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Research Article

Geometric Analysis of Reachability and Observability for Impulsive Systems on Complex Field

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Nowadays, quantum systems have become one of the focuses of the ongoing research and they are typical complex systems, whose state variables are defined on the complex field. In this paper, the issue of reachability and observability is addressed for a class of linear impulsive systems on complex field, for simplicity, complex linear impulsive systems. This kind of time-driven impulsive systems allows free impulsive instants, which leads to the limitation of using traditional definitions of reachability and observability directly. New notations about the span reachable set and unobservable set are proposed. Sufficient and necessary conditions for span reachability and observability of such systems are established. Moreover, the explicit characterization of span reachable set and unobservable set is presented by geometric analysis. It is pointed out that the geometric conditions are equivalent to the algebraic ones in known results for special cases. Numerical examples are also presented to show the effectiveness of the proposed methods.

1. Introduction

Recent years have witnessed growing interest in investigating the control theory of hybrid systems and most progress has been made in the stability and stabilization of hybrid systems, see [1–5] and the references therein. Impulsive dynamical systems are an important class of hybrid systems which exhibit continuous evolutions described by ordinary differential equations and abrupt changes at some instants or impulses. Examples of these systems include evolution processes, optimal control models in economics, stimulated neural networks, frequency-modulated systems, and some motions of missiles or aircrafts. In view of both theoretical and practical significance, much attention has been paid on the analysis and synthesis of impulsive systems, or impulsive control systems, see [6–11] and the references therein.

Closely related to the pole assignment, structural decomposition, quadratic optimal control and observer design, the controllability, reachability, and observability play a significant role in the control theory and engineering [12–14]. The controllability and observability of various hybrid systems have been extensively investigated using different approaches such as geometric analysis [7–9], algebraic characterization [10, 11], functional analysis [15, 16], and differential geometric method [3]. Particularly, research efforts have been made on the controllability and observability for impulsive systems. By proposing algebraic rank conditions, the state controllability and observability of linear time-varying impulsive systems were investigated in [10, 11]. For impulsive functional differential systems, the controllability is considered with the help of fixed-point theorems [15, 16]. References [7–9] presented the geometric analysis of reachability, controllability and observability for (switched) impulsive systems. Geometric analysis is effective in providing easily verifiable conditions for the controllability and observability based on the explicit characterization of controllable and observable sets in terms of invariant sets of systems. Hence, it provides an effective and simple method to investigate the fundamental properties of hybrid systems.

However, in the above-mentioned works, the state space of the considered systems is always n-dimensional real vector space, that is, $\mathbb{R}^n$, except few reports on the issue of controllability for complex systems [10, 17]. Nowadays, control of complex systems, especially quantum systems, has attracted considerable attention [18–22]. It should be noticed that quantum system models are typical complex dynamical systems whose states evolve in Banach (Hilbert) space on the field of complex number, which are much more complicated than real systems. In view of this, complex dynamics systems have many potential applications ranging from science to engineering. Therefore, it is important and necessary to study the control theory of a special class of complex dynamical systems, complex linear impulsive systems. This motivates us to consider the reachability and observability of complex linear impulsive systems by geometric analysis. The impulsive system considered in the current paper has uncertainty in the impulsive instants which can be regarded as time-driven impulsive systems. This kind of more general systems exists in many practical applications [7]. Due to the novel properties of reachable set and unobservable set for this kind of systems, traditional geometric analysis may be limited to characterize them. Hence, new concepts on the reachable set and unobservable set are introduced. Based on these definitions, we generalize the geometric analysis approach for reachability and observability to complex linear impulsive systems. Specifically, sufficient and necessary criteria for reachability and observability are derived, and explicit characterization of reachable set and unobservable set is proposed consequently. Moreover, it is proved that the span reachable set and unobservable set with free impulsive times are invariant subspaces of the complex impulsive system.

The rest of this paper is organized as follows. In Section 2, the complex linear impulsive systems to be dealt with are formulated and the solution expression for such systems is presented. In Sections 3 and 4, based on the geometric characterization of reachable set and unobservable set for complex linear impulsive systems, sufficient and necessary conditions for state reachability and state observability of complex linear impulsive systems are derived, respectively. Moreover, examples are discussed to illustrate the effectiveness of the proposed methods. Finally, some conclusions are drawn in Section 5.

2. Preliminaries

Consider the complex linear time-varying impulsive system described by

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \quad t \neq t_k, \\ \Delta x(t_k) &= E_k x(t_k) + F_k u_k, \\ y(t) &= C(t)x(t) + D(t)u(t), \\ x(t_0^+) &= x_0, \end{aligned} \tag{2.1}$$

