. The determining equations for multipliers of the form $\Lambda_1(t, x, u, v, u_x, v_x, u_{xx}, v_{xx})$ and $\Lambda_2(t, x, u, v, u_x, v_x, u_{xx}, v_{xx})$ from (2.6) are

$$\frac{\delta}{\delta u}\left[\Lambda_1\left(u_t - 3u^2u_x - 2uvv_x - v^2u_x - u_{xxx}\right) + \Lambda_2\left(v_t - 3v^2v_x - 2uvu_x - u^2v_x - v_{xxx}\right)\right] = 0,$$

$$\frac{\delta}{\delta v}\left[\Lambda_1\left(u_t - 3u^2u_x - 2uvv_x - v^2u_x - u_{xxx}\right) + \Lambda_2\left(v_t - 3v^2v_x - 2uvu_x - u^2v_x - v_{xxx}\right)\right] = 0. \tag{3.20}$$

Equation (3.20) finally results in the following overdetermined system:

$$\Lambda_{2xx} = 0, \qquad \Lambda_{1u_x} = 0, \qquad \Lambda_{2u_x} = 0, \qquad \Lambda_{1v_x} = 0, \qquad \Lambda_{2v_x} = 0,$$

$$\Lambda_{1u_{xx}} = \Lambda_{2v_{xx}}, \qquad \Lambda_{2u_{xx}} = 0, \qquad \Lambda_{1v_{xx}} = 0,$$

$$\Lambda_{1u_{xx}} = \Lambda_{2v_{xx}}, \qquad \Lambda_{2u_{xx}} = 0, \qquad \Lambda_{1v_{xx}} = 0,$$

$$\Lambda_{2vx} = \frac{\Lambda_{2x}}{v}, \qquad \Lambda_{2xv_{xx}} = 0, \qquad \Lambda_{2vv} = 6v\Lambda_{2v_{xx}},$$

$$\Lambda_{2vv_{xx}} = 0, \qquad \Lambda_{2v_{xx}v_{xx}} = 0,$$

$$\Lambda_{1t} = \frac{3}{v}\left(u_{xx} + u^3 + v^2 u\right)\Lambda_{2x}, \tag{3.21}$$

$$\Lambda_{2t} = \frac{3}{v}\left(\Lambda_{2x}v_{xx} + \Lambda_{2x}u^2 v + \Lambda_{2x}v^3\right),$$

$$\Lambda_{1x} = \frac{\Lambda_{2x}u}{v},$$

$$\Lambda_{1u} = 2\Lambda_{2v_{xx}}u^2 - 2\Lambda_{2v_{xx}}v^2 + \Lambda_{2v},$$
$$\Lambda_{2u} = 2\Lambda_{2v_{xx}}uv,$$

$$\Lambda_{1v} = 2\Lambda_{2v_{xx}}uv.$$

The solution of system (3.21) yields following five multipliers:

$$\Lambda_1^{(1)} = 0, \qquad \Lambda_2^{(1)} = 1,$$

$$\Lambda_1^{(2)} = 1, \qquad \Lambda_2^{(2)} = 0,$$

$$\Lambda_1^{(3)} = u, \qquad \Lambda_2^{(3)} = v,$$

$$\Lambda_1^{(4)} = u^3 + uv^2 + u_{xx}, \tag{3.22}$$

$$\Lambda_2^{(4)} = u^2 v + v^3 + v_{xx},$$

$$\Lambda_1^{(5)} = 3t\left(u^3 + uv^2 + u_{xx}\right) + xu,$$

$$\Lambda_2^{(5)} = 3t\left(v^3 + u^2 v + v_{xx}\right) + xv.$$

The corresponding conserved vectors conserved vectors are

$$T_1^{(1)} = v, \qquad T_2^{(1)} = -u^2 v - v^3 - v_{xx},$$

$$T_1^{(2)} = u, \qquad T_2^{(2)} = -u^3 - uv^2 - u_{xx},$$

$$T_1^{(3)} = \frac{u^2}{2} + \frac{v^2}{2},$$

$$T_2^{(3)} = \frac{u_x^2}{2} + \frac{v_x^2}{2} - uu_{xx} - vv_{xx} - \frac{3}{4}u^4 - \frac{3}{4}v^4 - \frac{3}{2}u^2 v^2,$$

