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Research Article

Existence and Numerical Simulation of Solutions for Fractional Equations Involving Two Fractional Orders with Nonlocal Boundary Conditions

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We study a boundary value problem for fractional equations involving two fractional orders. By means of a fixed point theorem, we establish sufficient conditions for the existence and uniqueness of solutions for the fractional equations. In addition, we describe the dynamic behaviors of the fractional Langevin equation by using the G_2 algorithm.

1. Introduction

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in various fields, such as physics, mechanics, chemical technology, population dynamics, biotechnology, and economics (see, e.g., [1–7]). As one of the important topics in the research on differential equations, the boundary value problem has attained a great deal of attention from many researchers (see [8–18]) and the references therein. As pointed out in [19], the nonlocal boundary condition can be more useful than the standard condition to describe some physical phenomena. There are several noteworthy papers (see [20–22]) dealing with nonlocal boundary value problems of fractional differential equations.

In [19], Benchohra et al. investigated the existence and uniqueness of the solutions for the differential equations with nonlocal conditions:

$$\begin{aligned} {}^{c}D^{\alpha}u(t) + f(t,u(t)) = 0, \quad & 1 < \alpha \le 2,\ 0 < t < T, \\ u(0) = g(u), \quad & u(T) = u_T, \end{aligned} \tag{1}$$

where ${}^{c}D^{\alpha}$ denotes Caputo's fractional derivative of order α with the lower limit zero.

In [22], Zhong and Lin studied the existence and uniqueness of solutions in the nonlocal and multiple-point boundary value problem for fractional differential equation:

$$\begin{aligned} & {}^{c}D^{q}u(t) + f(t,u(t)) = 0, \quad 0 < t < 1,\ 1 < q \le 2, \\ & u(0) = u_0 + g(u), \qquad u'(1) = u_1 + \sum_{i=1}^{m-2} b_i u'(\xi_i), \end{aligned} \tag{2}$$

where ${}^{c}D^{q}$ denotes Caputo's fractional derivative of order q with the lower limit zero.

In this paper we will study the fractional Langevin equation where the fractional derivative is in Caputo sense. In 1908 the French physicist Langevin introduced the concept of the equation of motion with a random variable, which reads as

$$m\frac{d^2x(t)}{dt^2} = -\gamma\frac{dx(t)}{dt} + F(t) + \xi(t), \tag{3}$$

where m is the mass of the particle, γ is the coefficient of viscosity, $F(x)$ is the external force, and $\xi(t)$ is the random force. The Langevin equation is always regarded as the first stochastic differential equation.

Langevin equation has been widely used to describe the evolution of physical phenomena in fluctuating environm-

ents [23–25]. For instance, Brownian motion is well described by the Langevin equation when the random fluctuation force is assumed to be white noise. In case the random fluctuation force is not white noise, the motion of the particle is described by the generalized Langevin equation [26]. For systems in complex media, ordinary Langevin equation does not provide the correct description of the dynamics. Various generalizations of Langevin equations have been proposed to describe dynamical processes in a fractal medium. One such generalization is the generalized Langevin equation [27–32] which incorporates the fractal and memory properties with a dissipative memory kernel into the Langevin equation.

Fractional order models are more accurate than integer-order models as fractional order models allow more degrees of freedom. The presence of memory term in such models not only takes into account the history of the process involved but also carries its impact to present and future development of the process. Fractional differential equations are also regarded as an alternative model to nonlinear differential equations [33]. In consequence, the subject of fractional differential equations is gaining much importance and attention. For some recent work on fractional differential equations, see [1, 34–46].

In [47], Ahmad et al. studied nonlinear Langevin equation involving two fractional orders in different intervals:

$$\begin{aligned} {}^{c}D^{\beta}\left({}^{c}D^{\alpha}+\lambda\right)x(t) &= f\left(t,x(t),x'(t)\right), \\ 0<t<1,\quad 1<\alpha\le 2,\quad 0<\beta\le 1,& \\ x(0)=0,\quad x(\eta)=0,\quad x(1)=0,\quad 0<\eta<1,& \end{aligned} \tag{4}$$

where ${}^{c}D^{\alpha}$ and ${}^{c}D^{\beta}$ denote Caputo's fractional derivative of order α and β with the lower limit zero.

In [48], A. Chen and Y. Chen studied existence of solutions to nonlinear Langevin equation involving two fractional orders with boundary value conditions:

$$\begin{aligned} {}^{c}D^{\beta}\left({}^{c}D^{\alpha}+\lambda\right)u(t) &= f\left(t,u(t),u'(t)\right), \\ 0<t<T,\quad 0<\alpha\le 1,\quad 1<\beta\le 2,& \\ u(0)=-u(T),\qquad u'(0)=u'(T)=0,& \end{aligned} \tag{5}$$

where ${}^{c}D^{\alpha}$ and ${}^{c}D^{\beta}$ denote Caputo's fractional derivative of order α and β with the lower limit zero.

The fractional calculus has been studied for more than three hundred years. In recent few decades, the fractional calculus has been widely used in many fields such as chaotic dynamics, viscoelasticity, acoustics, and physical chemistry. In [49], Guo studied the numerical solution of fractional partial differential equations. In [50], Guo studied the numerical simulation of the fractional Langevin equation.

As far as we know, there are no papers discussing the existence and numerical simulation of solutions for fractional equations involving two fractional orders with nonlocal boundary conditions.

Motivated by the works mentioned above, in this paper, we establish the existence and uniqueness of solutions by the fixed point theorem and use G_2 algorithm to describe the dynamic behaviors for the following problem:

$$\begin{aligned} {}^{c}D^{\beta}\left({}^{c}D^{\alpha}+\lambda\right)u(t) &= f\left(t,u(t),u'(t)\right), \\ 0<t<1,\quad 1<\alpha\le 2,\quad 0<\beta\le 1,& \\ \alpha_1 u(0)+\beta_1 u(1)=g_1(u),& \\ \alpha_2 u'(0)+\beta_2 u'(1)=g_2(u),& \\ u(0)=\eta u'(0),\quad \eta\neq 0,& \end{aligned} \tag{6}$$

where ${}^{c}D^{\alpha}$ and ${}^{c}D^{\beta}$ denote Caputo's fractional derivative of order α and β with the lower limit zero, $f:[0,1]\times R^2\to R$ is a given continuous function and λ is a real number, and $g_1, g_2 : C([0,1],R)\to R$ are two continuous functions, $\alpha\beta_2[(\alpha_1+\beta_1)\eta+\beta_1]\neq\beta_1(\alpha_2+\beta_2)$. Evidently, problem (6) not only includes boundary value problems mentioned above but also extends them to a much wider case.

The organization of this paper is as follows. In Section 2, we will give some lemmas which are essential to prove our main results. In Section 3, main results are given. In Section 4, we will give the numerical simulation for the fractional Langevin equation.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts. Throughout this paper, set $\mathscr{C}=C([0,1],R)$ denotes the Banach space of all continuous functions from $[0,1]\to R$ with the norm $\|x\|_{\mathscr{C}}=\sup_{t\in[0,1]}|x(t)|$. We also introduce the Banach space $u\in C^1([0,1],R)$ endowed with the norm defined by $\|u\|_{C^1}=\max\{\sup_{t\in[0,1]}|u(t)|, \sup_{t\in[0,1]}|u'(t)|\}$.

For the convenience of the readers, let us recall the following useful definitions and fundamental facts of fractional calculus theory.

Definition 1 (see [1, 6]). The Riemann-Liouville derivative of order γ with the lower limit zero for a function $f:[0,\infty)\to R$ can be written as

$${}^{L}D^{\gamma}f(t)=\frac{1}{\Gamma(n-\gamma)}\frac{d^n}{dt^n}\int_0^t\frac{f(s)}{(t-s)^{\gamma+1-n}}ds, \quad t>0,\quad n-1<\gamma<n. \tag{7}$$

Definition 2 (see [1, 6]). The fractional (arbitrary) order integral of the function $f:[0,\infty)\to R$ of order $p>0$ is defined by

$$I^{p}f(x)=\frac{1}{\Gamma(p)}\int_0^x(x-s)^{p-1}f(s)\,ds. \tag{8}$$

Definition 3 (see [1]). Let $\alpha\ge 0$, $n=[\alpha]+1$. If $f\in AC^n[a,b]$, the Caputo fractional derivative of order α of f is defined by

$${}^{c}D^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t\frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}}ds, \quad t>0,\quad n-1<\alpha<n. \tag{9}$$

Definition 4 (see [6]). Let $p \in (n-1, n]$, $n \in N$ and the the Grünwald-Letnikov fractional derivative of order p of f defined by

$$D^p f(t) = \lim_{\substack{h \to 0 \\ nh = t-a}} \frac{1}{h^p} \sum_{r=0}^{n} (-1)^r \binom{p}{r} f(t - rh), \tag{10}$$

where $\binom{p}{r} = p(p-1)(p-2)\cdots(p-r+1)/r!$.

Lemma 5 (see [1]). *Let $p \in (m-1, m]$, $m \in N$ and the Caputo derivative of order p for a function $f : [0, \infty) \to R$. If for $t \in [0,1]$, $f \in AC^m[0,1]$ or $f \in C^m[0,1]$,*

$$I^p {}^c D^p f(t) = f(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} f^{(k)}(0). \tag{11}$$

We also easily prove the following lemmas.