where $k = 1, 2, \ldots, A(t), B(t), C(t)$, and $D(t)$ are known $n\times n$, $n\times m$, $p\times n$, and $p\times m$ continuous-time complex-valued matrices, $x \in \mathbb{C}^n$ is the state vector, $u \in \mathbb{C}^m$ is the control input, E_k and F_k are complex $n \times n$ and $n \times m$ constant matrices, respectively, $y \in \mathbb{C}^p$ is the output, $J = [t_0, +\infty)$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, where $x(t_k^+) = \lim_{h\to 0^+} x(t_k + h)$, $x(t_k) = x(t_k^-) = \lim_{h\to 0^+} x(t_k - h)$ with discontinuity points $t_0 < t_1 < t_2 < \cdots < t_k < \cdots, \lim_{k\to\infty} t_k = \infty$, which implies that the solution of system (2.1) is left-continuous at t_k. It should be noticed that the impulsive instants t_k can be chosen freely in this paper. We know that $x(t) : \mathbb{R} \to \mathbb{C}^n$, and $\mathbb{C}^n$ is a Banach space on the complex field $\mathbb{C}$. $A(t) : \mathbb{R} \to \mathfrak{U}$ where $\mathfrak{U} = \mathfrak{L}(\mathbb{C}^n, \mathbb{C}^n)$ is the bounded $\mathbb{C}^n$-linear continuous map. Hence complex impulsive system (2.1) is a special differential-difference equation in Banach space defined on the complex number field $\mathbb{C}$. Let $A^* = \overline{A}^\top$ be the conjugated transpose of the complex matrix A. $\prod_{i=k-1}^{1} A_i$ stands for the matrix product $A_{k-1}A_{k-2}\cdots A_1$.

Corresponding to system (2.1), consider the following complex differential equation:

$$\dot{x}(t) = A(t)x(t). \tag{2.2}$$

Suppose that $X(t)$ is the fundamental solution matrix of system (2.2). Then $X(t,s) := X(t)X^{-1}(s)$, $(t, s \in J)$ is the transition matrix associated with the matrix $A(t)$. It is clear that $X(t,t) = I$, $X(t,\tau)\, X(\tau,s) = X(t,s)$ and $X(t,s) = X^{-1}(s,t)$. Now we present the solution expression of complex impulsive system (2.1) which was proved in [10] using ordinary differential equations theory in the complex field.

Lemma 2.1 (see [10]). *For $t \in (t_{k-1}, t_k]$, $k = 1, 2, \ldots$, the solution of system* (2.1) *is given by*

$$\begin{aligned} x(t) = X(t, t_{k-1}) \Bigg[&\prod_{j=k-1}^{1} (I + E_j) X(t_j, t_{j-1}) x_0 + \sum_{i=1}^{k-1} \prod_{j=k-1}^{i} (I + E_j) X(t_j, t_{j-1}) \\ &\times \int_{t_{i-1}}^{t_i} X(t_{i-1}, s) B(s) u(s)\, ds + \sum_{i=2}^{k-1} \prod_{j=k-1}^{i} (I + E_j) X(t_j, t_{j-1}) F_{i-1} u_{i-1} + F_{k-1} u_{k-1} \Bigg] \\ &+ \int_{t_{k-1}}^{t} X(t,s) B(s) u(s)\, ds. \end{aligned} \tag{2.3}$$

In the remainder of this paper, we focus our attention on the reachability of time-invariant version of system (2.1) with respect to the continuous-time input $u(t)$ and observability with respect to the continuous-time output $y(t)$. The complex linear impulsive system is given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad t \neq t_k, \\ x(t_k^+) &= Ex(t_k), \\ y(t) &= Cx(t), \\ x(t_0^+) &= x_0, \end{aligned} \tag{2.4}$$

where A, B, C, E are known $n \times n$, $n \times m$, $p \times n$, $n \times n$ constant complex matrices. Let $\mathcal{T} = \{t_1, t_2, \ldots.\}$ be a countable set of impulse times.

Remark 2.2. System (2.4) is a class of more general linear impulsive systems in the complex field with time-driven impulsive behavior. The system parameter matrices are all complex matrices. It should be noticed that the impulse times could be chosen freely, allowing for a richer interaction between the continuous-time dynamics and the impulsive effects. Hence, with inherent uncertainties, system (2.4) has interesting features in reachability and observability different from that of common impulsive systems. This motivates our current work.

Given an initial time t_0 and final time t_f, Lemma 2.1 gives the solution of (2.4) as follows:

$$\begin{aligned} x(t_f) = e^{Ah_M} \Bigg\{ \prod_{m=M-1}^{1} E e^{Ah_m} x(t_0) + \sum_{m=1}^{M-2} \left[\prod_{j=M-1}^{m+1} \left(E e^{Ah_j} \right) E \int_{t_{m-1}}^{t_m} e^{A(t_m - s)} Bu(s) ds \right] \\ + E \int_{t_{M-2}}^{t_{M-1}} e^{A(t_{M-1}-s)} Bu(s) ds \Bigg\} + \int_{t_{M-1}}^{t_M} e^{A(t_M - s)} Bu(s) ds, \end{aligned} \tag{2.5}$$

where $t_f = t_M$ and $h_i = t_i - t_{i-1}$, $i = 1, 2, \ldots, M$. In the subsequent, we proceed to investigate the reachability and observability criteria of complex linear impulsive system (2.4).

3. Geometric Analysis of Reachability

In this section, the main purpose is to characterize the geometric properties of reachability of complex linear impulsive system (2.4) and establish the equivalence between algebraic criteria in known results [10] and the geometric ones obtained here. To discuss the geometric property of reachable set for complex impulsive system (2.4), we first introduce the concept of invariant subspace of complex linear systems.

