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Research Article

About Projections of Solutions for Fuzzy Differential Equations

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In this paper we propose the concept of *fuzzy projections* on subspaces of $\mathcal{F}(\mathbb{R}^n)$, obtained from Zadeh's extension of canonical projections in $\mathbb{R}^n$, and we study some of the main properties of such projections. Furthermore, we will review some properties of fuzzy projection solution of fuzzy differential equations. As we will see, the concept of fuzzy projection can be interesting for the graphical representation of fuzzy solutions.

1. Introduction

Consider the set $U \subset \mathbb{R}^n$. Denote by $\mathcal{F}(U)$ the set formed by the fuzzy subsets of U whose subsets have support compacts in U. Some properties for metrics $\mathcal{F}(U)$ can be found in [1]. If A is a subset of U, we will use the notation χ_A to indicate a membership function for the fuzzy set called membership function or crisp of U.

Consider the autonomous equation defined by

$$\frac{dx}{dt} = f(x), \tag{1}$$

where $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ is a sufficiently smooth function. For each $x_o \in U$, denote by $\varphi_t(x_o)$ the deterministic solution (1) with initial condition x_o. Here we are assuming that the solution is defined for all $t \in \mathbb{R}_+$. The function $\varphi_t : U \to U$ will be called *deterministic flow*.

To consider initial conditions with inaccuracies modeled by fuzzy sets [2], consider the proposed Zadeh's extension φ_t, the application $\widehat{\varphi}_t : \mathcal{F}(U) \to \mathcal{F}(U)$, which takes the fuzzy set $\mathbf{x}_o \in \mathcal{F}(U)$ and the fuzzy set $\widehat{\varphi}_t(\mathbf{x}_o)$. In the context of this paper we call the application $\widehat{\varphi}_t$ of *fuzzy flow*. Given $\mathbf{x}_o \in \mathcal{F}(U)$, we say $\widehat{\varphi}_t(\mathbf{x}_o)$ is a *fuzzy solution* to (1) whose initial condition is the fuzzy set $\mathbf{x}_o$.

The conditions for existence of fuzzy equilibrium points and the nature of the stability of such spots were first presented in [2]. The concepts of stability and asymptotic stability for fuzzy equilibrium points are similar to those of equilibrium points of deterministic solutions, and stability conditions for fuzzy equilibrium points can be found in [2]. Conditions for the existence of periodic fuzzy solutions and the stability of such solutions can be found in [3].

In this paper, we propose the concept of *fuzzy projections* on subspaces of $\mathcal{F}(\mathbb{R}^n)$, obtained from Zadeh's extension defined canonical projections in $\mathbb{R}^n$, and study some of the main properties of such projections. Furthermore, we review some properties of fuzzy projection solution of fuzzy differential equations. As we will see, the concept of fuzzy projection can be interesting for the graphical representation of fuzzy solutions.

2. Projections in Fuzzy Metric Spaces

We restrict our analysis to the set $\mathcal{F}(X)$ whose elements are subsets of a fuzzy set X whose α-levels are compact and nonempty subsets in X. The fuzzy subsets that are $\mathcal{F}(X)$ will be denoted by bold lowercase letters to differentiate the elements X. So $\mathbf{x} \in \mathcal{F}(X)$ if and only if $[\mathbf{x}]^\alpha$ is compact and nonempty subset for all $\alpha \in [0, 1]$.

We can define a structure of metric spaces in $\mathcal{F}(X)$ by the Hausdorff metric for compact subsets of X. Let $\mathcal{K}(X)$ be the set formed by nonempty compact subsets of the metric space

(X,d). Given two sets A,B in $\mathscr{K}(X)$, the distance between them can be defined by

$$\text{dist}\,(A,B)=\sup_{a\in A}\inf_{b\in B}d\,(a,b)\,. \tag{2}$$

The distance between sets defined above is a *pseudometric* to $\mathscr{K}(X)$ since $\text{dist}\,(A,B)=0$ if and only if $A\subseteq B$, not necessarily equal value. However, *Hausdorff distance* between $A,B\in\mathscr{K}(X)$ defined by

$$\begin{aligned}d_H(A,B)&=\max\left\{\sup_{a\in A}\inf_{b\in B}d\,(a,b)\,,\sup_{b\in B}\inf_{a\in A}d\,(a,b)\right\}\\&=\max\left\{\text{dist}\,(A,B)\,,\text{dist}\,(B,A)\right\}\end{aligned} \tag{3}$$

is a metric for all $\mathscr{K}(X)$. so that $(\mathscr{K}(X),d_H)$ is a metric space. It is also worth that (X,d) is a complete metric space, so $(\mathscr{K}(X),d_H)$ is also a complete metric space [4].

Through the Hausdorff metric d_H, we can define a metric for all $\mathscr{F}(X)$. Here we denote it by d_∞. Given two points $\mathbf{u},\mathbf{v}\in\mathscr{F}(X)$, the distance between $\mathbf{u},\mathbf{v}$ is defined by

$$d_\infty(\mathbf{u},\mathbf{v})=\sup_{\alpha\in[0,1]}d_H\left([\mathbf{u}]^\alpha,[\mathbf{v}]^\alpha\right). \tag{4}$$

It is not difficult to show that the distance defined above satisfies the properties of a metric and thus $(\mathscr{F}(X),d_\infty)$ is a metric space.

Nguyen's theorem provides an important link between α-levels image of fuzzy subsets and the image of his α-levels by a function $f:X\times Y\to Z$. According to [5], if $X\subseteq\mathbb{R}^n$ and $Y\subseteq\mathbb{R}^m$ and $f:X\to Y$ is continuous, then Zadeh's extension $\widehat{f}:\mathscr{F}(X)\to\mathscr{F}(Y)$ is well defined and is worth

$$\left[\widehat{f}(\mathbf{u})\right]^\alpha=f\left([\mathbf{u}]^\alpha\right) \tag{5}$$

for all $\alpha\in[0,1]$ and $\mathbf{u}\in\mathscr{F}(X)$.

2.1. Projections Fuzzy. Consider the application $P_n:\mathbb{R}^{n+m}\to\mathbb{R}^n$ that for each $(x,y)\in\mathbb{R}^{n+m}$ associates point $P_n(x,y)=x\in\mathbb{R}^n$.

Provided that $\mathbb{R}^n$ can be characterized as a subset of $\mathbb{R}^{n+m}$ by identifying it with the subset $\mathbb{R}^n\times\{0\}$, then the application P_n can be seen as the projection of $\mathbb{R}^{n+m}$ on the set $\mathbb{R}^n$. For this reason, we say that x is the projection in $\mathbb{R}^n$; the point $(x,y)\in\mathbb{R}^{n+m}$.

Notice that a point (u,v) is in the image of P_n if and only if $v=0$. Furthermore, $P_n(x,y)=x$ for all $y\in\mathbb{R}^m$. Thus, given a point $\mathbf{z}\in\mathscr{F}(\mathbb{R}^{n+m})$, with membership function $\mu_\mathbf{z}:\mathbb{R}^{n+m}\to[0,1]$, the image $\widehat{P}_n(\mathbf{z})$, obtained by Zadeh's extension projection P_n, has the membership function

$$\mu_{\widehat{P}_n(\mathbf{z})}(x)=\sup_{v\in\mathbb{R}^m}\mu_\mathbf{z}(x,v)\,. \tag{6}$$

The application $\widehat{P}_n:\mathscr{F}(\mathbb{R}^{n+m})\to\mathscr{F}(\mathbb{R}^n)$, obtained by Zadeh's extension of P_n, that for each $\mathbf{z}\in\mathscr{F}(\mathbb{R}^{n+m})$ associates the point $\widehat{P}_n(\mathbf{z})\in\mathscr{F}(\mathbb{R}^n)$ can be seen as a projection of $\mathscr{F}(\mathbb{R}^{n+m})$ in $\mathscr{F}(\mathbb{R}^n)$, as it can be identified with the subset $\mathscr{F}(\mathbb{R}^n)\times\chi_{\{0\}}$. Similarly the projection P_n satisfies:

$$\widehat{P}_n\left(\widehat{P}_n(\mathbf{z})\right)=\widehat{P}_n(\mathbf{z})\,. \tag{7}$$

Based on this, we can define the *projection of fuzzy* $\mathbf{z}\in\mathscr{F}(\mathbb{R}^{n+m})$ in $\mathscr{F}(\mathbb{R}^n)$ as the point $\mathbf{x}\in\mathscr{F}(\mathbb{R}^n)$ with a membership function

$$\mu_\mathbf{x}(x)=\sup_{a\in\mathbb{R}^m}\mu_\mathbf{z}(x,a)\,. \tag{8}$$

We also consider the function $P_m:\mathbb{R}^{n+m}\to\mathbb{R}^m$ that for all $(x,y)\in\mathbb{R}^{n+m}$ associates the point $P_n(x,y)=y\in\mathbb{R}^m$. In this case, the image of a point $\mathbf{z}\in\mathscr{F}(\mathbb{R}^{n+m})$, with the membership function $\mu_\mathbf{z}:\mathbb{R}^{n+m}\to[0,1]$, is a point $\mathbf{y}\in\mathscr{F}(\mathbb{R}^m)$ with the membership function

$$\mu_\mathbf{y}(y)=\sup_{a\in\mathbb{R}^n}\mu_\mathbf{z}(a,y) \tag{9}$$

which we call *fuzzy projection* $\mathbf{z}$ in $\mathscr{F}(\mathbb{R}^m)$. Thus the application $\widehat{P}_m:\mathscr{F}(\mathbb{R}^{n+m})\to\mathscr{F}(\mathbb{R}^m)$ can be viewed as a *fuzzy projection* $\mathscr{F}(\mathbb{R}^{n+m})$ in $\mathscr{F}(\mathbb{R}^m)$.