Lemma 6. *Let $\sigma \in L^q([0,1], R)$, $q > 1/(\alpha+\beta)$. ${}^c D^\alpha u \in C^1([0,1], R)$, $u \in C^2([0,1], R)$ satisfying the following differential equation:*

$$\begin{aligned} &{}^c D^\beta \left({}^c D^\alpha + \lambda \right) u(t) = \sigma(t), \\ &0 < t < 1, \quad 1 < \alpha \le 2, \quad 0 < \beta \le 1, \\ &\alpha_1 u(0) + \beta_1 u(1) = g_1(u), \\ &\alpha_2 u'(0) + \beta_2 u'(1) = g_2(u), \\ &u(0) = \eta u'(0), \quad \eta \ne 0, \end{aligned} \tag{12}$$

is a solution of the following integral equation:

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \\ &\quad \times \left[\frac{1}{\Gamma(\beta)} \int_0^\tau (\tau - s)^{\beta-1} \sigma(s)\, ds - \lambda u(\tau) \right] d\tau \\ &\quad + \frac{\alpha\beta_2(\eta + t) - t^\alpha(\alpha_2 + \beta_2)}{\alpha\beta_2[(\alpha_1 + \beta_1)\eta + \beta_1] - \beta_1(\alpha_2 + \beta_2)} g_1(u) \\ &\quad - \frac{\beta_1(\eta + t) - t^\alpha[(\alpha_1 + \beta_1)\eta + \beta_1]}{\alpha\beta_2[(\alpha_1 + \beta_1)\eta + \beta_1] - \beta_1(\alpha_2 + \beta_2)} g_2(u) \\ &\quad + \frac{\beta_1 t^\alpha(\alpha_2 + \beta_2) - \alpha\beta_1\beta_2(\eta + t)}{\Gamma(\alpha)[\alpha\beta_2((\alpha_1 + \beta_1)\eta + \beta_1) - \beta_1(\alpha_2 + \beta_2)]} \\ &\quad \times \int_0^1 (1-\tau)^{\alpha-1} \\ &\quad \times \left[\frac{1}{\Gamma(\beta)} \int_0^\tau (\tau - s)^{\beta-1} \sigma(s)\, ds - \lambda u(\tau) \right] d\tau \\ &\quad + \frac{\beta_2[\beta_1(\eta + t) - t^\alpha((\alpha_1 + \beta_1)\eta + \beta_1)]}{\Gamma(\alpha - 1)[\alpha\beta_2((\alpha_1 + \beta_1)\eta + \beta_1) - \beta_1(\alpha_2 + \beta_2)]} \\ &\quad \times \int_0^1 (1-\tau)^{\alpha-2} \\ &\quad \times \left[\frac{1}{\Gamma(\beta)} \int_0^\tau (\tau - s)^{\beta-1} \sigma(s)\, ds - \lambda u(\tau) \right] d\tau. \end{aligned} \tag{13}$$

Proof. According to Lemma 5 and applying the operator I^β to both sides of (12), for some constants c_0, c_1, and c_2, we get

$$I^\beta {}^c D^\beta \left({}^c D^\alpha + \lambda \right) u(t) = I^\beta \sigma(t); \tag{14}$$

then the above equation can be written as

$$\left({}^c D^\alpha + \lambda \right) u(t) = I^\beta \sigma(t) + c_0, \tag{15}$$

and applying the operator I^α to both sides of the above equation, we obtain

$$I^\alpha D^\alpha u(t) = I^\alpha \left[I^\beta \sigma(t) - \lambda u(t) \right] + I^\alpha c_0; \tag{16}$$

then the above equation can be written as

$$u(t) = I^\alpha \left[I^\beta \sigma(t) - \lambda u(t) \right] + I^\alpha c_0 + c_1 + c_2 t, \tag{17}$$

that can be written as (13). The proof is completed. □

Definition 7. The function $u \in C^1([0,1], R)$ satisfying (13) is a generalized solution of the nonlocal boundary value problem (6).

Lemma 8 (Krasnoselskii). *Let $\mathfrak{B}$ be a closed convex and nonempty subset of X. Suppose that $\mathscr{L}$ and $\mathscr{N}$ are general nonlinear operators which map $\mathfrak{B}$ into X such that*

(1) *$\mathscr{L}x + \mathscr{N}y \in \mathfrak{B}$ whenever $x, y \in \mathfrak{B}$;*

(2) *$\mathscr{L}$ is a contraction mapping;*

(3) *$\mathscr{N}$ is compact and continuous.*

Then there exists $z \in \mathfrak{B}$ such that $z = \mathscr{L}z + \mathscr{N}z$.

3. Main Results

In order to apply Lemma 8 to prove our main results, we first give F, S, T as follows. Let $\overline{\Omega}_r = \{u \in C^1([0,1], R) : \|u\|_{C^1} \le r\}$, $r > 0$.

Define an operator $F : C^1 \to C^1$ by

$$(Fu)(t) = (Su)(t) + (Tu)(t),$$

$$\begin{aligned}
&(Su)(t) \\
&= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \\
&\quad \times \left[\frac{1}{\Gamma(\beta)} \int_0^\tau (\tau - s)^{\beta-1} f\left(s, u(s), u'(s)\right) ds \right. \\
&\quad \left. -\lambda u(\tau) \right] d\tau,
\end{aligned}$$

$$\begin{aligned}
&(Tu)(t) \\
&= \frac{\alpha\beta_2(\eta+t) - t^\alpha(\alpha_2+\beta_2)}{\alpha\beta_2[(\alpha_1+\beta_1)\eta+\beta_1] - \beta_1(\alpha_2+\beta_2)} g_1(u) \\
&\quad - \frac{\beta_1(\eta+t) - t^\alpha[(\alpha_1+\beta_1)\eta+\beta_1]}{\alpha\beta_2[(\alpha_1+\beta_1)\eta+\beta_1] - \beta_1(\alpha_2+\beta_2)} g_2(u) \\
&\quad + \frac{\beta_1 t^\alpha(\alpha_2+\beta_2) - \alpha\beta_1\beta_2(\eta+t)}{\Gamma(\alpha)[\alpha\beta_2((\alpha_1+\beta_1)\eta+\beta_1) - \beta_1(\alpha_2+\beta_2)]} \\
&\quad \times \int_0^1 (1-\tau)^{\alpha-1} \\
&\qquad \times \left[\frac{1}{\Gamma(\beta)} \int_0^\tau (\tau - s)^{\beta-1} f\left(s, u(s), u'(s)\right) ds \right. \\
&\qquad \left. -\lambda u(\tau) \right] d\tau \\
&\quad + \frac{\beta_2[\beta_1(\eta+t) - t^\alpha((\alpha_1+\beta_1)\eta+\beta_1)]}{\Gamma(\alpha-1)[\alpha\beta_2((\alpha_1+\beta_1)\eta+\beta_1) - \beta_1(\alpha_2+\beta_2)]} \\
&\quad \times \int_0^1 (1-\tau)^{\alpha-2} \\
&\qquad \times \left[\frac{1}{\Gamma(\beta)} \int_0^\tau (\tau - s)^{\beta-1} f\left(s, u(s), u'(s)\right) ds \right. \\
&\qquad \left. -\lambda u(\tau) \right] d\tau.
\end{aligned} \tag{18}$$

Lemma 9. *The function $u \in C^1([0,1], R)$ is a generalized solution of the nonlocal boundary value problem* (6) *if $Fu(t) = u(t)$, for all $t \in [0,1]$.*

Proof. Firstly, we show that $u \in C^1$.

Assuming $u \in C^1$ is a generalized solution of the problem (6), there exist three constants c_0, c_1, and c_2. Equation (13) can be written as

$$\begin{aligned}
u(t) &= I^{\alpha+\beta} f\left(t, u(t), u'(t)\right) - \lambda I^\alpha u(t) \\
&\quad + \frac{c_0 t^\alpha}{\Gamma(1+\alpha)} + c_1 + c_2 t,
\end{aligned} \tag{19}$$

and differentiating both sides of the above equation, we get

$$\begin{aligned}
u'(t) &= I^{\alpha+\beta-1} f\left(t, u(t), u'(t)\right) - \lambda I^{\alpha-1} u(t) \\
&\quad + \frac{c_0 t^{\alpha-1}}{\Gamma(\alpha)} + c_2.
\end{aligned} \tag{20}$$

It is clear that every term of the above equation belongs to C; then $u \in C^1$.

Secondly, we show that u is the generalized solution of the problem (6).