Consider the following complex linear system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \\ x(t_0^+) &= x_0. \end{aligned} \tag{3.1}$$

For the complex constant matrix $B \in \mathbb{C}^{n \times m}$, denote $\operatorname{Im} B$ as the range of B spanned by the columns of B, that is, $\mathcal{B} \triangleq \operatorname{Im} B = \{y \mid y = Bu, \forall u \in \mathbb{C}^m\}$. For a given matrix $A \in \mathbb{C}^{n \times n}$ and a linear space $\mathcal{W} \subseteq \mathbb{C}^n$, let $\langle A \mid \mathcal{W}\rangle$ be the minimal A-invariant subspace containing $\mathcal{W}$, that is, $\langle A \mid \mathcal{W}\rangle = \sum_{i=0}^{n-1} A^i \mathcal{W}$. For simplicity, we denote $\langle A \mid B\rangle = \langle A \mid \operatorname{Im} B\rangle$. By [13], for any complex matrices A and B, we have $\{x : x = \int_{t_0}^{t} e^{A(t-\tau)} B u(\tau) d\tau,\ \text{for some p.c.v. function } u, t > t_0\} = \langle A \mid B\rangle$, which is the reachable subspace of complex linear system (3.1). Moreover, the equivalence between the algebraic condition for controllability and the geometric one is given as follows. rank $(B\ AB \cdots A^{n-1}B) = n$ is equivalent to $\operatorname{Im}(B) + A\operatorname{Im}(B) + \cdots + A^{n-1}\operatorname{Im}(B) = \mathbb{C}^n$.

In view of the special structure of the system considered here, definitions about the reachability are introduced first. The state space for complex impulsive system (2.4) is denoted by $\mathcal{X}$.

Definition 3.1 (Reachable set with fixed final time and fixed impulse times). For complex linear impulsive system (2.4), a nonzero state x_f is said to be reachable from zero with fixed final time and fixed impulse times, if given $t_0, t_f > t_0$ and a set of impulse times $\mathcal{T}$, there exists a piecewise continuous input $u(t), t \in [t_0, t_f]$, such that the system is driven from $x(t_0) = 0$ to $x(t_f) = x_f$. The set of reachable states with fixed final time t_f and fixed impulse times $\mathcal{T}$ is denoted by $\mathcal{R}_{\text{fixed}}(t_0, t_f, \mathcal{T})$.

Definition 3.2 (Reachable set with free final time and free impulse times). For system (2.4), a nonzero state x_f is said to be reachable from zero with free final time and free impulse times, if given t_0, there exists $t_f > t_0$, a set of impulse times $\mathcal{T}$ and a piecewise continuous input $u(t), t \in [t_0, t_f]$, such that the system is driven from $x(t_0) = 0$ to $x(t_f) = x_f$. The set of reachable states with free final time t_f and free impulse times $\mathcal{T}$ is denoted by $\mathcal{R}_{\text{free}}$.

From the definitions, we obtain $\mathcal{R}_{\text{free}} = \bigcup_{\mathcal{T}} \bigcup_{t_f > t_0} \mathcal{R}_{\text{fixed}}(t_0, t_f, \mathcal{T})$. Given an impulse times set $\mathcal{T}$, by (2.5) and Definition 3.1, $\mathcal{R}_{\text{fixed}}(\mathcal{T})$ is given by

$$\mathcal{R}_{\text{fixed}} = e^{Ah_M}\left\{\prod_{m=M-1}^{1} Ee^{Ah_m}x(t_0) + \sum_{m=1}^{M-2}\left[\prod_{j=M-1}^{m+1}\left(Ee^{Ah_j}\right)E\langle A \mid B\rangle\right] + E\langle A \mid B\rangle\right\} + \langle A \mid B\rangle. \tag{3.2}$$

In [7], it was pointed out that for real impulsive systems, the reachable set does not necessarily constitute a subspace. Thus, for complex linear impulsive system (2.4), $\mathcal{R}_{\text{free}}$ may be a subset instead of subspace of the state space. This fact will be clarified in the following example.

Example 3.3. Consider complex linear impulsive system (2.4) with

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{pmatrix} 0 \\ 1+2i \\ 0 \\ 0 \end{pmatrix}, \qquad E = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1+i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{3.3}$$

It is clear that $\langle A \mid B\rangle = \operatorname{Im}(B)$. For the case $t_0 < t_1 < t_f$,

$$\mathcal{R}_{\text{fixed}}(t_0, t_f, \mathcal{T}) = \begin{bmatrix} (1+i)(1+2i)h_1 & 0 \\ 0 & 1+2i \\ (1+i)(1+2i) & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.4}$$

For any even number $k \geq 2$ and $t_0 < t_1 < \cdots < t_k < t_f$, it yields that

$$\mathcal{R}_{\text{fixed}}(t_0, t_f, \mathcal{T}) = \begin{bmatrix} (1+i)^{k-1}(1+2i) & 0 & 0 \\ 0 & (1+i)^{k-2}(1+2i) & 1+2i \\ (1+i)^k(1+2i)h_k & 0 & 0 \\ 0 & (1+i)^{k-2}(1+2i)h_{k-1} & 0 \end{bmatrix}. \tag{3.5}$$