Here are some examples.

Example 1. Let $\mathbf{a}\in\mathscr{F}(\mathbb{R}^n)$ and $\mathbf{b}\in\mathscr{F}(\mathbb{R}^m)$. We can define $\mathbf{z}=(\mathbf{a},\mathbf{b})\in\mathscr{F}(\mathbb{R}^{n+m})$ with membership function

$$\mu_\mathbf{z}(x,y)=\min\left\{\mu_\mathbf{a}(x)\,,\mu_\mathbf{b}(y)\right\}. \tag{10}$$

The image of $\mathbf{z}$ by applying $\widehat{P}_n$, in this case, has a membership function:

$$\mu_{\widehat{P}_n(\mathbf{z})}(x)=\sup_{v\in\mathbb{R}^m}\min\left\{\mu_\mathbf{a}(x)\,,\mu_\mathbf{b}(v)\right\}. \tag{11}$$

Since $\min\{\mu_\mathbf{a}(x),\mu_\mathbf{b}(v)\}\le\mu_\mathbf{a}(x)$, so,

$$\sup_{v\in\mathbb{R}^m}\min\left\{\mu_\mathbf{a}(x)\,,\mu_\mathbf{b}(v)\right\}\le\mu_\mathbf{a}(x)\,. \tag{12}$$

As $\mathbf{b}\in\mathscr{F}(\mathbb{R}^m)$, so $v\in\mathbb{R}^m$ so that $\mu_\mathbf{b}(v)=1$. So, the fuzzy projection $\mathbf{x}$ of $\mathbf{z}$ about $\mathscr{F}(\mathbb{R}^n)$ has a membership function:

$$\mu_\mathbf{x}(x)=\sup_{v\in\mathbb{R}^m}\min\left\{\mu_\mathbf{a}(x)\,,\mu_\mathbf{b}(v)\right\}=\mu_\mathbf{a}(x)\,. \tag{13}$$

In Figure 1, the membership functions of $\mathbf{z}\in\mathscr{F}(\mathbb{R}^2)$, defined from $\mathbf{a}$ and $\mathbf{b}\in\mathscr{F}(\mathbb{R})$ and your fuzzy projection in $\mathscr{F}(\mathbb{R})$, respectively, can be seen. In this figure,

$$\mu_\mathbf{a}(x)=\mu_\mathbf{b}(x)=\max\left\{1-x^2,0\right\}. \tag{14}$$

With similar argument, we can show that $\mathbf{b}\in\mathscr{F}(\mathbb{R}^m)$ is a fuzzy projection of $\mathbf{z}$ in $\mathscr{F}(\mathbb{R}^m)$.

We can also define $\mathbf{x}=(\mathbf{a},\mathbf{b})\in\mathscr{F}(\mathbb{R}^{n+m})$ through the *t-norm* product, that is,

$$\mu_\mathbf{z}(x,y)=\mu_\mathbf{a}(x)\,\mu_\mathbf{b}(y)\,. \tag{15}$$

The projection of $\mathbf{z}$ in $\mathscr{F}(\mathbb{R}^n)$ has a membership function:

$$\sup_{v\in\mathbb{R}^m}\mu_\mathbf{x}(x,v)=\sup_{v\in\mathbb{R}^m}\mu_\mathbf{a}(x)\,\mu_\mathbf{b}(v)=\mu_\mathbf{a}(x)\,. \tag{16}$$

Moreover, the projection $\mathscr{F}(\mathbb{R}^m)$ has a membership function:

$$\sup_{u\in\mathbb{R}^n} \mu_{\mathbf{x}}(u, y) = \sup_{u\in\mathbb{R}^n} \mu_{\mathbf{a}}(u)\, \mu_{\mathbf{b}}(y) = \mu_{\mathbf{b}}(y). \quad (17)$$

Similarly, we can show that fuzzy projections $\mathbf{z} = (\mathbf{a}, \mathbf{b})$ in $\mathscr{F}(\mathbb{R}^n)$ and $\mathscr{F}(\mathbb{R}^m)$ for all *t-norm* Δ are, respectively, $\mathbf{a}$ and $\mathbf{b}$. First, for any *t-norm* Δ, we have

$$\Delta(\mu_{\mathbf{a}}(x), \mu_{\mathbf{b}}(y)) \le \Delta(\mu_{\mathbf{a}}(x), 1) = \mu_{\mathbf{a}}(x). \quad (18)$$

So,

$$\sup_{v\in\mathbb{R}^m} \Delta(\mu_{\mathbf{a}}(x), \mu_{\mathbf{b}}(v)) \le \mu_{\mathbf{a}}(x). \quad (19)$$

But the ultimate is reached if we take $v \in \mathbb{R}^n$ so that $\mu_{\mathbf{b}}(v) = 1$. Then, the projection of $\mathbf{z} = (\mathbf{a}, \mathbf{b})$ in $\mathscr{F}(\mathbb{R}^n)$ has membership function

$$\mu_{\mathbf{x}}(x) = \sup_{v\in\mathbb{R}^m} \Delta(\mu_{\mathbf{a}}(x), \mu_{\mathbf{b}}(v)) = \mu_{\mathbf{a}}(x), \quad (20)$$

for all *t-norm* Δ.

Example 2. Consider $\mathbf{z} \in \mathscr{F}(\mathbb{R}^2)$ determined by membership function

$$\mu_{\mathbf{z}}(x, y) = \max\left\{1 - x^2 - 2y^2, 0\right\}. \quad (21)$$

For this case, we have the fuzzy projections $\mathbf{x}$ and $\mathbf{y}$ on $\mathscr{F}(\mathbb{R})$, respectively, determined by

$$\begin{aligned} \mu_{\mathbf{x}}(x) &= \sup_{v\in\mathbb{R}^m} \mu_{\mathbf{z}}(x, v) = \max\left\{1 - x^2, 0\right\}, \\ \mu_{\mathbf{y}}(y) &= \sup_{u\in\mathbb{R}^n} \mu_{\mathbf{z}}(u, y) = \max\left\{1 - 2y^2, 0\right\}. \end{aligned} \quad (22)$$

In Figure 2 we can see the membership functions $\mathbf{z}$ and $\mathbf{x}$, respectively.

Proposition 3. *Let* $\overline{\mathbf{x}} = \widehat{P}_n(\mathbf{x})$ *and* $\overline{\mathbf{y}} = \widehat{P}_n(\mathbf{y})$, *with* $\mathbf{x}$ *and* $\mathbf{y} \in \mathscr{F}(\mathbb{R}^{n+m})$. *The distance between the fuzzy projections* $\overline{\mathbf{x}}$ *and* $\overline{\mathbf{y}}$ *is always limited by the distance between* $\mathbf{x}$ *and* $\mathbf{y}$.

Proof. In fact, for all $\alpha \in [0, 1]$ we have

$$\begin{aligned} &\operatorname{dist}\left([\mathbf{x}]^\alpha, [\mathbf{y}]^\alpha\right) \\ &\quad = \sup_{a\in[\mathbf{x}]^\alpha} \inf_{b\in[\mathbf{y}]^\alpha} \|a - b\| \\ &\quad = \sup_{(a_1,a_2)\in[\mathbf{x}]^\alpha} \inf_{(b_1,b_2)\in[\mathbf{y}]^\alpha} \sqrt{\|a_1 - b_1\|^2 + \|a_2 - b_2\|^2} \\ &\quad \ge \sup_{(a_1,a_2)\in[\mathbf{x}]^\alpha} \inf_{(b_1,b_1)\in[\mathbf{y}]^\alpha} \sqrt{\|a_1 - b_1\|^2} \\ &\quad = \sup_{a_1\in[\overline{\mathbf{x}}]^\alpha} \inf_{b_1\in[\overline{\mathbf{y}}]^\alpha} \sqrt{\|a_1 - b_1\|^2} \\ &\quad = \operatorname{dist}\left([\overline{\mathbf{x}}]^\alpha, [\overline{\mathbf{y}}]^\alpha\right). \end{aligned} \quad (23)$$