$$T_1^{(4)} = \frac{1}{4}\left[u^4 + v^4 + 2u^2 v^2 + 2uu_{xx} + 2vv_{xx}\right],$$

$$T_2^{(4)} = -\frac{1}{2}\left[u^6 + v^6 + u_{xx}^2 + v_{xx}^2 + 2uv^2 u_{xx} + 2u^2 vv_{xx} - v_x v_t\right.$$

$$\left. -u_x u_t + 2u^3 u_{xx} + 3u^4 v^2 + 3u^2 v^4 + 2v^3 v_{xx} + vv_{tx} + uu_{tx}\right],$$

(3.23)

$$T_1^{(5)} = \frac{3t}{4}\left[u^4 + v^4 + 2u^2 v^2 + 2uu_{xx} + 2vv_{xx}\right] + \frac{x}{2}u^2 + \frac{x}{2}v^2,$$

$$T_2^{(5)} = -\frac{3t}{2}\left[u^6 + v^6 + u_{xx}^2 + v_{xx}^2 + 2uv^2 u_{xx} + 2u^2 vv_{xx} - v_x v_t\right.$$

$$\left. -u_x u_t + 2u^3 u_{xx} + 3u^4 v^2 + 3u^2 v^4 + 2v^3 v_{xx} + vv_{tx} + uu_{tx}\right]$$

$$-\frac{1}{2}uu_x - \frac{1}{2}vv_x + x\left[\frac{1}{2}u_x^2 + \frac{1}{2}v_x^2 - \frac{3}{4}u^4 - \frac{3}{4}v^4 - uu_{xx} - vv_{xx} - \frac{3}{2}u^2 v^2\right].$$

3.4. Conservation Laws for Ito Integrable System

Consider the following integrable Ito coupled system [20]:

$$u_t = v_x,$$

$$v_t = -2(v_{xxx} + 3uv_x + 3vu_x) - 12ww_x,$$

(3.24)

$$w_t = w_{xxx} + 3uw_x,$$

where $u(t,x), v(t,x)$ and $w(t,x)$. For simplicity, consider multipliers of the form $\Lambda_1 = \Lambda_1(t,x,u,v,w), \Lambda_2 = \Lambda_2(t,x,u,v,w)$, and $\Lambda_3 = \Lambda_3(t,x,u,v,w)$.

The determining equations for multipliers Λ_1, Λ_2, and Λ_3 from (2.6) are

$$\frac{\delta}{\delta u}\left[\Lambda_1(u_t - v_x) + \Lambda_2(v_t + 2v_{xxx} + 6uv_x + 6vu_x + 12ww_x) + \Lambda_3(w_t - w_{xxx} - 3uw_x)\right] = 0,$$

(3.25)

$$\frac{\delta}{\delta v}\left[\Lambda_1(u_t - v_x) + \Lambda_2(v_t + 2v_{xxx} + 6uv_x + 6vu_x + 12ww_x) + \Lambda_3(w_t - w_{xxx} - 3uw_x)\right] = 0,$$

(3.26)

$$\frac{\delta}{\delta w}\left[\Lambda_1(u_t - v_x) + \Lambda_2(v_t + 2v_{xxx} + 6uv_x + 6vu_x + 12ww_x) + \Lambda_3(w_t - w_{xxx} - 3uw_x)\right] = 0,$$

(3.27)

where the standard Euler operators $\delta/\delta u$, $\delta/\delta v$, and $\delta/\delta w$ can be computed from (2.2). Separating (3.25)–(3.27), after expansion, according to different combinations of derivatives of u, v, and w and after some simplification following system of equations for $\Lambda_1, \Lambda_2, \Lambda_3$ is obtained:

$$\Lambda_{1xx} = 0, \qquad \Lambda_{1t} = 0, \qquad \Lambda_{1u} = 0, \qquad \Lambda_{1v} = 0, \qquad \Lambda_{1w} = 0,$$