Let u be a generalized solution of the problem (6) and

$$\begin{aligned}
u(t) &= I^{\alpha+\beta} f\left(t, u(t), u'(t)\right) - \lambda I^\alpha u(t) + \frac{c_0 t^\alpha}{\Gamma(1+\alpha)} \\
&\quad + c_1 + c_2 t.
\end{aligned} \tag{21}$$

Applying the operator ${}^cD^\alpha$ to both sides of the above equation, we obtain

$$\begin{aligned}
{}^cD^\alpha u(t) &= {}^cD^\alpha \left[I^{\alpha+\beta} f\left(t, u(t), u'(t)\right) - \lambda I^\alpha u(t) \right. \\
&\qquad \left. + \frac{c_0 t^\alpha}{\Gamma(1+\alpha)} + c_1 + c_2 t \right] \\
&= {}^cD^\alpha I^{\alpha+\beta} f\left(t, u(t), u'(t)\right) \\
&\quad - \lambda\, {}^cD^\alpha I^\alpha u(t) + {}^cD^\alpha \frac{c_0 t^\alpha}{\Gamma(1+\alpha)} \\
&\quad + {}^cD^\alpha c_1 + {}^cD^\alpha c_2 t \\
&= I^\beta f\left(t, u(t), u'(t)\right) - \lambda u(t), \\
\left({}^cD^\alpha + \lambda\right) u(t) &= I^\beta f\left(t, u(t), u'(t)\right),
\end{aligned} \tag{22}$$

and then applying the operator ${}^cD^\beta$ to both sides of the above equation, we obtain

$$\begin{aligned}
{}^cD^\beta \left({}^cD^\alpha + \lambda\right) u(t) &= {}^cD^\beta I^\beta f\left(t, u(t), u'(t)\right) \\
&= f\left(t, u(t), u'(t)\right).
\end{aligned} \tag{23}$$

By simple calculations, it is clear that u satisfies conditions (6); then it is a generalized solution for the problem (6). The proof is completed. □

For convenience, let us set

$$\begin{aligned}
\Lambda_1 &= (\alpha_1+\beta_1)\eta+\beta_1,\\
\Lambda_2 &= \frac{1}{\alpha\beta_2\left[(\alpha_1+\beta_1)\eta+\beta_1\right]-\beta_1(\alpha_2+\beta_2)},\\
\Lambda_3 &= \int_0^1 (1-\tau)^{\alpha-1}\\
&\quad\times\left[\frac{1}{\Gamma(\beta)}\int_0^\tau(\tau-s)^{\beta-1}f\left(s,u(s),u'(s)\right)ds\right.\\
&\quad\left.-\lambda u(\tau)\right]d\tau,\\
\Lambda_4 &= \int_0^1 (1-\tau)^{\alpha-2}\\
&\quad\times\left[\frac{1}{\Gamma(\beta)}\int_0^\tau(\tau-s)^{\beta-1}f\left(s,u(s),u'(s)\right)ds\right.\\
&\quad\left.-\lambda u(\tau)\right]d\tau,\\
\Lambda_5 &= \int_0^1 (1-\tau)^{\alpha-1}\\
&\quad\times\left[\frac{1}{\Gamma(\beta)}\int_0^\tau(\tau-s)^{\beta-1}f\left(s,v(s),v'(s)\right)ds\right.\\
&\quad\left.-\lambda v(\tau)\right]d\tau,\\
\Lambda_6 &= \int_0^1 (1-\tau)^{\alpha-2}\\
&\quad\times\left[\frac{1}{\Gamma(\beta)}\int_0^\tau(\tau-s)^{\beta-1}f\left(s,v(s),v'(s)\right)ds\right.\\
&\quad\left.-\lambda v(\tau)\right]d\tau.
\end{aligned}\tag{24}$$

Clearly, for any $t\in[0,1]$,

$$\begin{aligned}
&(Su)'(t)\\
&=\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-\tau)^{\alpha-2}\\
&\quad\times\left[\frac{1}{\Gamma(\beta)}\int_0^\tau(\tau-s)^{\beta-1}f\left(s,u(s),u'(s)\right)ds\right.\\
&\quad\left.-\lambda u(\tau)\right]d\tau,
\end{aligned}$$

$$\begin{aligned}
&(Tu)'(t)\\
&=\left(\alpha\beta_2-\alpha t^{\alpha-1}(\alpha_2+\beta_2)\right)\Lambda_2 g_1(u)\\
&\quad+\left(-\beta_1+\alpha t^{\alpha-1}\Lambda_1\right)\Lambda_2 g_2(u)\\
&\quad+\frac{\left[\alpha\beta_1 t^{\alpha-1}(\alpha_2+\beta_2)-\alpha\beta_1\beta_2\right]\Lambda_2\Lambda_3}{\Gamma(\alpha)}\\
&\quad+\frac{\beta_2\left(\beta_1-\alpha t^{\alpha-1}\Lambda_1\right)\Lambda_2\Lambda_4}{\Gamma(\alpha-1)}.
\end{aligned}\tag{25}$$

Now, we make the following hypotheses.

(H1) There exist two real-valued functions $g\in L^{1/r}([0,1],R)$ for some $r\in(0,1)$, such that

$$\begin{aligned}
&\left|f\left(t,u,u'\right)-f\left(t,v,v'\right)\right|\\
&\quad\le 2g(t)\max\left\{|u-v|,\left|u'-v'\right|\right\},
\end{aligned}\tag{26}$$

for almost all $t\in[0,1]$, $u,v\in R$.

(H2) There exist two positive constants l_1, l_2 such that $l_1+l_2=L<1$. Moreover, $g_1(0)=0$, $g_2(0)=0$ and

$$\begin{aligned}
&\left|g_1(u)-g_1(v)\right|\le l_1\|u-v\|_{C^1},\\
&\left|g_2(u)-g_2(v)\right|\le l_2\|u-v\|_{C^1},\\
&\qquad\forall u,v\in C^1([0,1],R).
\end{aligned}\tag{27}$$

Theorem 10. *Let $f:[0,1]\times R\times R\to R$ be a jointly continuous function and the assumptions (H1) and (H2) hold. In addition, assume that*

$$\begin{aligned}
\Lambda \triangleq\ &\max\{\Upsilon_1+(\eta+1)\\
&\times(\alpha\beta_2\Lambda_2 L+\beta_1\Lambda_2 L+\beta_1\beta_2\Lambda_2\Upsilon_2)\\
&+\beta_1(\alpha_2+\beta_2)\Lambda_2\Upsilon_1,\Upsilon_2\\
&+(\alpha_2+\beta_2)\alpha\beta_1\Lambda_2\Upsilon_1+\alpha\beta_2\Lambda_2 L\\
&+\alpha\Lambda_1\Lambda_2 L+\beta_1\beta_2\Lambda_2\Upsilon_2\}<1,
\end{aligned}\tag{28}$$

where $p\in(0,1)$, $1<\alpha\le 2$, $0<\beta\le 1$, $\eta\neq 0$, $g^*=(\int_0^1(g(s))^{1/(1-p)}ds)^{1-p}$.

Then the problem (6) has at most one solution.

Proof. The proof will be given in two steps.

Step 1. F is bounded.

Now we show that $F\overline{\Omega}_r\subset\overline{\Omega}_r$.

Let $M = \sup_{s\in[0,1]}|f(s,0,0)|$. For any $u \in \overline{\Omega}_r$, we have

$$
\begin{aligned}
|\Lambda_3| = {} & \left| \int_0^1 (1-\tau)^{\alpha-1} \right. \\
& \times \left[\frac{1}{\Gamma(\beta)} \int_0^\tau (\tau - s)^{\beta-1} f\left(s, u(s), u'(s)\right) ds \right. \\
& \left. \left. -\lambda u(\tau) \right] d\tau \right| \\
\le {} & \int_0^1 (1-\tau)^{\alpha-1} \\
& \times \left[\frac{1}{\Gamma(\beta)} \int_0^\tau (\tau - s)^{\beta-1} \right. \\
& \times \left(\left| f\left(s, u(s), u'(s)\right) - f(s,0,0) \right| \right. \\
& \left. \left. + \left| f(s,0,0) \right| \right) ds \right. \\
& \left. + |\lambda u(\tau)| \right] d\tau \\
\le {} & \int_0^1 (1-\tau)^{\alpha-1} \\
& \times \left[\frac{1}{\Gamma(\beta)} \int_0^\tau (\tau - s)^{\beta-1} \right. \\
& \times \left(2g(s) \max\left\{ |u(s)|, \left|u'(s)\right| \right\} + M \right) ds \\
& \left. + |\lambda| r \right] d\tau \\
\le {} & \int_0^1 (1-\tau)^{\alpha-1} \frac{1}{\Gamma(\beta)} \int_0^\tau (\tau - s)^{\beta-1} 2g(s) r ds\, d\tau \\
& + \int_0^1 (1-\tau)^{\alpha-1} \frac{1}{\Gamma(\beta)} \int_0^\tau (\tau - s)^{\beta-1} M ds\, d\tau \\
& + \int_0^1 (1-\tau)^{\alpha-1} |\lambda| r d\tau \\
\le {} & \frac{2r}{\Gamma(\beta)} \int_0^1 (1-\tau)^{\alpha-1} \int_0^\tau (\tau - s)^{\beta-1} g(s)\, ds\, d\tau \\
& + \frac{M}{\Gamma(\beta)} \int_0^1 (1-\tau)^{\alpha-1} \int_0^\tau (\tau - s)^{\beta-1} ds\, d\tau \\
& + |\lambda| r \int_0^1 (1-\tau)^{\alpha-1} d\tau \\
\le {} & \frac{2r}{\Gamma(\beta)} \int_0^1 (1-\tau)^{\alpha-1} \\
& \times \left[\left(\int_0^\tau (\tau - s)^{(\beta-1)/p} ds \right)^p \right. \\
& \left. \times \left(\int_0^\tau \left(g(s)\right)^{1/(1-p)} ds \right)^{1-p} \right] d\tau \\
& + \frac{M}{\Gamma(\beta)} \int_0^1 (1-\tau)^{\alpha-1} \int_0^\tau (\tau - s)^{\beta-1} ds\, d\tau \\
& + |\lambda| r \int_0^1 (1-\tau)^{\alpha-1} d\tau \\
\le {} & \frac{2rg^*}{\Gamma(\beta)} \int_0^1 (1-\tau)^{\alpha-1} \left(\frac{p\tau^{(\beta+p-1)/p}}{\beta+p-1} \right)^p d\tau \\
& + \frac{MB(\beta+1,\alpha)}{\beta\Gamma(\beta)} + \frac{|\lambda| r}{\alpha} \\
\le {} & \frac{2rg^* p^p B(\beta+p,\alpha)}{\Gamma(\beta)(\beta+p-1)^p} + \frac{M\Gamma(\alpha)}{\Gamma(\alpha+\beta+1)} + \frac{|\lambda| r}{\alpha} \triangleq \Phi_3.
\end{aligned}
\tag{29}
$$