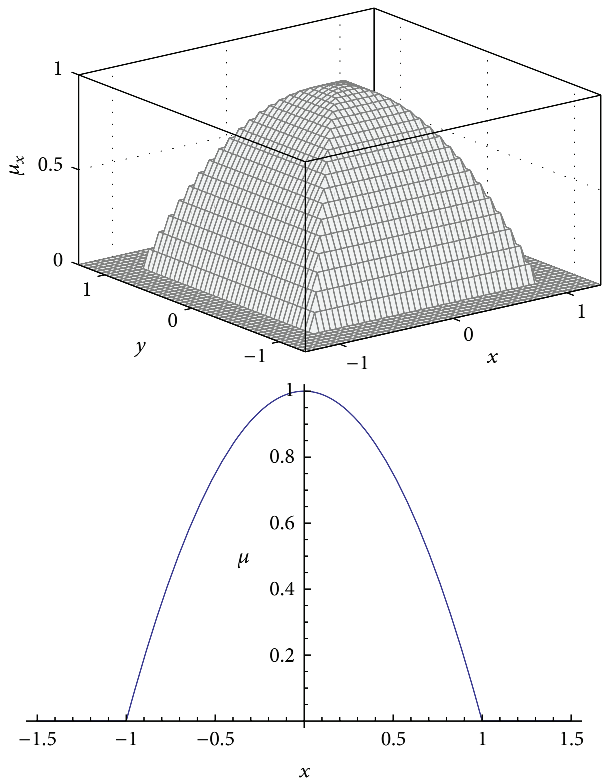


Figure 1: Membership function of $\mathbf{z}$ and $\mathbf{a}$ respectively.

We can prove that $\operatorname{dist}([\mathbf{y}]^\alpha, [\mathbf{x}]^\alpha) \ge \operatorname{dist}([\overline{\mathbf{y}}]^\alpha, [\overline{\mathbf{x}}]^\alpha)$. Therefor,

$$d_\infty(\overline{\mathbf{x}}, \overline{\mathbf{y}}) \le d_\infty(\mathbf{x}, \mathbf{y}). \quad (24)$$

□

The fuzzy projection $\overline{\mathbf{p}} \in \mathscr{F}(\mathbb{R}^n)$ to a point $\mathbf{p} \in \mathscr{F}(\mathbb{R}^{n+m})$ satisfies another important property of the projections. Namely, the projection $\overline{\mathbf{p}}$ is the point that minimizes the distance between the point $\mathbf{p} \in \mathscr{F}(\mathbb{R}^{n+m})$ and the set $\mathscr{F}(\mathbb{R}^n)$, the latter set is considered as a subset of $\mathscr{F}(\mathbb{R}^{n+m})$.

Proposition 4. *The fuzzy projection* $\overline{\mathbf{p}}$ *in* $\mathscr{F}(\mathbb{R}^n)$ *of* $\mathbf{p} \in \mathscr{F}(\mathbb{R}^{n+m})$ *is such that*

$$d_\infty(\mathbf{p}, \overline{\mathbf{p}}) = \inf_{\mathbf{z}\in\mathscr{F}(\mathbb{R}^n)} d_\infty(\mathbf{p}, \mathbf{z}). \quad (25)$$

Proof. First, let us note the abuse of notation in the statement. The term $d_\infty(\mathbf{p}, \mathbf{z})$ only makes sense because we can see $\mathscr{F}(\mathbb{R}^n)$ as a subset of $\mathscr{F}(\mathbb{R}^{n+m})$. Provided that $[\mathbf{p}]^\alpha \subset \mathbb{R}^{n+m}$ and $[\overline{\mathbf{p}}]^\alpha \subset \mathbb{R}^n$, for $x \in \mathbb{R}^n$ and $y = (y_1, y_2) \in \mathbb{R}^{n+m}$, we have

$$\|x - y\| = \sqrt{\|x - y_1\|^2 + \|y_2\|^2} \quad (26)$$

since

$$\begin{aligned} \operatorname{dist}\left([\mathbf{p}]^\alpha, [\overline{\mathbf{p}}]^\alpha\right) &= \sup_{y\in[\mathbf{p}]^\alpha} \inf_{x\in[\overline{\mathbf{p}}]^\alpha} \sqrt{\|y_1 - x\|^2 + \|y_2\|^2} \\ &= \sup_{y\in[\mathbf{p}]^\alpha} \|y_2\|. \end{aligned} \quad (27)$$

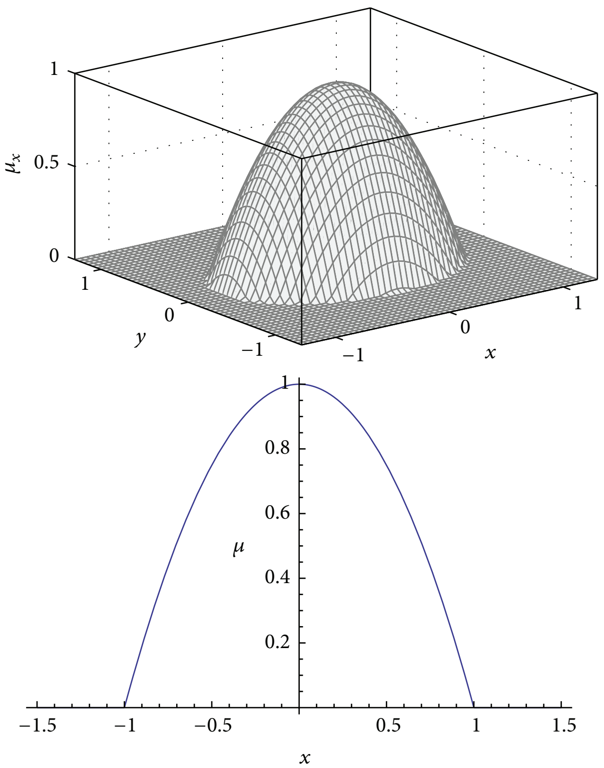


Figure 2: Membership function of **z** and **x**, respectively.

Moreover, we have

$$\operatorname{dist}\left([\overline{\mathbf{p}}]^{\alpha},[\mathbf{p}]^{\alpha}\right)=\sup_{x\in[\overline{\mathbf{p}}]^{\alpha}}\inf_{y\in[\mathbf{p}]^{\alpha}}\sqrt{\left\|y_1-x\right\|^2+\left\|y_2\right\|^2}. \tag{28}$$

Now, since $x\in[\overline{\mathbf{p}}]^{\alpha}$, so, $(x,z)\in[\mathbf{p}]^{\alpha}$ for some $z\in\mathbb{R}^m$, where we have the inequality

$$\begin{aligned}\operatorname{dist}\left([\overline{\mathbf{p}}]^{\alpha},[\mathbf{p}]^{\alpha}\right)&=\sup_{x\in[\overline{\mathbf{p}}]^{\alpha}}\inf_{y\in[\mathbf{p}]^{\alpha}}\sqrt{\left\|y_1-x\right\|^2+\left\|y_2\right\|^2}\\&\le\|z\|\le\sup_{y\in[\mathbf{p}]^{\alpha}}\left\|y_2\right\|.\end{aligned} \tag{29}$$

Thus, the Hausdorff distance between $[\mathbf{p}]^{\alpha}$ and $[\overline{\mathbf{p}}]^{\alpha}$ in this case is

$$\begin{aligned}d_H\left([\mathbf{p}]^{\alpha},[\overline{\mathbf{p}}]^{\alpha}\right)&=\max\left\{\operatorname{dist}\left([\mathbf{p}]^{\alpha},[\overline{\mathbf{p}}]^{\alpha}\right),\operatorname{dist}\left([\overline{\mathbf{p}}]^{\alpha},[\mathbf{p}]^{\alpha}\right)\right\}\\&=\operatorname{dist}\left([\mathbf{p}]^{\alpha},[\overline{\mathbf{p}}]^{\alpha}\right).\end{aligned} \tag{30}$$

Let $\mathbf{q}\in\mathcal{F}(\mathbb{R}^n)$ such that $\mathbf{q}\neq\overline{\mathbf{p}}$. This implies that $[\mathbf{q}]^{\alpha}\neq[\overline{\mathbf{p}}]^{\alpha}$, for some $\alpha\in[0,1]$. Consequently, there $y=(y_1,y_2)\in[\mathbf{p}]^{\alpha}$ such that $y_1\notin[\mathbf{q}]^{\alpha}$ or exists $z_1\in[\mathbf{q}]^{\alpha}$ such that $z=(z_1,z_2)\notin[\mathbf{p}]^{\alpha}$, for all $z_2\in\mathbb{R}^m$. Namely, $z_1\notin[\overline{\mathbf{p}}]^{\alpha}$. For the first case, we have

$$\sqrt{\left\|x-y_1\right\|^2+\left\|y_2\right\|^2}>\left\|y_2\right\| \tag{31}$$

for all $x\in[\mathbf{q}]^{\alpha}$. The second property follows directly from the projection inequality

$$\left\|z_1-y\right\|=\sqrt{\left\|z_1-y_1\right\|^2+\left\|y_2\right\|^2}>\left\|y_2\right\|, \tag{32}$$

for all $y=(y_1,y_2)\in[\mathbf{p}]^{\alpha}$. Thus in both cases we have to

$$\begin{aligned}\operatorname{dist}\left([\mathbf{p}]^{\alpha},[\mathbf{q}]^{\alpha}\right)&=\sup_{y\in[\mathbf{p}]^{\alpha}}\inf_{x\in[\overline{\mathbf{p}}]^{\alpha}}\sqrt{\left\|y_1-x\right\|^2+\left\|y_2\right\|^2}\\&\ge\sup_{y\in[\mathbf{p}]^{\alpha}}\left\|y_2\right\|\\&=\operatorname{dist}\left([\mathbf{p}]^{\alpha},[\overline{\mathbf{p}}]^{\alpha}\right).\end{aligned} \tag{33}$$