$$\Lambda_{2x} = 0, \qquad \Lambda_{2u} = 0, \qquad \Lambda_{2v} = 0, \qquad \Lambda_{2w} = 0, \qquad (3.28)$$

$$\Lambda_{2t} = \Lambda_{1x}, \qquad \Lambda_3(t, x, u, v, w) = 0.$$

The system of determining equation (3.28) yields

$$\Lambda_1 = c_1 + c_2 x, \qquad \Lambda_2 = c_3 + c_2 t, \qquad \Lambda_3 = 0, \qquad (3.29)$$

where c_1, c_2, c_3 are arbitrary constants and we have three multipliers

$$\Lambda_1^{(1)} = 1, \qquad \Lambda_2^{(1)} = 0, \qquad \Lambda_3^{(1)} = 0,$$

$$\Lambda_1^{(2)} = x, \qquad \Lambda_2^{(2)} = t, \qquad \Lambda_3^{(2)} = 0, \qquad (3.30)$$

$$\Lambda_1^{(3)} = 0, \qquad \Lambda_2^{(3)} = 1, \qquad \Lambda_3^{(3)} = 0.$$

From (2.5), the conservation laws associated with multipliers given in (3.30) are

$$T_1^{(1)} = u, \qquad T_2^{(1)} = -v,$$

$$T_1^{(2)} = ux + vt, \qquad T_2^{(2)} = 6uvt + 6w^2 t - vx + 2tv_{xx}, \qquad (3.31)$$

$$T_1^{(3)} = v, \qquad T_2^{(3)} = 6uv + 6w^2 + 2v_{xx}.$$

The multiplier approach gave three nontrivial conservation laws for Ito system (3.24).

3.5. Conservation Laws Navier-Stokes Equations for the Compressible Flow of a Heat-Conducting, Viscous Fluid for the Ideal Gas Case

The one-dimensional compressible flow of a heat-conducting, viscous fluid for the ideal gas case is represented by the Navier-stokes equations [21]

$$E_1 = \rho_t + \rho_x u + \rho u_x = 0,$$

$$E_2 = u_t + u u_x + R\frac{\theta \rho_x}{\rho} + R\theta_x - \mu\frac{u_{xx}}{\rho} = 0 \qquad (3.32)$$

$$E_3 = \theta_t + u\theta_x + \frac{R}{c_v}\theta u_x - \frac{\mu}{c_v}\frac{u_x^2}{\rho} - \frac{k}{c_v}\frac{\theta_{xx}}{\rho} = 0,$$

where $u(t,x)$, $\rho(t,x)$, and $\theta(t,x)$ are the density, velocity, and temperature, $\mu > 0$ is the viscosity coefficient, $k > 0$ is the heat conductivity coefficient, $R > 0$ is called the ideal gas constant, $c_v > 0$ represents the specific heat at constant volume. Consider multipliers of the form $\Lambda_1 = \Lambda_1(t,x,u,\rho,\theta)$, $\Lambda_2 = \Lambda_2(t,x,u,\rho,\theta)$, and $\Lambda_3 = \Lambda_3(t,x,u,\rho,\theta)$. The determining equations for multipliers Λ_1, Λ_2, and Λ_3 from (2.6) are

$$\frac{\delta}{\delta u}[\Lambda_1 E_1 + \Lambda_2 E_2 + \Lambda_3 E_3] = 0,$$

$$\frac{\delta}{\delta \rho}[\Lambda_1 E_1 + \Lambda_2 E_2 + \Lambda_3 E_3] = 0, \tag{3.33}$$

$$\frac{\delta}{\delta \theta}[\Lambda_1 E_1 + \Lambda_2 E_2 + \Lambda_3 E_3] = 0,$$

where the standard Euler operators $\delta/\delta u$, $\delta/\delta \rho$, and $\delta/\delta \theta$ can be calculated from (2.2). Expanding (3.33) and then separating according to different combinations of derivatives of u, ρ, and θ and after some simplification following system of equations for $\Lambda_1, \Lambda_2, \Lambda_3$ is obtained:

$$\Lambda_{1xx} = 0, \qquad \Lambda_{1t} = -u\Lambda_{1x}, \qquad \Lambda_{2t} = -\Lambda_{1x}\rho,$$