Clearly, we also can get

$$
\Lambda_4 \le \frac{2rg^* p^p B(\beta+p,\alpha-1)}{\Gamma(\beta)(\beta+p-1)^p} + \frac{M\Gamma(\alpha-1)}{\Gamma(\alpha+\beta)} + \frac{|\lambda| r}{\alpha-1} \triangleq \Phi_4,
$$

$$
\begin{aligned}
& |(Fu)(t)| \\
& = \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \right. \\
& \quad \times \left[\frac{1}{\Gamma(\beta)} \int_0^\tau (\tau - s)^{\beta-1} \right. \\
& \quad \left. \times f\left(s, u(s), u'(s)\right) ds \right. \\
& \quad \left. -\lambda u(\tau) \right] d\tau \\
& \quad + \left(\alpha\beta_2(\eta+t) - t^\alpha(\alpha_2+\beta_2)\right)\Lambda_2 g_1(u) \\
& \quad - \left(\beta_1(\eta+t) - t^\alpha\Lambda_1\right)\Lambda_2 g_2(u) \\
& \quad + \frac{\left(\beta_1 t^\alpha(\alpha_2+\beta_2) - \alpha\beta_1\beta_2(\eta+t)\right)\Lambda_2\Lambda_3}{\Gamma(\alpha)} \\
& \quad \left. + \frac{\beta_2\left[\beta_1(\eta+t) - t^\alpha\Lambda_1\right]}{\Gamma(\alpha-1)}\Lambda_2\Lambda_4 \right| \\
& \le \frac{1}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} \\
& \quad \times \left[\frac{1}{\Gamma(\beta)} \int_0^\tau (\tau - s)^{\beta-1} \right. \\
& \quad \left. \times f\left(s, u(s), u'(s)\right) ds \right. \\
& \quad \left. -\lambda u(\tau) \right] d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+\alpha\beta_2(\eta+1)\Lambda_2 l_1\|u\|+\Lambda_1\Lambda_2 l_2\|u\|\\
&+\frac{\beta_1(\alpha_2+\beta_2)\Lambda_2\Lambda_3}{\Gamma(\alpha)}+\frac{\beta_2\beta_1(\eta+1)}{\Gamma(\alpha-1)}\Lambda_2\Lambda_4\\
&\le\frac{\Lambda_3}{\Gamma(\alpha)}+\alpha\beta_2(\eta+1)\Lambda_2 l_1 r+\Lambda_1\Lambda_2 l_2 r\\
&+\frac{\beta_1(\alpha_2+\beta_2)\Lambda_2\Lambda_3}{\Gamma(\alpha)}+\frac{\beta_2\beta_1(\eta+1)}{\Gamma(\alpha-1)}\Lambda_2\Lambda_4\\
&\le\frac{\Phi_3}{\Gamma(\alpha)}+\left(\alpha\beta_2(\eta+1)+\Lambda_1\right)\Lambda_2 Lr\\
&+\frac{\beta_1(\alpha_2+\beta_2)\Lambda_2\Phi_3}{\Gamma(\alpha)}+\frac{\beta_2\beta_1(\eta+1)}{\Gamma(\alpha-1)}\Lambda_2\Phi_4,
\end{aligned}
$$

$$
\begin{aligned}
&\left|(Fu)'(t)\right|\\
&=\left|\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-\tau)^{\alpha-2}\right.\\
&\qquad\times\left[\frac{1}{\Gamma(\beta)}\int_0^\tau(\tau-s)^{\beta-1}\right.\\
&\qquad\qquad\left.\times f\left(s,u(s),u'(s)\right)ds\right.\\
&\qquad\left.-\lambda u(\tau)\right]d\tau\\
&\quad\times\left(\alpha\beta_2-\alpha t^{\alpha-1}(\alpha_2+\beta_2)\right)\Lambda_2 g_1(u)\\
&\quad+\left(-\beta_1+\alpha t^{\alpha-1}\Lambda_1\right)\Lambda_2 g_2(u)\\
&\quad+\frac{\left[\alpha\beta_1 t^{\alpha-1}(\alpha_2+\beta_2)-\alpha\beta_1\beta_2\right]\Lambda_2\Lambda_3}{\Gamma(\alpha)}\\
&\quad\left.+\frac{\beta_2\left[\beta_1-\alpha t^{\alpha-1}\Lambda_1\right]\Lambda_2\Lambda_4}{\Gamma(\alpha-1)}\right|\\
&\le\frac{1}{\Gamma(\alpha-1)}\int_0^1(1-\tau)^{\alpha-2}\\
&\qquad\times\left[\frac{1}{\Gamma(\beta)}\int_0^\tau(\tau-s)^{\beta-1}\right.\\
&\qquad\qquad\left.\times f\left(s,u(s),u'(s)\right)ds\right.\\
&\qquad\left.-\lambda u(\tau)\right]d\tau\\
&\quad+\alpha\beta_2\Lambda_2 l_1\|u\|+\alpha\Lambda_1\Lambda_2 l_2\|u\|\\
&\quad+\frac{\alpha\beta_1(\alpha_2+\beta_2)\Lambda_2\Lambda_3}{\Gamma(\alpha)}+\frac{\beta_2\beta_1\Lambda_2\Lambda_4}{\Gamma(\alpha-1)}\\
&\le\frac{\Lambda_4}{\Gamma(\alpha-1)}+\alpha\beta_2\Lambda_2 l_1 r+\alpha\Lambda_1\Lambda_2 l_2 r\\
&\quad+\frac{\alpha\beta_1(\alpha_2+\beta_2)\Lambda_2\Lambda_3}{\Gamma(\alpha)}+\frac{\beta_2\beta_1\Lambda_2\Lambda_4}{\Gamma(\alpha-1)}\\
&\le\frac{\Phi_4}{\Gamma(\alpha-1)}+\alpha\left(\beta_2+\Lambda_1\right)\Lambda_2 Lr\\
&\quad+\frac{\alpha\beta_1(\alpha_2+\beta_2)\Lambda_2\Phi_3}{\Gamma(\alpha)}+\frac{\beta_2\beta_1\Lambda_2\Phi_4}{\Gamma(\alpha-1)}.
\end{aligned}
\tag{30}
$$

For convenience, we let

$$
\begin{aligned}
\psi=\ &\max\left\{\frac{\Phi_3}{\Gamma(\alpha)}+\left(\alpha\beta_2(\eta+1)+\Lambda_1\right)\Lambda_2 Lr\right.\\
&+\frac{\beta_1(\alpha_2+\beta_2)\Lambda_2\Phi_3}{\Gamma(\alpha)}\\
&+\frac{\beta_2\beta_1(\eta+1)}{\Gamma(\alpha-1)}\Lambda_2\Phi_4,\frac{\Phi_4}{\Gamma(\alpha-1)}\\
&+\alpha\left(\beta_2+\Lambda_1\right)\Lambda_2 Lr\\
&\left.+\frac{\alpha\beta_1(\alpha_2+\beta_2)\Lambda_2\Phi_3}{\Gamma(\alpha)}+\frac{\beta_2\beta_1\Lambda_2\Phi_4}{\Gamma(\alpha-1)}\right\},
\end{aligned}
\tag{31}
$$

where we have used the Hölder inequality and the following equalities:

$$
B(\beta+1,\alpha)=\int_0^1(1-\tau)^{\alpha-1}\tau^\beta d\tau=\frac{\Gamma(\beta+1)\Gamma(\alpha)}{\Gamma(\alpha+\beta+1)}.
\tag{32}
$$

Therefore, $\|(Fu)(t)\|\le\psi$.

Step 2. F is a contraction operator.