Therefore, we have $d_H([\mathbf{p}]^{\alpha},[\mathbf{q}]^{\alpha})\ge d_H([\mathbf{p}]^{\alpha},[\overline{\mathbf{p}}]^{\alpha})$. Thus, we can conclude that, for all $\mathbf{q}\in\mathcal{F}(\mathbb{R}^n)$, $d_\infty(\mathbf{p},\mathbf{q})\ge d_\infty(\mathbf{p},\overline{\mathbf{p}})$, which proves the assertion. □

We can also define fuzzy projections $\mathbf{z}\in\mathcal{F}(U\times P)$ in $\mathcal{F}(U)$ and $\mathcal{F}(P)$, where $U\subset\mathbb{R}^n$ and $P\subset\mathbb{R}^m$. In this case, the supremum in membership functions (8) and (9) is taken on the sets P and U, respectively, and properties shown above metrics remain valid.

We can also consider the projection $\pi_i:\mathbb{R}^n\to\mathbb{R}$ from a point $x=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ in ith coordinate axis; that is, $\pi_i(x)=x_i$. As shown before, the projection of Zadeh's extension π_i defines the application $\widehat{\pi}_i:\mathcal{F}(\mathbb{R}^n)\to\mathcal{F}(\mathbb{R})$ that we call for the ith *fuzzy projection* of $\mathcal{F}(\mathbb{R}^n)$ on $\mathcal{F}(\mathbb{R})$. Thus, given a point $\mathbf{x}\in\mathcal{F}(\mathbb{R})$, the ith fuzzy projection of $\mathbf{x}$ on $\mathcal{F}(\mathbb{R})$ is a point $\mathbf{x}_i$ with membership function given by

$$\mu_{\mathbf{x}_i}(a)=\sup_{\substack{x\in\mathbb{R}^n\\x_i=a}}\mu_{\mathbf{x}}(x). \tag{34}$$

Again, if $\mathbf{x}=(\mathbf{a}_1,\mathbf{a}_2,\ldots,\mathbf{a}_n)$ is defined by fuzzy Cartesian product, then ith fuzzy projection of $\mathbf{x}\in\mathcal{F}(\mathbb{R}^n)$ in $\mathcal{F}(\mathbb{R})$ is a point $\mathbf{a}_i$. For simplicity, consider $\mathbf{x}\in\mathbb{R}^3$ defined by

$$\mu_{\mathbf{x}}(x,y,z)=\Delta\left(\Delta\left(\mu_{\mathbf{a}_1}(x),\mu_{\mathbf{a}_2}(y)\right),\mu_{\mathbf{a}_3}(z)\right). \tag{35}$$

By the properties of *t-norm*, it follows that

$$\begin{aligned}&\Delta\left(\Delta\left(\mu_{\mathbf{a}_1}(x),\mu_{\mathbf{a}_2}(y)\right),\mu_{\mathbf{a}_3}(z)\right)\\&\quad\le\Delta\left(\Delta\left(\mu_{\mathbf{a}_1}(x),\mu_{\mathbf{a}_2}(y)\right),1\right)\\&\quad=\Delta\left(\mu_{\mathbf{a}_1}(x),\mu_{\mathbf{a}_2}(y)\right)\\&\quad\le\mu_{\mathbf{a}_2}(y),\end{aligned} \tag{36}$$

for all $x,y,z\in\mathbb{R}$.

Thus, the second fuzzy projection $\mathbf{x}$ on $\mathcal{F}(\mathbb{R})$ is the point $\mathbf{x}_2$ where the membership function is

$$\mu_{\mathbf{x}_2}(a)=\sup_{\substack{x\in\mathbb{R}^3\\x_2=a}}\mu_{\mathbf{x}}(x). \tag{37}$$

For the previous inequality, we have

$$\mu_{\mathbf{x}_2}(a)=\sup_{\substack{x\in\mathbb{R}^3\\x_2=a}}\mu_{\mathbf{x}}(x)\le\mu_{\mathbf{a}_2}(a). \tag{38}$$

Taking $\overline{x}$ and $\overline{z}$ such that $\mu_{\mathbf{a}_1}(\overline{x}) = \mu_{\mathbf{a}_3}(\overline{z}) = 1$, equality is attained in the supremum, and hence,

$$\mu_{\mathbf{x}_2}(a) = \sup_{\substack{x\in\mathbb{R}^3 \\ x_2=a}} \mu_{\mathbf{x}}(x) = \mu_{\mathbf{a}_2}(a). \tag{39}$$

Induction proves the general case in which $\mathbf{x} \in \mathcal{F}(\mathbb{R}^n)$.

Through expression (8), we can determine the α-levels of fuzzy projection $\mathbf{x} \in \mathcal{F}(\mathbb{R}^n)$ to a point $\mathbf{z} \in \mathcal{F}(\mathbb{R}^{n+m})$. Indeed, if $\mu_{\mathbf{x}}(x) \geq \alpha$, so, $y \in \mathbb{R}^m$ such that $\mu_{\mathbf{z}}(x, y) \geq \alpha$ so that $(x, y) \in [\mathbf{z}]^\alpha$. The reciprocal is also true, because if $\mu_{\mathbf{z}}(x, y) \geq \alpha$, then by (8), $\mu_{\mathbf{x}}(x) \geq \alpha$. Thus, we conclude that:

$$x \in [\mathbf{x}]^\alpha \Longleftrightarrow (x, y) \in [\mathbf{z}]^\alpha \quad \text{for some } y \in \mathbb{R}^m, \tag{40}$$

or

$$[\mathbf{x}]^\alpha = \{x \in \mathbb{R}^n : (x, y) \in [\mathbf{z}]^\alpha\}. \tag{41}$$

Since applying π_i is continuous, we can use the equality (5) to show that the ith fuzzy projection $\mathbf{x}_i \in \mathcal{F}(\mathbb{R})$ of $\mathbf{x} \in \mathcal{F}(\mathbb{R}^n)$ has α-levels:

$$[\mathbf{x}_i]^\alpha = \{a \in \mathbb{R} : x \in [\mathbf{x}]^\alpha, x_i = a\}. \tag{42}$$

3. Projection of Fuzzy Solutions

3.1. Projection on the Coordinate Axes. Consider the flow $\varphi_t : U \subset \mathbb{R}^n \to U$ generated by the autonomous equation

$$\frac{dx}{dt} = f(x), \tag{43}$$

where $\varphi_t^{(i)} : U \to \mathbb{R}$ is the projection of the deterministic flow ith coordinate axis; that is, $\varphi_t^{(i)}(x_o)$ is the ith solution component $\varphi_t(x_o)$, or even $\varphi_t^{(i)}(x_o)$ is the solution of the equation

$$\frac{dx_i}{dt} = f_i(x), \quad x(0) = x_o. \tag{44}$$

By applying Zadeh's extension to $\varphi_t^{(i)}$, we have the application $\widehat{\varphi}_t^{(i)} : \mathcal{F}(U) \to \mathcal{F}(\mathbb{R})$ that for each $\mathbf{x}_o \in \mathcal{F}(U)$ associates the image $\widehat{\varphi}_t^{(i)}(\mathbf{x}_o) \in \mathcal{F}(\mathbb{R})$. As in the deterministic case, we show that the application $\widehat{\varphi}_t(i) : \mathcal{F}(U) \to \mathcal{F}(\mathbb{R})$ ia an ith fuzzy projection to fuzzy flow $\widehat{\varphi}_t : \mathcal{F}(U) \to \mathcal{F}(U)$ on $\mathcal{F}(\mathbb{R})$.