Therefore, when the final time and the impulse times are fixed, the system can reach at most a three-dimensional complex subspace of the state space. It follows that when only two impulse times are required, that is, $0 < t_1 < t_2 < t_f$, $\mathcal{R}_{\text{free}}$ can be characterized as follows:

$$\mathcal{R}_{\text{free}} = \bigcup_{h_1,h_2>0} \mathcal{R}_{\text{fixed}}(t_0, t_f, \mathcal{T}) = \bigcup_{h_1,h_2>0} \operatorname{Im} \begin{bmatrix} (1+i)(1+2i) & 0 & 0 \\ 0 & (1+2i) & 1+2i \\ 2i(1+2i)h_2 & 0 & 0 \\ 0 & (1+2i)h_1 & 0 \end{bmatrix}. \tag{3.6}$$

For a vector given by $x = [\,a\ b\ c\ d\,]^T (a, b, c, d \neq 0)$, it can be represented by the linear combination of elements in $\mathcal{R}_{\text{free}}$ while it is not included in $\mathcal{R}_{\text{free}}$. It should be noticed that the subspace spanned by the reachable set with free impulse times is the entire complex state space.

Example 3.3 motivates us to present a new concept, span reachability, for complex impulsive system (2.4).

Definition 3.4 (Span Reachability). For complex impulsive system (2.4), the subspace spanned by the elements of $\mathcal{R}_{\text{free}}$ is denoted by $\mathcal{R}_{\text{span}}$. A complex impulsive system for which $\mathcal{R}_{\text{span}} = \mathbb{C}^n$ is said to be span reachable.

In the following, the explicit construction of span reachable set is proposed and its property is discussed. Denote the following subspaces sequences:

$$\begin{gathered} \mathcal{W}_0 = \langle A \mid B \rangle, \qquad \mathcal{W}_m = \langle A \mid E\mathcal{W}_{m-1} \rangle, \quad m \geq 1, \\ \mathcal{V}_m = \sum_{i=0}^{m} \mathcal{W}_i, \quad m = 0, 1, 2, \ldots. \end{gathered} \tag{3.7}$$

It is clear that $\mathcal{V}_0 \subseteq \mathcal{V}_1 \subseteq \cdots \subseteq \mathcal{V}_{m-1} \subseteq \mathcal{V}_m$, $\dim \mathcal{V}_m < \infty$. If there exists an integer $m > 0$ such that $\mathcal{V}_m = \mathcal{V}_{m+1}$, by the construction of $\mathcal{V}_m$, it is easy to verify $\mathcal{V}_m = \mathcal{V}_{m+1} = \mathcal{V}_{m+2} = \ldots$. This implies that the sequence $\{\mathcal{V}_m, m = 1, 2, \ldots\}$ converges to $\mathcal{V}_n$. For the proof of the main results, a Lemma is presented first. The proof is similar to that of Lemma 2 in [12]. Thus, we omit it here.

Lemma 3.5. *Given a complex matrix $A \in \mathbb{C}^{n\times n}$, for almost $T \in \mathbb{R}$, one has $\langle A \mid \mathcal{W} \rangle = \langle e^{AT} \mid \mathcal{W} \rangle$.*

Theorem 3.6. *For complex linear impulsive system* (2.4)*, one has*

$$\mathcal{R}_{\text{span}} = \mathcal{V}_n. \tag{3.8}$$

Proof. For any $x_f \in \mathcal{R}_{\text{free}}$, by (3.2) and the property of invariant subspace, we have $x_f \in \mathcal{V}_n$. Then $\mathcal{R}_{\text{span}} \subseteq \mathcal{V}_n$. Next we prove the reverse inclusion. From Lemma 3.5, there exists $h > 0$ such that sequence (3.7) can be redefined as follows:

$$\mathcal{W}_0 = \langle A \mid B \rangle, \quad \mathcal{W}_m = \langle e^{Ah} \mid E\mathcal{W}_{m-1} \rangle, \quad m \geq 1. \tag{3.9}$$

Using the property of invariant subspace, (3.9) can be rewritten as

$$\mathcal{W}_0 = \langle A \mid B \rangle, \quad \mathcal{W}_m = e^{Ah}\langle e^{Ah} \mid E\mathcal{W}_{m-1} \rangle, \quad m \geq 1, \tag{3.10}$$

which implies that $\mathcal{V}_n$ has the following form:

$$\mathcal{V}_n = \langle A \mid B \rangle + \sum_{m=1}^{n} \sum_{l_m,\dots,l_2,l_1 \in \{1,2,\dots,n\}} \left[e^{Ah}\right]^{l_m} E \cdots \left[e^{Ah}\right]^{l_1} E \langle A \mid B \rangle. \tag{3.11}$$