$$\Lambda_{3t} = 0, \qquad \Lambda_{2x} = 0, \qquad \Lambda_{3x} = 0, \qquad \Lambda_{1u} = \frac{\Lambda_2}{\rho},$$

$$\Lambda_{2u} = \frac{\Lambda_3}{c}, \qquad \Lambda_{3u} = 0, \qquad \Lambda_{1\rho} = 0, \qquad \Lambda_{2\rho} = \frac{\Lambda_2}{\rho}, \tag{3.34}$$

$$\Lambda_{3\rho} = \frac{\Lambda_3}{\rho}, \qquad \Lambda_{1\theta} = \frac{\Lambda 3}{\rho}, \qquad \Lambda_{2\theta} = 0, \qquad \Lambda_{3\theta} = 0,$$

$$R + c_v \neq 0, \qquad -R + 2c_v \neq 0.$$

The adiabatic constant $\gamma = 1 + R/c_v$ has physical applications in region $(1,(5/3)]$ (see e.g., [21, 22]). If $R + c_v = 0$ or $-R + 2c_v = 0$ then the value of γ will be outside the region of physical significance. The assumptions $R + c_v \neq 0$ and $-R + 2c_v \neq 0$ are necessary for having a true physical model. The solution of system (3.34) results in the following four multipliers:

$$\Lambda_1^{(1)} = 1, \qquad \Lambda_2^{(1)} = 0, \qquad \Lambda_3^{(1)} = 0,$$

$$\Lambda_1^{(2)} = u, \qquad \Lambda_2^{(2)} = \rho, \qquad \Lambda_3^{(2)} = 0,$$

$$\Lambda_1^{(3)} = x - tu, \qquad \Lambda_2^{(3)} = -t\rho, \qquad \Lambda_3^{(3)} = 0, \tag{3.35}$$

$$\Lambda_1^{(4)} = \theta + \frac{u^2}{2c_v}, \qquad \Lambda_2^{(4)} = \frac{u\rho}{c_v}, \qquad \Lambda_3^{(4)} = \rho.$$

The expressions for the fluxes are

$$T_1^{(1)} = \rho, \qquad T_2^{(1)} = u\rho,$$

$$T_1^{(2)} = u\rho, \qquad T_2^{(2)} = u^2\rho - \mu u_x + R\rho\theta,$$

$$T_1^{(3)} = \rho(x - tu), \qquad T_2^{(3)} = xu\rho - tu^2\rho + \mu t u_x - Rt\rho\theta,$$

$$T_1^{(4)} = \rho\theta + \frac{u^2\rho}{2c_v}, \qquad T_2^{(4)} = \frac{1}{2c_v}\Big[2c_v u\rho\theta + u^3\rho + 2Ru\rho\theta - 2\mu u u_x - 2k\theta_x\Big].$$

(3.36)

We will get same results if we consider multipliers of the form $\Lambda_i = \Lambda_i(t, x, u, \rho, \theta, u_x, \rho_x, \theta_x)$, $i = 1, 2, 3$ or $\Lambda_i = \Lambda_i(t, x, u, \rho, \theta, u_t, \rho_t, \theta_t)$, $i = 1, 2, 3$.

4. Conclusions

The conservation laws for the integrable coupled kdv type system, complexly coupled kdv system, coupled system arising from complex-valued KDV in magnetized plasma, Ito integrable system, and Navier stokes equations of gas dynamics were computed by multipliers approach. The multipliers having dependence on dependent variables, independent variables, and derivatives of dependent variables up to some fixed order were constructed. After computing multipliers, the conservation laws fluxes were derived.

First of all, we considered integrable coupled kdv-type system and multiplier approach yielded four local conserved vectors. For the complexly coupled KDV system, total eight multipliers were obtained. The two new conserved vectors corresponding to second-order multipliers were obtained and were not found in [18]. The multiplier approach on coupled system arising from complex-valued KDV in magnetized plasma gave five conserved vectors. For Ito integrable system three and for Navier stokes equations of gas dynamics four, nontrivial conserved vectors were derived.

The conserved vectors derived here can be used in constructing the solutions of underlying PDE systems and will be considered in the future work.

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