Proposition 5. *The application $\widehat{\varphi}_t^{(i)} : \mathcal{F}(U) \to \mathcal{F}(\mathbb{R})$ is ith fuzzy projection of fuzzy flow $\widehat{\varphi}_t : \mathcal{F}(U) \to \mathcal{F}(U)$ on $\mathcal{F}(\mathbb{R})$.*

Proof. Let $\mathbf{x}_o \in \mathcal{F}(U)$. By definition, ith fuzzy projection $\widehat{\varphi}_t(\mathbf{x}_o)$ on $\mathcal{F}(\mathbb{R})$ is the point $\mathbf{x}_i = \widehat{\pi}_i(\widehat{\varphi}_t(\mathbf{x}_o))$. Since the projection is a continuous map, then it is worth

$$\begin{aligned} [\mathbf{x}_i]^\alpha &= \pi_i\left(\varphi_t\left([\mathbf{x}_o]^\alpha\right)\right) = \left\{\pi_i\left(\varphi_t\left(x_o\right)\right) : x_o \in [\mathbf{x}_o]^\alpha\right\} \\ &= \left\{\varphi_t^{(i)}\left(x_o\right) : x_o \in [\mathbf{x}_o]^\alpha\right\} \\ &= \varphi_t^{(i)}\left([\mathbf{x}_o]^\alpha\right). \end{aligned} \tag{45}$$

Then, $[\widehat{\varphi}_t^{(i)}(\mathbf{x}_o)]^\alpha = [\mathbf{x}_i]^\alpha$ for all $\alpha \in [0, 1]$ and the assertion is proved. □

We showed in [3] that the equilibrium point x_e deterministic flow $\varphi_t : U \to U$ depends on the initial condition $x_o \in U$; then the equilibrium point for flow fuzzy $\widehat{\varphi}_t : \mathcal{F}(U) \to \mathcal{F}(U)$ is obtained by Zadeh's extension $x_e : U \to U$. Let $x_e^{(i)}(x_o)$ be an ith coordinated of equilibrium point x_e. Similarly, we can prove that ith projected of the equilibrium point fuzzy $\mathbf{x}_e = \widehat{x}_e(\mathbf{x}_o) \in \mathcal{F}(U)$ is the point $\overline{\mathbf{x}}_i = \widehat{x}_e^{(i)}(\mathbf{x}_o) \in \mathcal{F}(\mathbb{R})$ where $\widehat{x}_e^{(i)} : \mathcal{F}(U) \to \mathcal{F}(\mathbb{R})$ is a Zadeh's extension of $x_e^{(i)} : U \to \mathbb{R}$. More briefly, for $\mathbf{x}_o \in \mathcal{F}(U)$, the equality holds following:

$$\mu_{\overline{\mathbf{x}}_i}(x) = \mu_{\widehat{x}_e^{(i)}(\mathbf{x}_o)}(x), \tag{46}$$

where $\overline{\mathbf{x}}_i$ is ith fuzzy projection of the fuzzy equilibrium point $\mathbf{x}_e$.

Consider just a few examples of the results presented previously.

Example 6. The autonomous equation

$$\begin{aligned} \frac{dx_1}{dt} &= x_2, \quad x_1(0) = x_{01}, \\ \frac{dx_2}{dt} &= -x_1, \quad x_2(0) = x_{02}, \end{aligned} \tag{47}$$

determines the flow two-dimensional $\varphi_t : \mathbb{R}^2 \to \mathbb{R}^2$, $\varphi_t = (\varphi_t^{(1)}, \varphi_t^{(2)})$, given by

$$\begin{aligned} \varphi_t^{(1)}\left(x_{01}, x_{02}\right) &= x_{01}\cos t + x_{02}\operatorname{sen} t, \\ \varphi_t^{(2)}\left(x_{01}, x_{02}\right) &= x_{02}\cos t - x_{01}\operatorname{sen} t. \end{aligned} \tag{48}$$

We have already shown in [3] that the fuzzy solution $\widehat{\varphi}_t(\mathbf{x}_o)$ this equation is periodic for any choice of initial condition $\mathbf{x}_o \in \mathcal{F}(\mathbb{R}^2)$. According to the previous proposition, projections of fuzzy $\widehat{\varphi}_t : \mathcal{F}(\mathbb{R}^2) \to \mathcal{F}(\mathbb{R}^2)$ on $\mathcal{F}(\mathbb{R})$ are obtained by taking extensions of components Zadeh $\varphi_t^{(1)}$ and $\varphi_t^{(2)}$.

Figure 3 shows the time evolution of the fuzzy projection of $\widehat{\varphi}_t(\mathbf{x}_o)$ on x and y, respectively. Take the initial condition x_0 defined by the membership function.

$$\mu_{\mathbf{x}_o}(x, y) = \max\left\{1 - (x-3)^2 - y^2, 0\right\}. \tag{49}$$

Example 7. Consider the epidemiological model SI defined by equations

$$\begin{aligned} \frac{dS}{dt} &= -rSI, \quad S(0) = S_o > 0, \\ \frac{dI}{dt} &= rSI, \quad I(0) = I_o > 0, \\ S &= I + N. \end{aligned} \tag{50}$$

The solution of the model SI, defined by functions

$$\begin{aligned} \varphi_t^{(1)}\left(S_o, I_o\right) &= N_o\left(1 - \frac{I_o}{I_o + S_o e^{N_o rt}}\right), \\ \varphi_t^{(2)}\left(S_o, I_o\right) &= \frac{-N_o I_o}{I_o + S_o e^{-N_o rt}}, \end{aligned} \tag{51}$$

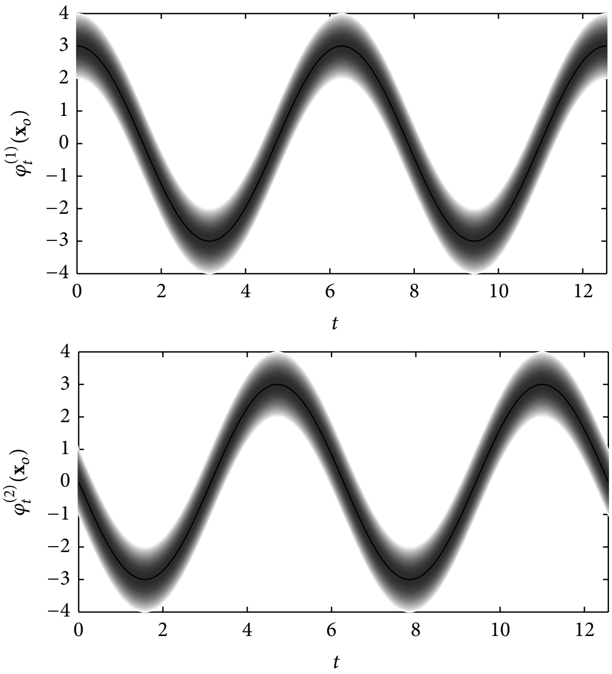


FIGURE 3: Time course of $\varphi_t^{(1)}(\mathbf{x}_o)$ and $\varphi_t^{(2)}(\mathbf{x}_o)$, respectively.

converges to the equilibrium point $x_e(S_o, I_o) = (0, S_o + I_o)$. According to what is discussed in [3], for all $\mathbf{x}_o \in \mathcal{F}(\mathbb{R}_+^2)$, the fuzzy solution $\widehat{\varphi}$ converges to the equilibrium point fuzzy $\mathbf{x}_e = \widehat{x}_e(\mathbf{x}_o)$.

According to the equality (46), projections of the equilibrium point $\mathbf{x}_e$ on the coordinate axis are obtained by extension of Zadeh components x_e. That is, the projections are fuzzy, respectively, $\overline{\mathbf{x}}_1 = \chi_{\{0\}}$ and $\overline{\mathbf{x}}$, whose membership function is given by

$$\mu_{\overline{\mathbf{x}}_2}(I) = \sup_{S_o+I_o=I} \mu_{\mathbf{x}_o}(S_o, I_o) = \sup_{S_o} \mu_{\mathbf{x}_o}(S_o, I - S_o). \tag{52}$$

By Proposition 5, fuzzy projections of fuzzy solution $\widehat{\varphi}_t(\mathbf{x}_o)$, on $\mathcal{F}(\mathbb{R})$, of model *SI* are obtained by extension of Zadeh, the components $\varphi_t^{(1)}$ and $\varphi_t^{(2)}$, given by

$$\begin{aligned} \varphi_t^{(1)}(S_o, I_o) &= N_o\left(1 - \frac{I_o}{I_o + S_o e^{N_o r t}}\right), \\ \varphi_t^{(2)}(S_o, I_o) &= \frac{N_o I_o}{I_o + S_o e^{N_o r t}}. \end{aligned} \tag{53}$$

To illustrate, suppose the *force infection* is $r = 0.01$, and we take the initial condition $\mathbf{x}_o \in \mathcal{F}(\mathbb{R}_+^2)$ defined by membership function

$$\mu_{\mathbf{x}_o}(S_o, I_o) = \max\left\{1 - 0.01(S_o - 80)^2 - 0.25(I_o - 5)^2, 0\right\}. \tag{54}$$

Figure 4 shows the evolution of applications $\widehat{\varphi}_t^{(1)}(\mathbf{x}_o)$ and $\widehat{\varphi}_t^{(2)}(\mathbf{x}_o)$ with the time evolution. Note that $\widehat{\varphi}_t^{(1)}(\mathbf{x}_o)$ converges to $\overline{\mathbf{x}}_1 = \chi_{\{0\}}$, whereas $\widehat{\varphi}_t^{(2)}(\mathbf{x}_o)$ converges to $\overline{\mathbf{x}}_2$ with the membership function given by (52).