Denote an impulse times set to be $\{l_1 h, l_2 h, \dots, l_n h\}$. It is easy to get that $\langle A \mid B \rangle + \sum_{m=1}^{n} [e^{Ah}]^{l_m} E \cdots [e^{Ah}]^{l_1} E \langle A \mid B \rangle \subseteq \mathcal{R}_{\text{fixed}}$. Hence, we obtain $\mathcal{V}_n \subseteq \mathcal{R}_{\text{free}}$. Since any element of $\mathcal{R}_{\text{span}}$ can be expressed as a linear combination of elements from $\mathcal{R}_{\text{free}}$, we conclude that $\mathcal{V}_n \subseteq \mathcal{R}_{\text{span}}$. This completes the proof. □

Remark 3.7. From Definition 3.4 and Theorem 3.6, it can be found that if $\mathcal{V}_n = \mathbb{C}^n$, system (2.4) is span reachable. For fixed final time and impulse times, if $\langle A \mid B \rangle = \mathbb{C}^n$, which implies that $\mathcal{R}_{\text{fixed}}$ constitutes the entire space, then we know that rank $(B, AB, \dots, A^{n-1}B) = n$. From Theorem 3 in [10], the above condition indicates that system (2.4) is controllable. Hence, when reduced to linear systems, the algebraic condition (3.11) in [10] and the geometric criterion $\langle A \mid B \rangle = \mathbb{C}^n$ are equivalent in checking the reachability and controllability of system (2.4). When reduced to complex linear impulsive systems with fixed impulse times and $AE = EA$, simple computation follows that the conditions for reachability and controllability are equivalent. While in this paper, we consider a more general system with time-driven impulses, and a new concept, span reachability is introduced. Hence, the derived conditions in this paper and the known literature [10] cannot be compared directly.

The concept of invariant subspace is fundamental to a geometric analysis of linear time-invariant systems. The invariance facilitates the investigation of system control problems such as disturbance decoupling, output stabilization, output regulation, and structure stability. Hence, we develop the invariant subspace characterization of the span reachable set $\mathcal{R}_{\text{span}}$ for complex linear time-driven impulsive systems. A follow-up question is that whether the span reachable set $\mathcal{R}_{\text{span}}$ is an invariant subspace of system (2.4). The invariant subspace of complex impulsive systems (2.4) is defined as follows.

Definition 3.8. For complex impulsive system (2.4) with $u(t) \equiv 0$, $\mathcal{V}$ is an invariant subspace if for any initial time t_0 and any set of impulse times $\mathcal{T}$, $x(0) \in \mathcal{V}$ implies $x(t) \in \mathcal{V}$, $\forall t \geq t_0$.

Generalizing Lemma 4.2 in [7] to the complex case, we conclude that for complex linear impulsive systems, $\mathcal{V}$ is an invariant subspace if and only if $A\mathcal{V} \subset \mathcal{V}$, $E\mathcal{V} \subset \mathcal{V}$. Now, for system (2.4), we relate $\mathcal{R}_{\text{span}}$ to the infimal invariant subspace $\langle A, E \mid B \rangle$ containing $\operatorname{Im} B$.

Theorem 3.9. *For complex linear impulsive system* (2.4)*, one has*

$$\mathcal{R}_{\text{span}} = \mathcal{V}_n = \langle A, E \mid B \rangle. \tag{3.12}$$

Proof. First, we prove that $\mathcal{V}_n \supseteq \langle A, E \mid B \rangle$. From (3.7), it is obvious that $\operatorname{Im} B \subseteq \mathcal{W}_0 \subseteq \mathcal{V}_n$, $A\mathcal{W}_0 = A\langle A \mid B \rangle \subseteq \langle A \mid B \rangle = \mathcal{W}_0$, $A\mathcal{W}_m = A\langle A \mid E\mathcal{W}_{m-1} \rangle \subseteq \langle A \mid E\mathcal{W}_{m-1} \rangle = \mathcal{W}_m$, $m \geq 1$; $E\mathcal{W}_m \subseteq \langle A \mid E\mathcal{W}_m \rangle = \mathcal{W}_{m+1} \subseteq \mathcal{V}_n$, $m \geq 0$. Thus $\mathcal{V}_n$ is an invariant subspace containing $\operatorname{Im} B$. Since $\langle A, E \mid B \rangle$ is the infimal one, we obtain that $\mathcal{V}_n \supseteq \langle A, E \mid B \rangle$.

Next, we prove that $\mathcal{V}_n \subseteq \langle A, E \mid B \rangle$. Since $\langle A, E \mid B \rangle$ is the infimal invariant subspace containing $\operatorname{Im} B$, we get $\operatorname{Im} B \subseteq \langle A, E \mid B \rangle$, $A^i \operatorname{Im} B \subseteq \langle A, E \mid B \rangle$, $i = 1, \ldots, n-1$. Then $\mathcal{W}_0 \subseteq \langle A, E \mid B \rangle$ and $E\mathcal{W}_0 \subseteq \langle A, E \mid B \rangle$. By the same reasoning, the fact that $A^i E\mathcal{W}_0 \subseteq \langle A, E \mid B \rangle$ implies that $\mathcal{W}_1 \subset \langle A, E \mid B \rangle$, $i = 0, \ldots, n-1$. Similarly, for $m > 1$, $\mathcal{W}_m \subseteq \langle A, E \mid B \rangle$, which means that $\sum_{i=0}^{n} \mathcal{W}_i = \mathcal{V}_n \subseteq \langle A, E \mid B \rangle$. The proof is completed. □