For convenience, we get

$$
\begin{aligned}
&|(Su)(t)-(Sv)(t)|\\
&=\left|\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\right.\\
&\qquad\times\left[\frac{1}{\Gamma(\beta)}\int_0^\tau(\tau-s)^{\beta-1}f\left(s,u(s),u'(s)\right)ds\right.\\
&\qquad\left.\left.-\lambda u(\tau)\right]d\tau\right.
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \\
&\qquad \times \left[\frac{1}{\Gamma(\beta)} \int_0^\tau (\tau-s)^{\beta-1} f\left(s, v(s), v'(s)\right) ds \right. \\
&\qquad \left. -\lambda v(\tau) \right] d\tau \Bigg| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} \\
&\qquad \times \left[\frac{1}{\Gamma(\beta)} \int_0^\tau (\tau-s)^{\beta-1} 2g(s) \right. \\
&\qquad \left. \times \max\left\{ |u-v|, \left|u'-v'\right| \right\} ds \right] d\tau \\
&\quad + \frac{|\lambda|}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} \|u-v\| \, d\tau \\
&\leq \|u-v\| \\
&\quad \times \left(\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 (1-\tau)^{\alpha-1} \int_0^\tau (\tau-s)^{\beta-1} 2g(s) \, ds \, d\tau \right. \\
&\quad \left. + \frac{|\lambda|}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} d\tau \right) \\
&\leq \|u-v\| \\
&\quad \times \left[\frac{2}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 (1-\tau)^{\alpha-1} \left(\int_0^\tau (\tau-s)^{(\beta-1)/p} ds \right)^p \right. \\
&\quad \left. \times \left(\int_0^\tau (g(s))^{1/(1-p)} ds \right)^{1-p} d\tau + \frac{|\lambda|}{\Gamma(\alpha+1)} \right] \\
&\leq \|u-v\| \left(\frac{2g^* p^p \Gamma(\beta+p)}{\Gamma(\beta)\Gamma(\alpha+\beta+p)(\beta+p-1)^p} + \frac{|\lambda|}{\Gamma(\alpha+1)} \right) \\
&\triangleq \Upsilon_1 \|u-v\| .
\end{aligned} \tag{33}
$$

Clearly, we can also get

$$
\begin{aligned}
&\left|(Su)'(t) - (Sv)'(t)\right| \\
&\quad \leq \|u-v\| \left(\frac{2g^* p^p \Gamma(\beta+p)}{\Gamma(\beta)\Gamma(\alpha+\beta+p-1)(\beta+p-1)^p} \right. \\
&\quad \left. + \frac{|\lambda|}{\Gamma(\alpha)} \right) \triangleq \Upsilon_2 \|u-v\| .
\end{aligned} \tag{34}
$$

For $u, v \in C^1([0,1], R)$ and for each $t \in [0,1]$, we obtain

$$
\begin{aligned}
&|(Fu)(t) - (Fv)(t)| \\
&= \Bigg| Su(t) + \left(\alpha\beta_2(\eta+t) - t^\alpha(\alpha_2+\beta_2)\right)\Lambda_2 g_1(u) \\
&\quad - \left(\beta_1(\eta+t) - t^\alpha\Lambda_1\right)\Lambda_2 g_2(u) \\
&\quad + \frac{\left(\beta_1 t^\alpha(\alpha_2+\beta_2) - \alpha\beta_1\beta_2(\eta+t)\right)\Lambda_2\Lambda_3}{\Gamma(\alpha)} \\
&\quad + \frac{\beta_2\left[\beta_1(\eta+t) - t^\alpha\Lambda_1\right]}{\Gamma(\alpha-1)}\Lambda_2\Lambda_4 \\
&\quad - \left[Sv(t) + \left(\alpha\beta_2(\eta+t) - t^\alpha(\alpha_2+\beta_2)\right)\Lambda_2 g_1(v) \right. \\
&\qquad - \left(\beta_1(\eta+t) - t^\alpha\Lambda_1\right)\Lambda_2 g_2(v) \\
&\qquad + \frac{\left(\beta_1 t^\alpha(\alpha_2+\beta_2) - \alpha\beta_1\beta_2(\eta+t)\right)\Lambda_2\Lambda_3}{\Gamma(\alpha)} \\
&\qquad \left. + \frac{\beta_2\left[\beta_1(\eta+t) - t^\alpha\Lambda_1\right]}{\Gamma(\alpha-1)}\Lambda_2\Lambda_4 \right] \Bigg| \\
&\leq |(Su)(t) - (Sv)(t)| \\
&\quad + \left|\left(\alpha\beta_2(\eta+t) - t^\alpha(\alpha_2+\beta_2)\right)\right|\Lambda_2 l_1 \|u-v\| \\
&\quad + \left|\left(\beta_1(\eta+t) - t^\alpha\Lambda_1\right)\right|\Lambda_2 l_2 \|u-v\| \\
&\quad + \left| \frac{\left(\beta_1 t^\alpha(\alpha_2+\beta_2) - \alpha\beta_1\beta_2(\eta+t)\right)\Lambda_2(\Lambda_3-\Lambda_5)}{\Gamma(\alpha)} \right. \\
&\qquad \left. + \frac{\beta_2\left[\beta_1(\eta+t) - t^\alpha\Lambda_1\right]}{\Gamma(\alpha-1)}\Lambda_2(\Lambda_4-\Lambda_6) \right| \\
&\leq |(Su)(t) - (Sv)(t)| \\
&\quad + \alpha\beta_2(\eta+t)\Lambda_2 l_1 \|u-v\| \\
&\quad + \beta_1(\eta+t)\Lambda_2 l_2 \|u-v\| \\
&\quad + \beta_1 t^\alpha(\alpha_2+\beta_2)\Lambda_2\Upsilon_1 \|u-v\| \\
&\quad + \beta_2\beta_1(\eta+t)\Lambda_2\Upsilon_2 \|u-v\| \\
&\leq \|u-v\| \left(\Upsilon_1 + (\eta+1)\right. \\
&\qquad \times \left(\alpha\beta_2\Lambda_2 L + \beta_1\Lambda_2 L + \beta_1\beta_2\Lambda_2\Upsilon_2\right) \\
&\qquad \left. + \beta_1(\alpha_2+\beta_2)\Lambda_2\Upsilon_1\right), \\
&\left|(Fu)'(t) - (Fv)'(t)\right| \\
&\leq \left|(Su)'(t) - (Su)'(t)\right| \\
&\quad + \left|\left(\alpha\beta_2 - \alpha t^{\alpha-1}(\alpha_2+\beta_2)\right)\Lambda_2\left(g_1(u) - g_1(v)\right)\right. \\
&\qquad \left. + \left(-\beta_1 + \alpha t^{\alpha-1}\Lambda_1\right)\Lambda_2\left(g_2(u) - g_2(v)\right)\right|
\end{aligned}
$$

$$+\left|\frac{\left[\alpha\beta_1 t^{\alpha-1}\left(\alpha_2+\beta_2\right)-\alpha\beta_1\beta_2\right]\Lambda_2\left(\Lambda_3-\Lambda_5\right)}{\Gamma(\alpha)}+\frac{\beta_2\left[\beta_1-\alpha t^{\alpha-1}\Lambda_1\right]\Lambda_2\left(\Lambda_4-\Lambda_6\right)}{\Gamma(\alpha-1)}\right|$$

$$\begin{aligned}&\le \Upsilon_2\|u-v\|+\alpha\beta_2\Lambda_2 l_1\|u-v\|\\&\quad+\alpha\Lambda_1\Lambda_2 l_2\|u-v\|\\&\quad+\alpha\beta_1 t^{\alpha-1}\left(\alpha_2+\beta_2\right)\Lambda_2\Upsilon_1\|u-v\|\\&\quad+\beta_2\beta_1\Lambda_2\Upsilon_2\|u-v\|\\&\le\|u-v\|\left(\Upsilon_2+\left(\alpha_2+\beta_2\right)\alpha\beta_1\Lambda_2\Upsilon_1\right.\\&\qquad\left.+\alpha\beta_2\Lambda_2 L+\alpha\Lambda_1\Lambda_2 L+\beta_1\beta_2\Lambda_2\Upsilon_2\right).\end{aligned}\tag{35}$$

Since $\Lambda < 1$, we have $\|F(u)-F(v)\| \le \Lambda\|u-v\|$; that is, F is a nonlinear contraction. Hence, by using Lemma 8, the conclusion of the theorem holds by Banach fixed point theorem.

The proof is completed. □

Theorem 11. *Let $f:[0,1]\times R\times R\to R$ be a jointly continuous function and the assumptions (H1) and (H2) hold. In addition,*

(H3) *assume that there exist a constant $l\in(0,1)$ and a real-valued function $m\in L^{1/l}([0,1],R^+)$ such that*

$$\left|f\left(t,u,u'\right)\right|\le m(t),\quad \forall\left(t,u,u'\right)\in[0,1]\times R\times R,\quad \text{with } \sup_{t\in[0,1]}|m(t)|=\|m\|.\tag{36}$$

Then the problem (6) *has at least one solution on* $[0,1]$ *if*

$$\begin{aligned}\xi\triangleq\ &\max\left\{\left(\eta+1\right)\left(\alpha\beta_2\Lambda_2 L+\beta_1\Lambda_2 L+\beta_1\beta_2\Lambda_2\Upsilon_2\right)\right.\\&+\beta_1\left(\alpha_2+\beta_2\right)\Lambda_2\Upsilon_1,\\&\left(\alpha_2+\beta_2\right)\alpha\beta_1\Lambda_2\Upsilon_1+\alpha\beta_2\Lambda_2 L\\&\left.+\alpha\Lambda_1\Lambda_2 L+\beta_1\beta_2\Lambda_2\Upsilon_2\right\}<1.\end{aligned}\tag{37}$$

Proof. Step 1. There exists a positive constant $r>0$ such that $Su+Tu\in\overline{\Omega}_r$.

For $u\in\overline{\Omega}_r$, by the same arguments of the first step of the proof in Theorem 10, we have $\|Su+Tu\|\le\psi$. In virtue of the definition of ψ and a simple calculation, we obtain

$$\psi\le M+\xi r,\tag{38}$$

where M is a constant. By the assumptions, $\xi<1$. Therefore, there exists a positive constant r large enough such that

$$\psi\le M+\xi r<r.\tag{39}$$

Hence, there exists a positive constant r such that $Su+Tu\in\overline{\Omega}_r$.

Step 2. T is a contraction operator.