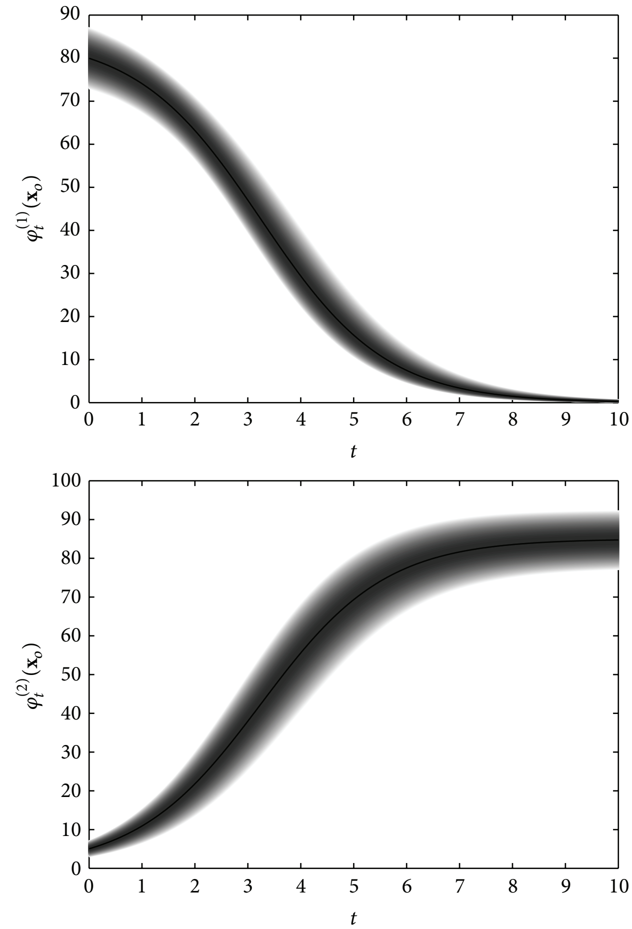


FIGURE 4: Time evolution of the fuzzy projection $\widehat{\varphi}_t(\mathbf{x}_o)$ on the axes S and I, respectively.

We also consider that the number of individuals in the population is known, say N. In this case, the variables S and R are related by equality $S + I = N$. Under this assumption, the deterministic solution converges to the point of equilibrium $x_e(S_o, I_o) = (0, N)$, and, therefore, the fuzzy solution converge to the equilibrium point fuzzy $\chi_{\{(0,N)\}}$. In this case, the projections $\widehat{\varphi}_t^{(1)}(\mathbf{x}_o)$ and $\widehat{\varphi}_t^{(2)}(\mathbf{x}_o)$ converges to $\chi_{\{0\}}$ and $\chi_{\{N\}}$, respectively.

In Figure 5, we plot the projections of the fuzzy solution $\varphi_t(\mathbf{x}_o)$, to the initial condition $I_o = 20$ and S_o given by fuzzy set

$$\mu_{\mathbf{x}_o}(S_o, I_o) = \begin{cases} \max\left\{1 - 0.01(S_o - 80)^2, 0\right\}, & \text{if } S_o + I_o = N, \\ 0 & \text{if } S_o + I_o \neq N. \end{cases} \tag{55}$$

The graphical representation of fuzzy projections of this work is established as follows: given an $\alpha \in [0, 1]$, the region in plan bounded by α-level $\widehat{\varphi}_{[0,T]}^{(i)}(\mathbf{x}_o)$ is filled with a shade of gray. If $\alpha = 0$, then the region bounded by $\widehat{\varphi}_{[0,T]}^{(i)}(\mathbf{x}_o)$ is filled with the white color, whereas if $\alpha = 1$, then the region

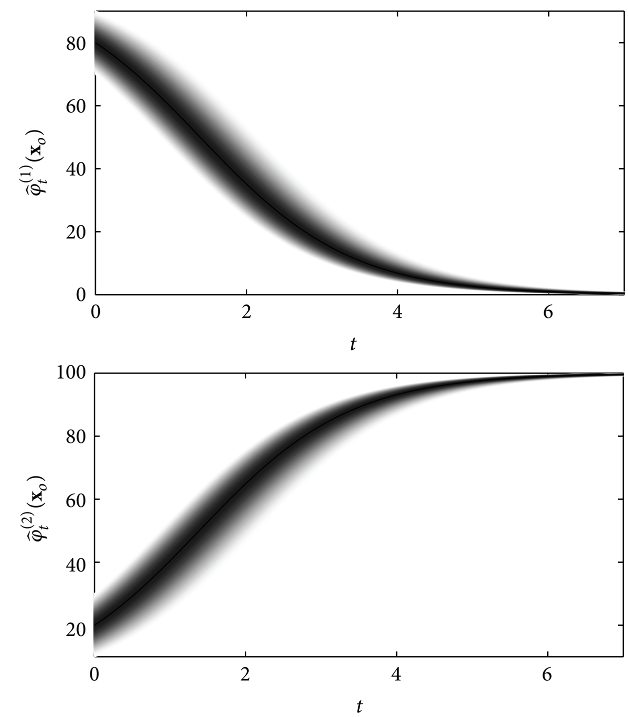


Figure 5: Time evolution of the fuzzy projection $\widehat{\varphi}_t(\mathbf{x}_o)$ on the axes S and I respectively.

bounded by $\widehat{\varphi}^{(i)}_{[0,T]}(\mathbf{x}_o)$ is filled with black. Thus, the larger the degree of membership of a point x, the darker its color.

4. Parameters and Initial Condition Fuzzy

In [2] the problem of uncertainty in the parameters of a given autonomous equation is solved using the strategy to consider such parameters as the initial condition of an equation with dimension higher than the original. More precisely, given an autonomous equation that depends on a parameter vector $p_o \in P \subset \mathbb{R}^m$

$$\frac{dx}{dt} = f(x, p_o), \quad x(0) = x_o \tag{56}$$

define the equation,

$$\begin{gathered}\frac{dx}{dt} = f(x, p), \quad x(0) = x_o, \\ \frac{dp}{dt} = 0, \quad p(0) = p_o,\end{gathered} \tag{57}$$

and thus, the parameter vector $p_o \in P \subset \mathbb{R}^m$ now is a part of the initial condition. Thus, Zadeh's extension $\widehat{\psi}_t : \mathcal{F}(U \times P) \to \mathcal{F}(U \times P)$ to the flow $\psi_t : U \times P \to U \times P$ generated by (57) incorporates the uncertainties of initial conditions and parameters of (56).

Once the solution $\varphi_t : U \times P \to U$ generated by (56) is continuous in the initial condition and parameters, Zadeh's extension $\widehat{\varphi}_t : \mathcal{F}(U \times P) \to \mathcal{F}(U)$ of $\varphi_t(x_o, p)$ is well defined, and according to (5), for all $\mathbf{y}_o \in \mathcal{F}(U \times P)$ we have:

$$[\widehat{\varphi}_t(\mathbf{y}_o)]^\alpha = \varphi_t([\mathbf{y}_o]^\alpha). \tag{58}$$

From the standpoint of applications, it is important to know the flow behavior of the deterministic phase space $U \subset \mathbb{R}^n$ of (56) instead of space $U \times P \subset \mathbb{R}^{n+m}$ to (57), since the flow components $\psi_t : U \times P \to U \times P$, that are $P \subset \mathbb{R}^m$, do not have any additional information. It is worth noting that, for all $y_o = (x_o, p_o)$, we have

$$P_n(\psi_t(y_o)) = P_n(\varphi_t(x_o, p_o), p_o) = \varphi_t(y_o). \tag{59}$$

Analogously to the deterministic case, we can also be interested only in the fuzzy flow behavior $\widehat{\psi}_t : \mathcal{F}(U \times P) \to \mathcal{F}(U \times P)$ on the phase space $\mathcal{F}(U)$. The fuzzy projections defined at the outset of this work can then be used to obtain the fuzzy flow behavior $\widehat{\psi}_t$ on the space $\mathcal{F}(U)$.

The following statement characterizes the relationship between the projection of fuzzy $\widehat{\psi}_t$ on the space $\mathcal{F}(U)$ and Zadeh's extension: $\widehat{\varphi}_t : \mathcal{F}(U \times P) \to \mathcal{F}(U)$ is a solution of (56).

Proposition 8. *The application $\widehat{\varphi}_t : \mathcal{F}(U \times P) \to \mathcal{F}(U)$, given by Zadeh's extension $\varphi_t : U \times P \to U$, is the fuzzy projection of fuzzy flow $\widehat{\psi}_t : \mathcal{F}(U \times P) \to \mathcal{F}(U \times P)$ on $\mathcal{F}(U)$.*

Proof. Let $\mathbf{y}_o \in \mathcal{F}(U \times P)$ and fix $t \geq 0$. To prove the claim, it suffices to show that $\mathbf{y}$ is the fuzzy projection of $\widehat{\psi}_t(\mathbf{y}_o)$ of $\mathcal{F}(U)$, then $\mathbf{y} = \widehat{\varphi}_t(\mathbf{y}_o)$.