Example 3.10. Consider complex linear impulsive system (2.4) with the same coefficient matrices as that in Example 3.3. Now we modify the matrix E as follows

$$E = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1+i & 0 & 0 \\ 0 & (1+i) & 0 & 0 \end{bmatrix}. \tag{3.13}$$

Using the construction proposed in (3.7), we have $\mathcal{W}_0 = [0 \;\; (1+2i) \;\; 0 \;\; 0]^T$, $\mathcal{W}_1 = \langle A \mid E\mathcal{W}_0 \rangle = [0 \;\; 0 \;\; 0 \;\; (1+2i)]^T$, $\mathcal{W}_2 = \langle A \mid E\mathcal{W}_1 \rangle = [(1+2i) \;\; 0 \;\; 0 \;\; 0]^T$ and $\mathcal{W}_3 = \langle A \mid E\mathcal{W}_2 \rangle = [0 \;\; 0 \;\; (1+2i) \;\; 0]^T$. It can be easily verified that $\mathcal{V}_4$ spans the entire complex state space $\mathbb{C}^4$. Hence, system (2.4) with the above matrices is span reachable. Moreover, form this example, we can find that the explicit construction (3.7) helps us to derive the span reachable set easily.

4. Geometric Characterization of Observability

In this section, we present the geometric characterization of the unobservability of complex linear impulsive system (2.4). For convenience, the unobservable set of complex linear systems and its geometric property are introduced first. For a matrix $C \in \mathbb{C}^{m\times n}$, let $\mathcal{K}$ be the kernel of C, that is, $\mathcal{K} \triangleq \operatorname{Ker} C = \{x \in \mathbb{C}^n \mid Cx = 0\}$. Given a matrix $A \in \mathbb{C}^{n\times n}$ and a linear space $\mathcal{M} \subseteq \mathbb{C}^n$, the largest A-invariant subspace contained in $\mathcal{M}$ is given by $\langle \mathcal{M} \mid A \rangle := \mathcal{M} \cap A^{-1}\mathcal{M} \cap A^{-2}\mathcal{M} \cap \cdots \cap A^{-n+1}\mathcal{M}$ which is the unobservable subspace for complex system (3.1) when $\mathcal{M} = \mathcal{K}$, where $A^{-1}\mathcal{M}$ denotes the inverse image of subspace $\mathcal{M}$. Also we have $\operatorname{Ker}(MA) = A^{-1}\operatorname{Ker}(M)$ [13]. We introduce the following definitions of unobservability.

Definition 4.1 (Unobservable set with finite intervals and fixed impulse times). For complex impulsive system (2.4), a state $x_0 \in \mathcal{X}$ is said to be unobservable on $[t_0, t_f]$ with fixed impulse times, if given $t_f > t_0$, impulse times set $\mathcal{T}$ and $x_0 = x(t_0)$, the output $y(t)$ is identically equal to zero for all $t \in [t_0, t_f]$. The set of unobservable states with finite interval and fixed impulse times is denoted by $\mathcal{Q}_{\text{fixed}}$.

Definition 4.2 (Unobservable set with free impulse times). For complex impulsive system (2.4), a state $x_0 \in \mathcal{X}$ is said to be unobservable on $[t_0, t_f]$ with free impulse time, if given t_0, $x_0 = x(t_0)$ yields a response $y(t)$ that is identically equal to zero for all $t \geq t_0$ and all impulse times sets $\mathcal{T}$. The set of these unobservable states is denoted by Q_{free}. System (2.4) is observable if $Q_{\text{free}} = \{0\}$.

By the above definitions, we have $Q_{\text{free}} = \bigcap_{\mathcal{T}} \bigcap_{t_f > t_0} Q_{\text{fixed}}(t_f, \mathcal{T})$.

It is easy to see from Definitions 4.1 and 4.2, the observability of complex linear impulsive system (2.4) is equivalent to that of zero-input complex impulsive system. In this way, given an impulse times set $\mathcal{T}$ and $x_0 \in \mathcal{X}$, the output $y(t)$ is given by

$$y(t) = \begin{cases} Ce^{A(t-t_0)}x_0, & t \in (t_0, t_1], \\ Ce^{A(t-t_{m-1})} \displaystyle\prod_{j=m-1}^{1} Ee^{Ah_j}x_0, & t \in (t_{m-1}, t_m], 2 \leq m \leq M. \end{cases} \tag{4.1}$$

Denote the following subspace sequences:

$$\begin{aligned} &\mathcal{O}_0 = \langle \mathcal{K} \mid A \rangle, \qquad \mathcal{O}_m = \left\langle E^{-1}\mathcal{O}_{m-1} \mid A \right\rangle, \quad m \geq 1, \\ &\mathcal{P}_m = \bigcap_{i=0}^{m} \mathcal{O}_i, \quad m = 0, 1, 2, \ldots. \end{aligned} \tag{4.2}$$

Similar to the discussion about $\mathcal{V}_m$, the sequence $\{\mathcal{P}_m, m = 0, 1, \ldots\}$ converges to $\mathcal{P}_n$.