For $u,v\in C^1([0,1],R)$ and for each $t\in[0,1]$, we obtain

$$\begin{aligned}&|(Tu)(t)-(Tv)(t)|\\&\quad\le\|u-v\|\left(\left(\eta+1\right)\left(\alpha\beta_2\Lambda_2 L+\beta_1\Lambda_2 L+\beta_1\beta_2\Lambda_2\Upsilon_2\right)\right.\\&\qquad\left.+\beta_1\left(\alpha_2+\beta_2\right)\Lambda_2\Upsilon_1\right),\\&\left|(Tu)'(t)-(Tv)'(t)\right|\\&\quad\le\|u-v\|\left(\left(\alpha_2+\beta_2\right)\alpha\beta_1\Lambda_2\Upsilon_1\right.\\&\qquad\left.+\alpha\beta_2\Lambda_2 L+\alpha\Lambda_1\Lambda_2 L+\beta_1\beta_2\Lambda_2\Upsilon_2\right).\end{aligned}\tag{40}$$

Since $\xi<1$, we have $\|T(u)-T(v)\|\le\xi\|u-v\|$; that is, T is a nonlinear contraction.

Step 3. S is continuous and compact.

Firstly, we show that the operator S is continuous. For $\{u_n\}\subset\overline{\Omega}_r$, $u_0\in\overline{\Omega}_r$ such that $u_n\to u_0$ in $\overline{\Omega}_r$; then

$$\begin{aligned}&\left|Su_n(t)-Su_0(t)\right|\\&=\left|\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\right.\\&\qquad\times\left[\frac{1}{\Gamma(\beta)}\int_0^\tau(\tau-s)^{\beta-1}f\left(s,u_n(s),u_n'(s)\right)ds\right.\\&\qquad\left.\left.-\lambda u_n(\tau)\right]d\tau\right.\\&\quad-\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\\&\qquad\times\left[\frac{1}{\Gamma(\beta)}\int_0^\tau(\tau-s)^{\beta-1}\right.\\&\qquad\left.\left.\times f\left(s,u_0(s),u_0'(s)\right)ds\right.\right.\\&\qquad\left.\left.-\lambda u_0(\tau)\right]d\tau\right|\\&\le\left|\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\right.\\&\qquad\times\frac{1}{\Gamma(\beta)}\int_0^\tau(\tau-s)^{\beta-1}\\&\qquad\times\left[f\left(s,u_n(s),u_n'(s)\right)\right.\\&\qquad\left.\left.-f\left(s,u_0(s),u_0'(s)\right)\right]ds\,d\tau\right.\\&\quad\left.+\frac{\lambda}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\left(u_n(\tau)-u_0(\tau)\right)d\tau\right|\\&\le\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\\&\qquad\times\frac{1}{\Gamma(\beta)}\int_0^\tau(\tau-s)^{\beta-1}\end{aligned}$$

$$\times 2g(s)\max\left\{\left|u_n(s)-u_0(s)\right|,\left|u_n'(s)-u_0'(s)\right|\right\}ds\,d\tau + \frac{|\lambda|}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left|u_n(\tau)-u_0(\tau)\right|d\tau$$
$$\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\times\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-s)^{\beta-1}2g(s)\left\|u_n-u_0\right\|ds\,d\tau + \frac{|\lambda|}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left\|u_n-u_0\right\|d\tau$$
$$\le \left\|u_n-u_0\right\|\left(\frac{2}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}\frac{1}{\Gamma(\beta)}\times\int_0^\tau (\tau-s)^{\beta-1}g(s)\,ds\,d\tau + \frac{|\lambda|}{\Gamma(\alpha+1)}\right)$$
$$\le \Upsilon_1\left\|u_n-u_0\right\|. \tag{41}$$

Similarly, we get

$$\left|Su_n'(t)-Su_0'(t)\right|\le\Upsilon_2\left\|u_n-u_0\right\|, \tag{42}$$

we get sequences $u_n(t)$ and $u_0(t)$, which converge on $[0,1]$ with $\lim_{n\to\infty}u_n(t)=u_0(t)$ and $\lim_{n\to\infty}u_n'(t)=u_0'(t)$.

Since

$$\left\|Su_n-Su_0\right\| = \max\left\{\sup_{t\in[0,1]}\left|Su_n(t)-Su_0(t)\right|,\ \sup_{t\in[0,1]}\left|(Su_n)'(t)-(Su_0)'(t)\right|\right\}. \tag{43}$$

Combining (41) and (42), we can get $\|Su_n-Su_0\|\to 0$. Thus S is continuous in $C^1([0,1],R)$.

Secondly, we show that the operator S is equicontinuous. Let $M_r=\max_{s\in[0,1]}\{|f(s,u,u')|,u\in\overline{\Omega}_r\}$. For any $u\in\overline{\Omega}_r$, for all $s_1,s_2\in[0,1]$, $0\le s_1<s_2\le 1$, we obtain

$$\left|(Su)(s_2)-(Su)(s_1)\right| = \left|\frac{1}{\Gamma(\alpha)}\int_0^{s_2}(s_2-\tau)^{\alpha-1}\times\left[\frac{1}{\Gamma(\beta)}\int_0^\tau(\tau-s)^{\beta-1}f\left(s,u(s),u'(s)\right)ds - \lambda u(\tau)\right]d\tau - \frac{1}{\Gamma(\alpha)}\int_0^{s_1}(s_1-\tau)^{\alpha-1}\times\left[\frac{1}{\Gamma(\beta)}\int_0^\tau(\tau-s)^{\beta-1}f\left(s,u(s),u'(s)\right)ds - \lambda u(\tau)\right]d\tau\right|$$
$$\le \left|\frac{1}{\Gamma(\alpha)}\int_0^{s_1}\left[(s_2-\tau)^{\alpha-1}-(s_1-\tau)^{\alpha-1}\right]\times\left[\frac{1}{\Gamma(\beta)}\int_0^\tau(\tau-s)^{\beta-1}\left|f\left(s,u(s),u'(s)\right)\right|ds + |\lambda|\,|u(\tau)|\right]d\tau + \frac{1}{\Gamma(\alpha)}\int_{s_1}^{s_2}(s_1-\tau)^{\alpha-1}\times\left[\frac{1}{\Gamma(\beta)}\int_0^\tau(\tau-s)^{\beta-1}\left|f\left(s,u(s),u'(s)\right)\right|ds + |\lambda|\,|u(\tau)|\right]d\tau\right|$$
$$\le \frac{1}{\Gamma(\alpha)}\int_0^{s_1}\left[(s_2-\tau)^{\alpha-1}-(s_1-\tau)^{\alpha-1}\right]\times\left(\frac{1}{\Gamma(\beta)}\int_0^\tau(\tau-s)^{\beta-1}M_r ds\right)d\tau + \frac{1}{\Gamma(\alpha)}\int_0^{s_1}\left[(s_2-\tau)^{\alpha-1}-(s_1-\tau)^{\alpha-1}\right]|\lambda|\,r\,d\tau + \frac{1}{\Gamma(\alpha)}\int_{s_1}^{s_2}(s_1-\tau)^{\alpha-1}\left(\frac{1}{\Gamma(\beta)}\int_0^\tau(\tau-s)^{\beta-1}M_r ds\right)d\tau + \frac{1}{\Gamma(\alpha)}\int_{s_1}^{s_2}(s_1-\tau)^{\alpha-1}|\lambda|\,r\,d\tau$$
$$\le \frac{1}{\Gamma(\alpha)}\int_0^{s_1}\left[(s_2-\tau)^{\alpha-1}-(s_1-\tau)^{\alpha-1}\right]\times\left(\frac{M_r}{\Gamma(\beta)}\int_0^\tau(\tau-s)^{\beta-1}ds\right)d\tau + \frac{|\lambda|\,r}{\Gamma(\alpha)}\int_0^{s_1}\left[(s_2-\tau)^{\alpha-1}-(s_1-\tau)^{\alpha-1}\right]d\tau + \frac{1}{\Gamma(\alpha)}\int_{s_1}^{s_2}(s_1-\tau)^{\alpha-1}\left(\frac{M_r}{\Gamma(\beta)}\int_0^\tau(\tau-s)^{\beta-1}ds\right)d\tau + \frac{|\lambda|\,r}{\Gamma(\alpha)}\int_{s_1}^{s_2}(s_1-\tau)^{\alpha-1}d\tau$$

$$
\begin{aligned}
&\le \frac{M_r}{\Gamma(\alpha)\Gamma(\beta+1)}\int_0^{s_1}(s_2-s_1)^{\alpha-1}\tau^{\beta}d\tau \\
&\quad+\frac{|\lambda|\,r\left(s_2^{\alpha}-s_1^{\alpha}-(s_2-s_1)^{\alpha}\right)}{\Gamma(\alpha+1)} \\
&\quad+\frac{M_r}{\Gamma(\alpha)\Gamma(\beta+1)}\int_{s_1}^{s_2}(s_1-\tau)^{\alpha-1}\tau^{\beta}d\tau \\
&\quad+\frac{|\lambda|\,r(s_2-s_1)^{\alpha}}{\Gamma(\alpha+1)} \\
&\le \frac{M_r(s_2-s_1)^{\alpha-1}}{\Gamma(\alpha)\Gamma(\beta+1)}\int_0^{1}\tau^{\beta}d\tau \\
&\quad+\frac{|\lambda|\,r\left(s_2^{\alpha}-s_1^{\alpha}-(s_2-s_1)^{\alpha}+(s_1-s_2)^{\alpha}\right)}{\Gamma(\alpha+1)} \\
&\quad+\frac{M_r}{\Gamma(\alpha)\Gamma(\beta+1)}\left(\int_{s_1}^{s_2}(s_1-\tau)^{(\alpha-1)/p}d\tau\right)^{p} \\
&\quad\times\left(\int_{s_1}^{s_2}\tau^{\beta/(1-p)}d\tau\right)^{1-p} \\
&\le \frac{M_r(s_2-s_1)^{\alpha-1}}{\Gamma(\alpha)\Gamma(\beta+2)} \\
&\quad+\left(M_r\left[p^{p}(s_1-s_2)^{\alpha+p-1}\right]\right. \\
&\qquad\left.\times\left[(1-p)^{1-p}\left(s_2^{(\beta+1-p)/(1-p)}-s_1^{(\beta+1-p)/(1-p)}\right)^{1-p}\right]\right) \\
&\quad\times\left(\Gamma(\alpha)\Gamma(\beta+1)(\alpha+p-1)^{p}(\beta+1-p)^{1-p}\right)^{-1} \\
&\quad+\frac{|\lambda|\,r\left(s_2^{\alpha}-s_1^{\alpha}-(s_2-s_1)^{\alpha}+(s_1-s_2)^{\alpha}\right)}{\Gamma(\alpha+1)}.
\end{aligned}
\tag{44}
$$