To simplify, let $\operatorname{Im}(\varphi_t)$ be the image set of $\varphi_t : U \times P \to U$. By definition, the membership function of $\widehat{\varphi}_t(\mathbf{y}_o)$ is given by

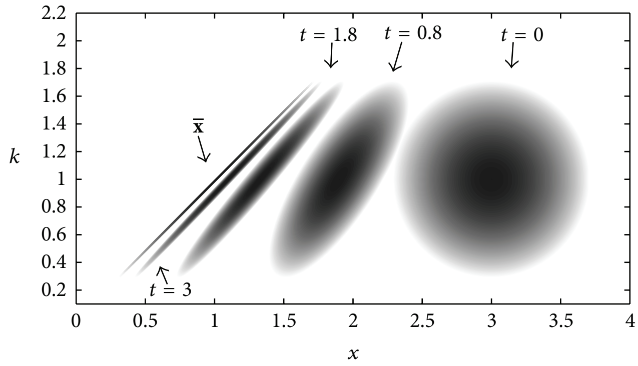
$$\mu_{\widehat{\varphi}_t(\mathbf{y}_o)}(x) = \begin{cases} \sup\limits_{\substack{(x_o, p_o) \\ \varphi_t(x_o, p_o) = x}} \mu_{\mathbf{y}_o}(x_o, p_o) & \text{if } x \in \operatorname{Im}(\varphi_t), \\ 0 & \text{if } x \notin \operatorname{Im}(\varphi_t). \end{cases} \tag{60}$$

Let $\mathbf{y} \in \mathcal{F}(U)$ be the projection of $\widehat{\psi}_t(\mathbf{y}_o)$ on $\mathcal{F}(U)$. By definition of fuzzy projection, the membership function of $\mathbf{y} \in \mathcal{F}(U)$ is given by

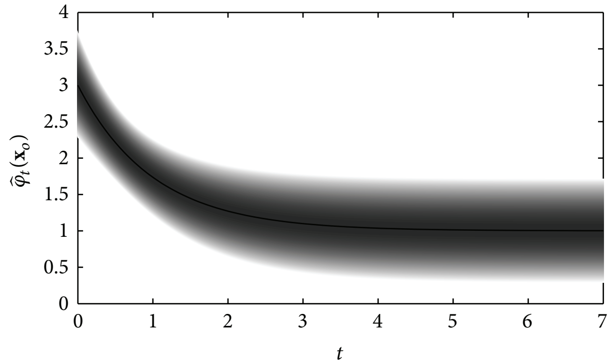
$$\mu_{\mathbf{y}}(x) = \sup_{p \in P} \mu_{\widehat{\psi}_t(\mathbf{y}_o)}(x, p). \tag{61}$$

Now, as $\psi_t(x_o, p_o) = (\varphi_t(x_o, p_o), p_o)$, so, $x \in \operatorname{Im}(\varphi_t)$ if and only if $(x, p) \in \operatorname{Im}(\psi_t)$ for some $p \in P$. So, for all $x \in \operatorname{Im}(\varphi_t)$, the membership function of $\widehat{\psi}_t(\mathbf{y}_o)$ is

$$\begin{aligned}\mu_{\widehat{\psi}_t(\mathbf{y}_o)}(x, p) &= \sup_{\psi_t(x_o, y) = (x, p)} \mu_{\mathbf{y}_o}(x_o, y) \\ &= \sup_{\substack{\varphi_t(x_o, y) = x \\ y = p}} \mu_{\mathbf{y}_o}(x_o, y) \\ &= \sup_{\substack{x_o \in U \\ \varphi_t(x_o, p) = x}} \mu_{\mathbf{y}_o}(x_o, p).\end{aligned} \tag{62}$$



(a)



(b)

FIGURE 6: (a) Evolution of $\widehat{\psi}_t(\mathbf{y}_o)$. (b) Evolution of fuzzy projection of $\widehat{\varphi}_t(\mathbf{y}_o)$.

If $x \notin \operatorname{Im}(\varphi_t)$, so, $(x,p) \notin \operatorname{Im}(\psi_t)$ for all $p \in P$, and so, $\mu_{\widehat{\psi}_t(\mathbf{y}_o)}(x,p) = 0$.

But the fuzzy projection of $\widehat{\psi}_t(\mathbf{y}_o)$ on $\mathcal{F}(U)$ has the membership function

$$\begin{aligned}\sup_{p\in P}\mu_{\widehat{\psi}_t(\mathbf{y}_o)}(x,p) &= \sup_{p\in P}\sup_{\substack{x_o\in U\\ \varphi_t(x_o,p)=x}}\mu_{\mathbf{y}_o}(x_o,p)\\ &= \sup_{\substack{(x_o,p)\\ \varphi_t(x_o,p)=x}}\mu_{\mathbf{y}_o}(x_o,p).\end{aligned} \tag{63}$$

Now, by definition, the point $\widehat{\varphi}_t(\mathbf{y}_o) \in \mathcal{F}(U)$ has the membership function

$$\mu_{\widehat{\varphi}_t(\mathbf{y}_o)}(x) = \sup_{\substack{(x_o,p)\\ \varphi_t(x_o,p)=x}}\mu_{\mathbf{y}_o}(x_o,p). \tag{64}$$

So, for all $x \in U$, the value equality is as follows:

$$\mu_{\mathbf{y}}(x) = \sup_{p\in P}\mu_{\widehat{\psi}_t(\mathbf{y}_o)}(x,p) = \mu_{\widehat{\varphi}_t(\mathbf{y}_o)}(x), \tag{65}$$

which proves the assertion. □

The proof of the proposition can also be made through the α-levels. In fact, we must show that

$$\widehat{\varphi}_t(\mathbf{y}_o) = \widehat{P}_n(\widehat{\psi}_t(\mathbf{y}_o)) \tag{66}$$

for all $\mathbf{y}_o \in \mathcal{F}(U\times P)$ and $t \in \mathbb{R}_+$. Using the continuity of applications P_n and ψ_t, we have

$$\begin{aligned}&\left[\widehat{P}_n(\widehat{\psi}_t(\mathbf{y}_o))\right]^\alpha\\ &\quad= P_n\left(\widehat{\psi}_t\left([\mathbf{y}_o]^\alpha\right)\right) = \left\{P_n(\widehat{\psi}_t(y_o)) : y_o \in [\mathbf{x}_o]^\alpha\right\}\\ &\quad= \left\{P_n(\varphi_t(x_o,p_o),p_o) : (x_o,p_o) \in [\mathbf{y}_o]^\alpha\right\}\\ &\quad= \left\{\varphi_t(x_o,p_o) : (x_o,p_o) \in [\mathbf{y}_o]^\alpha\right\}\\ &\quad= \varphi_t\left([\mathbf{y}_o]^\alpha\right),\end{aligned} \tag{67}$$

for all $\alpha \in [0,1]$. The previous equality concludes the proof proposition.

In contrast to [6, 7], when the equation depends on parameters such as (56), the fuzzy solution proposed by fuzzy Buckley and Feuring in [8] is obtained by Zadeh's extension flow deterministic $\varphi_t(x_o,p_o)$. This way, Proposition 8 ensures that the solution of fuzzy Buckley and Feuring is the fuzzy projection of the fuzzy solution proposed by [6, 7].

Consider that subjective parameters in (56) contributes to an increase in uncertainty. Set a parameter $\overline{p} \in P$, and given a fuzzy initial condition $\mathbf{x}_o$, the α-levels to the fuzzy flow generated by (56) are the sets

$$[\widehat{\varphi}_t(\mathbf{x}_o)]^\alpha = \left\{\varphi_t(x_o,\overline{p}) : x_o \in [\mathbf{x}_o]^\alpha\right\}. \tag{68}$$

On the other hand, if the α-levels of $\mathbf{p}_o \in \mathcal{F}(P)$ contain $\overline{p}$, so, by Proposition 8, we have

$$[\widehat{\varphi}_t(\mathbf{x}_o,\mathbf{p}_o)]^\alpha = \left\{\varphi_t(x_o,p_o) : (x_o,p_o) \in [\mathbf{x}_o]^\alpha\times[\mathbf{p}_o]^\alpha\right\}. \tag{69}$$

So, we have

$$[\widehat{\varphi}_t(\mathbf{x}_o)]^\alpha \subseteq [\widehat{\varphi}_t(\mathbf{x}_o,\mathbf{p}_o)]^\alpha. \tag{70}$$