Theorem 4.3. *For complex linear impulsive system* (2.4), *one has* $Q_{\text{free}} = \mathcal{P}_n$.

Proof. For an initial state $x_0 \in \mathcal{P}_n$ and a given impulsive times set $\mathcal{T} \cap (t_0, t_f) = \{t_0, t_1, \ldots, t_M\}$, it is obvious that from $x_0 \in \mathcal{O}_0 = \text{Ker}(C) \cap A^{-1}\,\text{Ker}(C) \cap \cdots \cap A^{-(n-1)}\,\text{Ker}(C)$, we have $CA^k x_0 = 0$, $k = 0, 1, \ldots, n-1$ which implies that $Ce^{A(t-t_0)}x_0 = 0$, $t \in (t_0, t_1]$. Since $x_0 \in \mathcal{O}_1 = \langle E^{-1}\langle \mathcal{K} \mid A \rangle \mid A \rangle$, the definition of the largest invariant subspace implies that $x_0 \in \bigcap_{k=0}^{n-1} A^{-k}E^{-1}\langle \mathcal{K} \mid A \rangle$. Then $EA^k x_0 \in \langle \mathcal{K} \mid A \rangle$, $k = 0, 1, \ldots, n-1$. From the property of matrix exponent, it follows that $Ee^{Ah_1}x_0 \in \langle \mathcal{K} \mid A \rangle$, which means that $Ce^{A(t-t_1)}Ee^{Ah_1}x_0 = 0, t \in (t_1, t_2]$. By the same reasoning, we get $Ce^{A(t-t_{m-1})}\prod_{j=m-1}^{1}(Ee^{Ah_j})x_0 = 0$, $t \in (t_{m-1}, t_m]$, $2 \leq m \leq M$. It means that the output $y(t) \equiv 0$, $t \in [t_0, t_f]$. From Definition 4.2, we conclude that $x_0 \in Q_{\text{free}}$ and $\mathcal{P}_n \subseteq Q_{\text{free}}$.

On the other hand, if $x_0 \in Q_{\text{free}}$, then for any impulse times set $\mathcal{T}$,

$$0 = y(t) = \begin{cases} Ce^{A(t-t_0)}x_0, & t \in (t_0, t_1], \\ Ce^{A(t-t_{m-1})} \displaystyle\prod_{j=m-1}^{1} Ee^{Ah_j}x_0, & t \in (t_{m-1}, t_m],\ 2 \leq m \leq M. \end{cases} \tag{4.3}$$

The first equation in (4.3) shows that $x_0 \in \langle \mathrm{Ker}(C) \mid A\rangle = \mathcal{O}_0$. If $m = 2$, (4.3) becomes $Ce^{A(t-t_1)}Ee^{Ah_1}x_0 = 0$, $t \in (t_1, t_2]$, then it follows from the definition of unobservable subspace that

$$\begin{aligned} x_0 \in \mathrm{Ker}\Big(Ce^{A(t-t_1)}Ee^{Ah_1}\Big) &= \Big\langle \mathrm{Ker}\Big(Ce^{A(t-t_1)}E\Big) \mid A\Big\rangle = \Big\langle E^{-1}\,\mathrm{Ker}\Big(Ce^{A(t-t_1)}\Big) \mid A\Big\rangle \\ &= \Big\langle E^{-1}\langle \mathcal{K} \mid A\rangle \mid A\Big\rangle. \end{aligned} \tag{4.4}$$

Repeating the same process, we obtain $x_0 \in \mathcal{O}_i$, $i \in \{0, 1, \ldots, n\}$. This means that $x_0 \in \mathcal{P}_n$ and $\mathcal{Q}_{\mathrm{free}} \subseteq \mathcal{P}_n$. The proof is completed. □

From Definition 4.2 and Theorem 4.3, we can see that if $\mathcal{P}_n = \{0\}$, system (2.4) is observable. Similarly, we aim to show the invariance of the unobservable set with free impulse times $\mathcal{Q}_{\mathrm{free}}$. Denote the supremal invariant subspace of system (2.4) contained in $\mathcal{K}$ to be $\langle \mathcal{K} \mid A, E\rangle$.

Theorem 4.4. *For complex linear impulsive system* (2.4)*, one has*

$$\mathcal{Q}_{\mathrm{free}} = \mathcal{P}_n = \langle \mathcal{K} \mid A, E\rangle. \tag{4.5}$$

Proof. First, we prove that $\mathcal{P}_n \supseteq \langle \mathcal{K} \mid A, E\rangle$. Given any $x_0 \in \langle \mathcal{K} \mid A, E\rangle$, since $\langle \mathcal{K} \mid A, E\rangle$ is the largest invariant subspace contained in $\mathrm{Ker}\,C$, we have $A^i x_0 \in \langle \mathcal{K} \mid A, E\rangle \subseteq \mathrm{Ker}\,C$, $i = 0, \ldots, n-1$, which means that $x_0 \in \mathcal{O}_0 = \bigcap_{i=0}^{n-1} A^{-i}\,\mathrm{Ker}\,C$. Furthermore, $A^j E A^i x_0 \in \langle \mathcal{K} \mid A, E\rangle \subseteq \mathrm{Ker}\,C$, $i, j = 0, \ldots, n-1$, which means that $x_0 \in \mathcal{O}_1 = \bigcap_{i=0}^{n-1} A^{-i}E^{-1}\mathcal{O}_0$. By the same deduction, $x_0 \in \mathcal{O}_m$ indicates that $x_0 \in \mathcal{P}_n$ by the definition of $\mathcal{P}_n$, $m > 1$. Then we have $\mathcal{P}_n \supseteq \langle \mathcal{K} \mid A, E\rangle$.