Clearly, we also easily get

$$
\begin{aligned}
&\left|(Su)'(s_2)-(Su)'(s_1)\right| \\
&\quad\le\frac{1}{\Gamma(\alpha-1)}\int_0^{s_1}\left[(s_2-\tau)^{\alpha-2}-(s_1-\tau)^{\alpha-2}\right] \\
&\qquad\times\left(\frac{M_r\tau^{\beta}}{\Gamma(\beta+1)}+|\lambda|\,r\right)d\tau \\
&\qquad+\frac{M_r(s_2-s_1)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}+\frac{M_r(s_2-s_1)^{\alpha-1}s_1^{\beta}}{\Gamma(\alpha)\Gamma(\beta+1)} \\
&\qquad+\frac{|\lambda|\,r(s_2-s_1)^{\alpha}}{\alpha\Gamma(\alpha-1)}.
\end{aligned}
\tag{45}
$$

Obviously the right hand side of the above inequality tends to zero independently of $u\in\overline{\Omega}_r$, as $t_2-t_1\to 0$; we get that S is relatively compact on $\overline{\Omega}_r$. Hence, by the Arzelá-Ascoli theorem, S is compact on $\overline{\Omega}_r$.

Thus, all the assumptions of Lemma 8 are satisfied and the conclusion of Lemma 8 implies that the boundary value problem (6) has at least one solution on $[0,1]$.

The proof is completed. □

4. Algorithm for the Fractional Langevin Equation and Examples

In this paper, we will give the numerical simulation for the fractional Langevin equation.

The definition of fractional order has many kinds; the different definitions will bring different algorithm forms and will cause different proof of the algorithm stability and different method of accuracy analysis. In the practical application, there are three kinds of fractional derivative definitions, such as Grünwald-Letnikov, Riemann-Liouvlle, and Caputo Fractional derivatives.

Remark 12 (see [51]). For $m-1<\alpha\le m$, $m\in N$, $f(t)\in C^m[a,b]$,

$$
{}^{\mathrm{GL}}D^{\alpha}f(t)={}^{\mathrm{RL}}D^{\alpha}f(t).
\tag{46}
$$

Remark 13 (see [49]). For $f^{(k)}(a)=0$, $k=0,1,\ldots,m$,

$$
\begin{aligned}
{}^{\mathrm{GL}}D^{\alpha}f(t)&=\sum_{k=0}^{m-1}\frac{f^{(k)}(a)}{\Gamma(k-a+1)}(t-a)^{k-\alpha} \\
&\quad+{}_{a}^{c}D^{\alpha}f(t)={}_{a}^{\mathrm{RL}}D^{\alpha}f(t).
\end{aligned}
\tag{47}
$$

In [52], shifted Grünwald-Letnikov formula is defined by

$$
\begin{aligned}
&{}^{\mathrm{GL}}D^{\alpha}f(t) \\
&\quad=\lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{k=0}^{[(t-a)/h+p]}w_k^{(\alpha)}f\left(t-(k-p)h\right),\quad \alpha>0.
\end{aligned}
\tag{48}
$$

We get the following approximation:

$$
\begin{aligned}
{}_{a}D^{\alpha}f(t)&\approx\frac{1}{h^{\alpha}}\sum_{k=0}^{[(t-a)/h+p]}w_k^{(\alpha)}f\left(t-(k-p)h\right) \\
&:=\left({}_{a}D^{\alpha}f(t)\right)_{G_{S(p)}}.
\end{aligned}
\tag{49}
$$

We put a call shifted Grünwald discrete format, simply "$G_{S(p)}$ algorithm" for short.

In addition, Oldham and Spanier [53] found the following approximation format in 1974:

$$
\begin{aligned}
{}_{a}D^{-1}f(t)&=\lim_{h\to 0}h\sum_{j=0}^{[(t-a)/h-1/2]}f\left(t-\left(j+\frac{1}{2}\right)h\right), \\
{}_{a}D^{1}f(t)&=\lim_{h\to 0}h^{-1}\sum_{j=0}^{[(t-a)/h+1/2]}(-1)^{j}f\left(t-\left(j-\frac{1}{2}\right)h\right).
\end{aligned}
\tag{50}
$$

The approximation format has the rapid convergence properties. So they put forward an improved Grünwald-Letnikov fractional derivative definition (take $p = \alpha/2$ to (48)):

$$ {}_aD^{\alpha} f(t) = \lim_{h\to 0} \frac{h^{-\alpha}}{\Gamma(-\alpha)} \times \sum_{j=0}^{[(t-a)/h+\alpha/2]} \frac{\Gamma(j-\alpha)}{\Gamma(j+1)} f\left(t-\left(j-\frac{1}{2}\alpha\right)h\right). \tag{51} $$

For $a = 0$, the above equation can be written as

$$ {}_aD^{\alpha} f(t) = \lim_{h\to 0} \frac{h^{-\alpha}}{\Gamma(-\alpha)} \times \sum_{j=0}^{[t/h+\alpha/2]} \frac{\Gamma(j-\alpha)}{\Gamma(j+1)} f\left(t-\left(j-\frac{1}{2}\alpha\right)h\right). \tag{52} $$

Therefore, they put forward "fractional center difference quotient" approximation format called in general "G_2 algorithm."

In this paper, we use the three-point interpolation formula:

$$ f\left(t-\left(j-\frac{1}{2}\alpha\right)h\right) \approx \left(\frac{\alpha}{4}+\frac{\alpha^2}{8}\right) f(t-(j-1)h) + \left(1-\frac{\alpha^2}{4}\right) f(t-jh) + \left(\frac{\alpha^2}{8}-\frac{\alpha}{4}\right) f(t-(j+1)h). \tag{53} $$

Then "G_2 algorithm" can be expressed as:

$$ \left({}_aD^{\alpha} f(t_n)\right)_{G_2} = h^{-\alpha}\sum_{j=0}^{n-1} w_j^{(\alpha)} \left\{ f_{n-j} + \frac{1}{4}\alpha\left(f_{n-j+1}-f_{n-j-1}\right) + \frac{1}{8}\alpha^2\left(f_{n-j+1}-2f_{n-j}+f_{n-j-1}\right)\right\}. \tag{54} $$

Remark 14 (see [49]). G_2 algorithm is based on Grünwald-Letnikov definition, not only used for numerical calculation of fractional derivative ($\alpha \geq 0$), but also used for numerical calculation of fractional integral ($\alpha \leq 0$).

As we all know, the fractional Langevin equation form is

$$ {}^cD^{\beta}\left({}^cD^{\alpha}+\lambda\right)u(t) = f(t)+\xi(t), \tag{55} $$

where $0 \leq t \leq 1$, $1 < \alpha \leq 2$, $0 < \beta \leq 1$, λ is a constant, $f(t)$ is an external force, and $\xi(t)$ is a random force.

The above equation can be written as

$$ {}^cD^{\beta}u(t) = f(t), \quad {}^cD^{\alpha}v(t) = u(t)-\lambda v(t). \tag{56} $$

According to G_2 algorithm, the Caputo fractional derivatives above can be written as

$$ {}^cD^{\beta}u(t) = \lim_{h\to 0} \frac{h^{-\beta}}{\Gamma(-\beta)} \sum_{j=0}^{[t/h+\beta/2]} \frac{\Gamma(j-\beta)}{\Gamma(j+1)} u\left(t-\left(j-\frac{\beta}{2}\right)h\right), \quad {}^cD^{\alpha}u(t) = \lim_{h\to 0} \frac{h^{-\alpha}}{\Gamma(-\alpha)} \sum_{j=0}^{[t/h+\alpha/2]} \frac{\Gamma(j-\alpha)}{\Gamma(j+1)} u\left(t-\left(j-\frac{\alpha}{2}\right)h\right). \tag{57} $$

The previous equations are approximated by the three-point interpolation formula and can be written as

$$ {}^cD^{\beta}u(t_n) = h^{-\beta}\sum_{j=0}^{n-1} w_j^{(\beta)} \left\{ u_{n-j} + \frac{1}{4}\beta\left(u_{n-j+1}-u_{n-j-1}\right) + \frac{1}{8}\beta^2\left(u_{n-j+1}-2u_{n-j}+u_{n-j-1}\right)\right\}, \quad {}^cD^{\alpha}v(t_n) = h^{-\alpha}\sum_{j=0}^{n-1} w_j^{(\alpha)} \left\{ u_{n-j} + \frac{1}{4}\alpha\left(u_{n-j+1}-u_{n-j-1}\right) + \frac{1}{8}\alpha^2\left(u_{n-j+1}-2u_{n-j}+u_{n-j-1}\right)\right\}, \tag{58} $$

where

$$ w_j^{(\beta)} = (-1)^j \binom{\beta}{j}, \quad (j = 0,1,2,\ldots), \quad \binom{\beta}{j} = \frac{\beta(\beta-1)\cdots(\beta-j+1)}{j!}, \quad w_j^{(\alpha)} = (-1)^j \binom{\alpha}{j}, \quad (j = 0,1,2,\ldots), \quad \binom{\alpha}{j} = \frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!}. \tag{59} $$

With the above algorithm we will give some examples.