Example 9. Consider the case where the parameter k_o in the equation

$$\frac{dx}{dt} = \beta(k_o - x) \tag{71}$$

is a fuzzy parameter. In the previous equation, the solution $\varphi_t : \mathbb{R}^2 \to \mathbb{R}$, in terms of x_o and k_o, is given by

$$\varphi_t(x_o,k_o) = k_o + (x_o - k_o)e^{-\beta t}, \tag{72}$$

and thus the flow 2-dimensional $\psi_t : \mathbb{R}^2 \to \mathbb{R}^2$, for the case in which the parameter is incorporated into the initial condition, is given by

$$\psi_t(x_o,k_o) = \left(k_o + (x_o - k_o)e^{-\beta t}, k_o\right). \tag{73}$$

According to Proposition 8, Zadeh's extension $\widehat{\varphi}_t : \mathcal{F}(\mathbb{R}^2) \to \mathcal{F}(\mathbb{R})$ of φ_t is the projection of $\mathcal{F}(\mathbb{R})$ fuzzy flow $\widehat{\psi}_t : \mathcal{F}(\mathbb{R}^2) \to \mathcal{F}(\mathbb{R}^2)$. To illustrate, consider $\mathbf{y}_o \in \mathcal{F}(\mathbb{R}^2)$. By definition, we have

$$\widehat{\varphi}_t\left([\mathbf{y}_o]^\alpha\right) = \left\{\varphi_t(x_o,p_o) : (x_o,p_o) \in [\mathbf{y}_o]^\alpha\right\}. \tag{74}$$

Moreover, the projection $\widehat{P}_1(\widehat{\psi}_t(\mathbf{y}_o))$ has α-levels given by

$$\begin{aligned}
&\left[\widehat{P}_n\left(\widehat{\psi}_t\left(\mathbf{y}_o\right)\right)\right]^\alpha \\
&\quad = P_1\left(\psi_t\left(\left[\mathbf{y}_o\right]^\alpha\right)\right) \\
&\quad = \left\{P_1\left(\psi_t\left(x_o, p_o\right)\right) : \left(x_o, p_o\right) \in \left[\mathbf{y}_o\right]^\alpha\right\} \\
&\quad = \left\{P_1\left(\varphi_t\left(x_o, p_o\right), p_o\right) : \left(x_o, p_o\right) \in \left[\mathbf{y}_o\right]^\alpha\right\} \\
&\quad = \left\{\varphi_t\left(x_o, p_o\right) : \left(x_o, p_o\right) \in \left[\mathbf{y}_o\right]^\alpha\right\},
\end{aligned} \tag{75}$$

from which we conclude that $[\widehat{P}_n(\widehat{\psi}_t(\mathbf{y}_o))]^\alpha = \widehat{\varphi}_t([\mathbf{y}_o]^\alpha)$, and consequently,

$$\widehat{P}_1\left(\widehat{\psi}_t\left(\mathbf{y}_o\right)\right) = \widehat{\varphi}_t\left(\mathbf{y}_o\right). \tag{76}$$

For any initial condition $\mathbf{y}_o \in \mathcal{F}(\mathbb{R}^2)$, we show that $\widehat{\psi}_t$ converges to the equilibrium points $\mathbf{y}_e$ which is Zadeh's extension $y_e : \mathbb{R}^2 \to \mathbb{R}^2$ given by $y_e(x_o, k_o) = (k_o, k_o)$. That is, the equilibrium point $\mathbf{y}_e$ has membership function

$$\mu_{\mathbf{y}_e}(x, k) = \min\left\{\chi_{\{k\}}(x), \sup_{x_o} \mu_{\mathbf{y}_o}(x_o, k)\right\}. \tag{77}$$

In particular, if $\mathbf{y}_o = (\mathbf{x}_o, \mathbf{k}_o)$ is the fuzzy cartesian product of $\mathbf{x}_o$ and $\mathbf{k}_o \in \mathcal{F}(\mathbb{R})$, then the membership function in this case is given by

$$\begin{aligned}
\mu_{\mathbf{y}_e}(x, k) &= \sup_{x_o} \mu_{\mathbf{y}_o}(x_o, k) = \sup_{x_o} \Delta\left(\mu_{\mathbf{x}_o}(x_o), \mu_{\mathbf{k}_o}(x)\right) \\
&= \mu_{\mathbf{k}_o}(x)
\end{aligned} \tag{78}$$

when $x = k$ and $\mu_{\mathbf{y}_e}(x, k) = 0$, when $x \neq k$.

The projection $\overline{\mathbf{x}} \in \mathcal{F}(\mathbb{R})$ for this equilibrium point has membership function

$$\mu_{\overline{\mathbf{x}}}(x) = \sup_{k \in \mathbb{R}} \mu_{\mathbf{y}_e}(x, k) = \mu_{\mathbf{k}_o}(x), \tag{79}$$

and we have $d_\infty(\widehat{\varphi}_t(\mathbf{y}_o), \overline{\mathbf{x}}) \to 0$ as $t \to \infty$.

In Figure 6, we have the graphical representation of the fuzzy solution $\widehat{\psi}_t(\mathbf{y}_o)$ and its fuzzy projection $\widehat{\varphi}_t(\mathbf{y}_o)$.

5. Conclusions

In this paper, we define the concept of fuzzy projections and study some of its main properties, in addition to establishing some results on projections of fuzzy differential equations. As we have seen, different concepts of fuzzy solutions of differential equations are related by fuzzy projections. Importantly, by means of fuzzy projections, we can analyze the evolution of fuzzy solutions over time.

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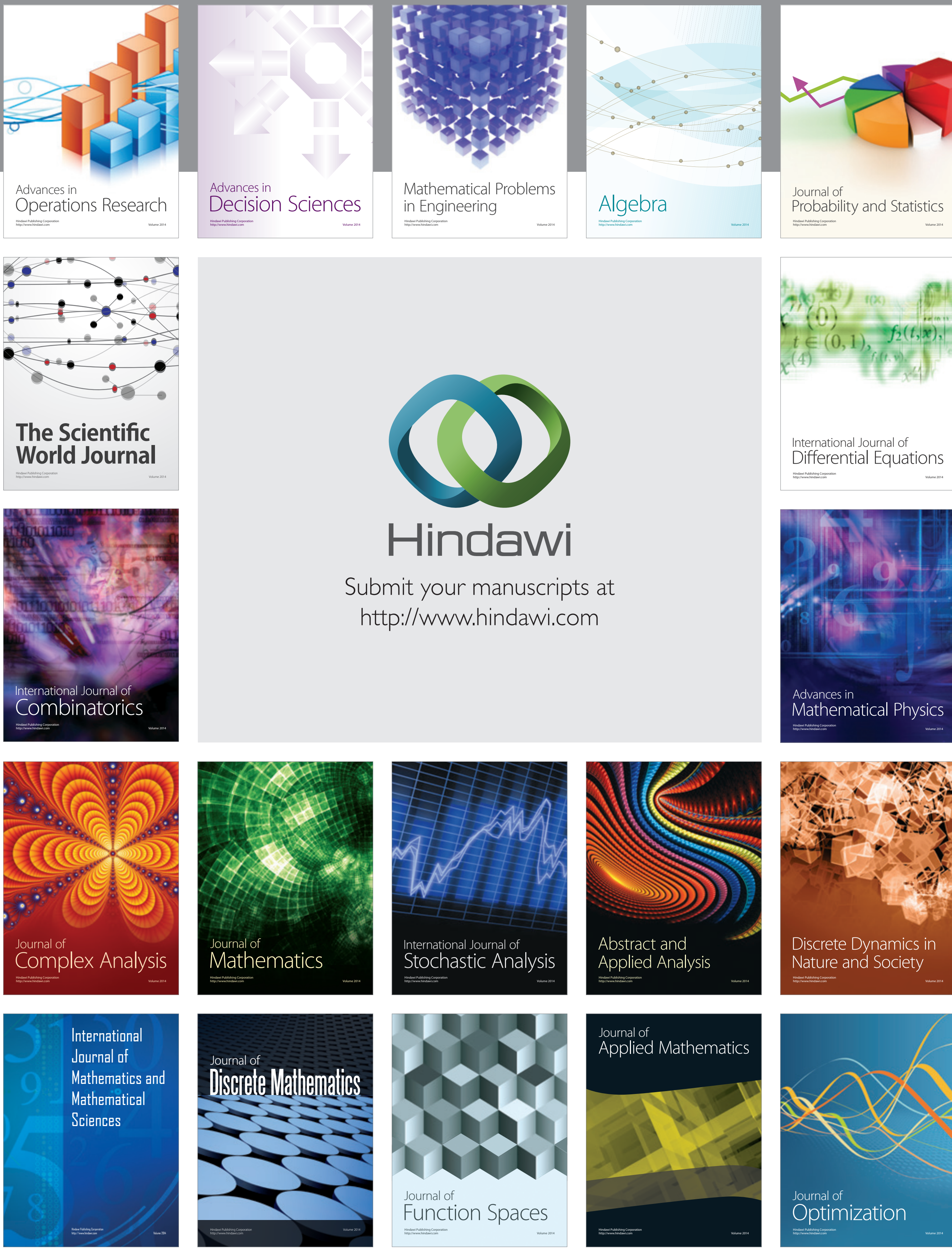
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