Next, we prove that $\mathcal{P}_n \subseteq \langle \mathcal{K} \mid A, E\rangle$. Given any $x_0 \in \mathcal{P}_n$, $x_0 \in \mathcal{O}_0 = \bigcap_{i=0}^{n-1} A^{-i}\,\mathrm{Ker}\,C \subseteq \mathrm{Ker}\,C$, then $\mathcal{P}_n \subseteq \mathrm{Ker}\,C$. Moreover, since $\mathcal{O}_m$ are A-invariant subspaces, we have $Ax_0 \in A\mathcal{O}_m \subseteq \mathcal{O}_m$, $m = 0, 1, \ldots, n$. It is clear that $Ax_0 \in \bigcap_{m=0}^{n} \mathcal{O}_m = \mathcal{P}_n$. On the other hand, the sequence $\{\mathcal{P}_m\}$ converges to $\mathcal{P}_n$, which implies that $x_0 \in \mathcal{P}_n = \mathcal{P}_{n+1}$ and $x_0 \in \mathcal{O}_m$, $m = 1, 2, \ldots, n+1$. Thus $Ex_0 \in E\mathcal{O}_m = \bigcap_{i=0}^{n-1} EA^{-i}E^{-1}\mathcal{O}_{m-1} \subseteq EE^{-1}\mathcal{O}_{m-1} = \mathcal{O}_{m-1}$, $m = 1, 2, \ldots, n+1$. This shows that $Ex_0 \in \mathcal{O}_m$, $m = 0, 1, \ldots, n$. In conclusion, we have $Ex_0 \in \mathcal{P}_n$ and $Ax_0 \in \mathcal{P}_n$. It means that $\mathcal{P}_n$ is an invariant subspace contained in $\mathcal{K}$ with respect to matrices A and E. Since $\langle \mathcal{K} \mid A, E\rangle$ is the largest one, we conclude that $\mathcal{P}_n \subset \langle \mathcal{K} \mid A, E\rangle$. This completes the proof. □

Remark 4.5. When the systems in this paper and [10] are reduced to complex linear systems, if $\langle \mathrm{Ker}(C) \mid A\rangle = \{0\}$, from [13], we know that $(\langle \mathrm{Ker}(C) \mid A\rangle)^{\perp} = \langle A^{\top} \mid C^{\top}\rangle = \mathbb{C}^n$, that is, $\mathrm{rank}(S) = n$, which means that system (2.4) is observable, where $S = [C^{\top} A^{\top} C^{\top} \cdots (A^{n-1})^{\top} C^{\top}]^{\top}$.

Thus the geometric condition is equivalent to the algebraic one in Theorem 5(i) in [10] for the observability of system (2.4). When reduced to complex linear impulsive systems with $AE = EA$, simple computation follows that the algebraic condition in [10] and the geometric one here for the observability are equivalent.

Example 4.6. Consider complex linear impulsive system (2.4) with

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad E = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}. \tag{4.6}$$

It is easy to get that $\mathcal{O}_0 = \ker(C) = \begin{bmatrix} 1+i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1+i & 0 \\ 0 & 0 & 1+i \end{bmatrix}$. Simple computations from (4.2) yield that $\mathcal{O}_1 = \bigcap_{i,j=1}^{3} \ker(CA^iEA^j) = \begin{bmatrix} 0 & 0 \\ 1+i & 0 \\ 0 & 0 \\ 0 & 1+i \end{bmatrix}$ and $\mathcal{O}_2 = \begin{bmatrix} 1+i & 0 & 0 \\ 0 & 1+i & 0 \\ 0 & 0 & 1+i \\ 0 & 0 & 0 \end{bmatrix}$. Then we have $\mathcal{O}_0 \cap \mathcal{O}_1 \cap \mathcal{O}_2 = \mathcal{Q}_{\text{free}} = \{0\}$, which means that the system is observable.

5. Conclusion

In this paper, the reachability and observability have been investigated for a class of time-driven complex linear impulsive systems which allow free impulsive times. It has been shown that traditional geometric approach may be not sufficient to study the reachability and observability for such systems. Hence, a new geometric analysis method is developed. New concepts of the reachability and observability have been introduced. Sufficient and necessary conditions for the span reachability and observability of such systems have been established. Moreover, geometric properties of span reachable set and unobservable set have been studied. The equivalence between the algebraic conditions in known results [10] and the geometric ones obtained here has been established. Numerical examples have been provided to show the explicit construction of the reachable subspace and unobservable subspace and easily-verifiable conditions for the reachability and observability of complex linear impulsive systems.

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