Example 1. Consider the following fractional differential equations:

$$ {}^cD^{0.8}\left({}^cD^{1.5}+0.125\right)u(t) = 1.1121u(t)+t^2, \quad 0 \leq t \leq 1, \quad u(0) = 0, \quad \left({}^cD^{1.5}+0.125\right)u(0) = 0. \tag{60} $$

Obviously, we get

$$ \left|f\left(t,u(t),u'(t)\right)\right| \leq 1.1121\|u\|+1, \quad \left|f\left(t,u(t),u'(t)\right)-f\left(t,v(t),v'(t)\right)\right| \leq g^*\|u-v\|. \tag{61} $$

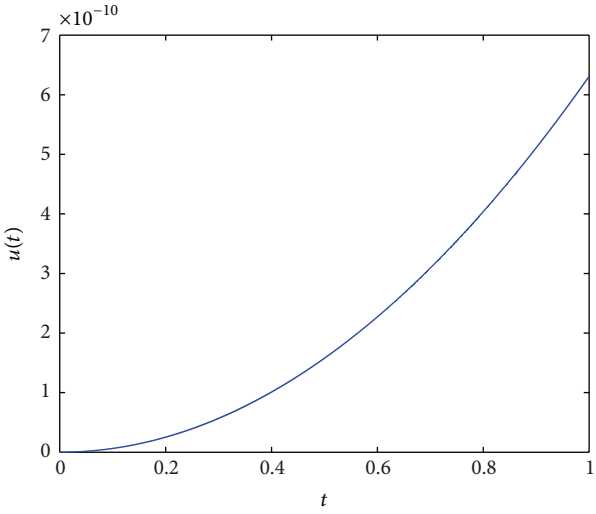

FIGURE 1: The displacement without a Wiener process, the numerical simulation of (60), where $\alpha = 1.5$, $\beta = 0.8$.

Letting $p = 0.9$, $g^* = 0.05$, $L = l_1 + l_2 = 0.01$, $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 2$, $\eta = 1.5$, and $g_1(u) = g_2(u) = 0$, we have

$$\begin{aligned}
&\alpha\beta_2\left[\left(\alpha_1+\beta_1\right)\eta+\beta_1\right] = 24 \neq \beta_1\left(\alpha_2+\beta_2\right) = 8,\\
&\Lambda \triangleq \max\{\Upsilon_1 + (\eta+1)\\
&\qquad \times\left(\alpha\beta_2 L + \beta_1\Lambda_2 L + \beta_1\beta_2\Lambda_2\Upsilon_2\right)\\
&\qquad + \beta_1\left(\alpha_2+\beta_2\right)\Lambda_2\Upsilon_1,\\
&\qquad \Upsilon_2 + \left(\alpha_2+\beta_2\right)\left(\alpha\Lambda_2 L + \alpha\beta_1\Lambda_2\Upsilon_1\right)\\
&\qquad + \alpha\Lambda_1\Lambda_2 L + \beta_2\beta_1\Lambda_2\Upsilon_2\}\\
&= \max\{0.3531, 0.3975\} = 0.3975 < 1.
\end{aligned} \tag{62}$$

Thus, by Theorem 10, we can get that the problem (60) has at most one solution.

With the above algorithm we get Figures 1 and 2.

Example 2. Consider the following fractional differential equations:

$$\begin{aligned}
&{}^cD^{0.723}\left({}^cD^{1.625} + 0.351\right)u(t)\\
&\quad = 1.1121e^{u(t)} + 1.3035t^3, \quad 0 \le t \le 1,\\
&u(0) = 0, \quad \left({}^cD^{1.625} + 0.351\right)u(0) = 0.
\end{aligned} \tag{63}$$

Obviously, we get

$$\begin{aligned}
&\left|f\left(t, u(t), u'(t)\right)\right| \le \|u\|^2,\\
&\left|f\left(t, u(t), u'(t)\right) - f\left(t, v(t), v'(t)\right)\right| \le g^*\|u - v\|.
\end{aligned} \tag{64}$$

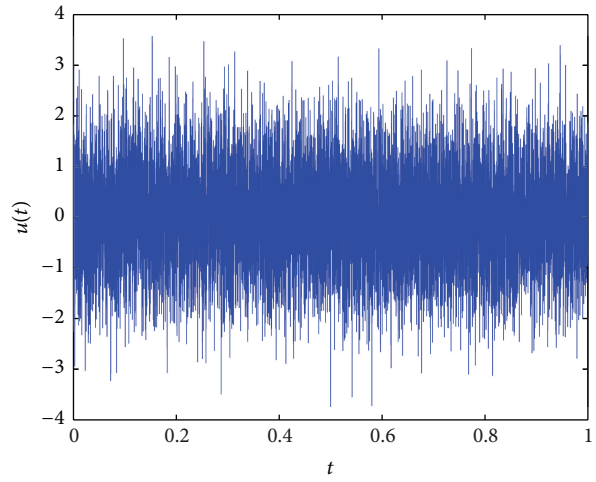

FIGURE 2: The displacement with a Wiener process, the numerical simulation of (60), where $\alpha = 1.5$, $\beta = 0.8$.

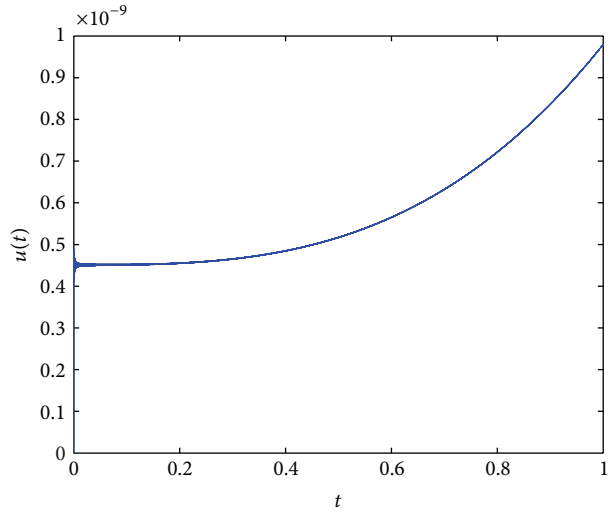

FIGURE 3: The displacement without a Wiener process, the numerical simulation of (63), where $\alpha = 1.625$, $\beta = 0.723$.

Letting $p = 0.9$, $g^* = 0.05$, $L = l_1 + l_2 = 0.01$, $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 1$, $\eta = 1$, and $g_1(u) = g_2(u) = 0$, we have

$$\begin{aligned}
&\alpha\beta_2\left[\left(\alpha_1+\beta_1\right)\eta+\beta_1\right] = 4.875 \neq \beta_1\left(\alpha_2+\beta_2\right) = 2,\\
&\Lambda \triangleq \max\{\Upsilon_1 + (\eta+1)\\
&\qquad \times\left(\alpha\beta_2 L + \beta_1\Lambda_2 L + \beta_1\beta_2\Lambda_2\Upsilon_2\right)\\
&\qquad + \beta_1\left(\alpha_2+\beta_2\right)\Lambda_2\Upsilon_1, \Upsilon_2\\
&\qquad + \left(\alpha_2+\beta_2\right)\left(\alpha\Lambda_2 L + \alpha\beta_1\Lambda_2\Upsilon_1\right)\\
&\qquad + \alpha\Lambda_1\Lambda_2 L + \beta_2\beta_1\Lambda_2\Upsilon_2\}\\
&= \max\{0.8257, 0.9842\} = 0.9842 < 1.
\end{aligned} \tag{65}$$

Thus, by Theorem 10, we can get that the problem (63) has at most one solution.

With the above algorithm we get Figures 3 and 4.

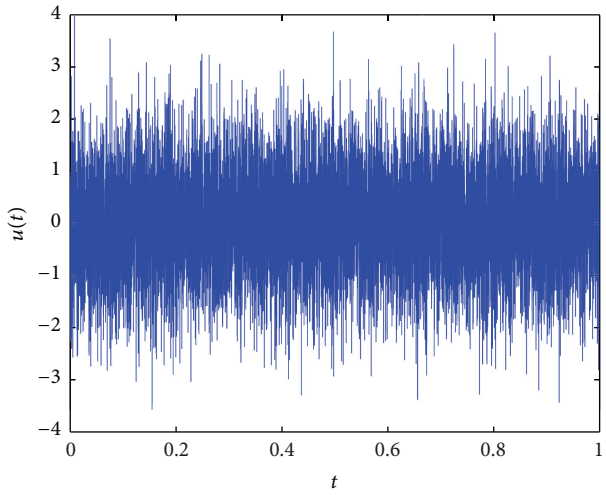

Figure 4: The displacement with a Wiener process, the numerical simulation of (63), where $\alpha = 1.625$, $\beta = 0.723$.

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