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Research Article

Delay-Dependent Finite-Time H_∞ Filtering for Markovian Jump Systems with Different System Modes

Yong Zeng,[1] Jun Cheng,[1] Shouming Zhong,[2] and Xiucheng Dong[3]

[1] *School of Automation Engineering, University of Electronic Science and Technology of China, Chengdu, Sichuan 611731, China*
[2] *School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu, Sichuan 611731, China*
[3] *Key Laboratory on Signal and Information Processing, Xihua University, Chengdu, Sichuan 610039, China*

Correspondence should be addressed to Jun Cheng; jcheng6819@126.com

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This paper is concerned with the problem of delay-dependent finite-time H_∞ filtering for Markovian jump systems with different system modes. By using the new augmented multiple mode-dependent Lyapunov-Krasovskii functional and employing the proposed integrals inequalities in the derivation of our results, a novel sufficient condition for finite-time boundness with an H_∞ performance index is derived. Particularly, two different Markov processes have been considered for modeling the randomness of system matrix and the state delay. Based on the derived condition, the H_∞ filtering problem is solved, and an explicit expression of the desired filter is also given; the system trajectory stays within a prescribed bound during a specified time interval. Finally, a numerical example is given to illustrate the effectiveness and the potential of the proposed techniques.

1. Introduction

Markovian jump systems were introduced by Krasovskiĭ and Lidskiĭ [1], which can be described by a set of systems with the transitions in a finite mode set. In the past few decades, there has been increasing interest in Markovian jump systems because this class of systems is appropriate to many physical systems which always go with random failures, repairs, and sudden environment disturbance [2–5]. Such class of systems is a special class of stochastic hybrid systems with finite operation modes, which may switch from one to another at different time, such as component failures, sudden environmental disturbance, and abrupt variations of the operating points of a nonlinear system. As a crucial factor, it is shown that such jumping can be determined by a Markovian chain [6]. For linear Markovian jumping systems, many important issues have been studied extensively such as stability, stabilization, control synthesis, and filter design [6–12]. In finite operation modes, Markovian jump systems are a special class of stochastic systems that can switch from one to another at different time.

It is worth pointing out that time delay is of interest to many researchers because of the fact that time delay is often encountered in various systems such as networked control systems, chemical processes, and communication systems. It is worth pointing out that time delay is one of the instability sources for dynamical systems and is a common phenomenon in many industrial and engineering systems. Hence, it is not surprising that much effort has been made to investigate Markovian jump systems with time delay during the last two decades [13–15]. The exponential stabilization of Markovian jump systems with time delay was firstly studied in [16] where the decay rate was estimated by solving linear matrix inequalities [17]. However, in the aforementioned works, the network-induced delays have been commonly assumed to be deterministic, which is fairly unrealistic since delays resulting from network transmissions are typically time varying [18–24].

Generally speaking, the delay-dependent criterions are less conservative than delay-independent ones, especially when the time delay is small enough in Markovian jump systems. Thus, recent efforts were devoted to the delay-dependent Markovian jump systems stability analysis by employing Lyapunov-Krasovskii functionals [25–33]. However, in most thesis, the time delay to be arbitrarily large are allowed in criterion, it always tends to be conservative.

Furthermore, though the decay rate can be computed, it is a fixed value that one cannot adjust to deduce if a larger decay rate is possible. Therefore, how to obtain the improved results without increasing the computational burden has greatly improved the current study. On the other hand, the practical problems which system described does not exceed a certain threshold over some finite time interval are considered. In finite-time interval, finite-time stability is investigated to address these transient performances of control systems. Recently, the concept of finite-time stability has been revisited in the light of linear matrix inequalities (LMIs) and Lyapunov function theory, and some results are obtained to ensure that systems are finite-time stability or finite-time boundness [34–50]. To the best of our knowledge, in most of the works about Markovian jump systems with mode-dependent delay, the delay mode is always assumed to be the same as the system matrices mode. However, in real systems, the delay mode may not be the same as that for jump in other system parameters. In other words, variations of delay usually depend on phenomena which may not cause abrupt changes in other systems parameters. Therefore, the work of Markovian jump systems with different system modes is not only theoretically interesting and challenging, but also very important in practical applications.

Motivated by the previous above discussions, in this paper, we present a new augmented Lyapunov functional for a class of Markovian jump systems with different system modes; in order to reduce the possible conservativeness and computational burden, some slack matrices are introduced [32]. Several sufficient conditions are derived to guarantee the finite-time stability and boundedness of the resulting closed-loop system. We find that finite-time stability is an independent concept from Lyapunov stability and always can be affected by switching behavior significantly, and the finite-time boundness criteria can be tackled in the form of LMIs. Finally, a numerical example is presented to illustrate the effectiveness of the developed techniques.

Notations. Throughout this paper, we let $P > 0$ ($P \geq 0$, $P < 0$, and $P \leq 0$) denote a symmetric positive definite matrix P (positive semidefinite, negative definite, and negative semidefinite). For any symmetric matrix P, $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote the maximum and minimum eigenvalues of matrix P, respectively. $\mathscr{R}^n$ denotes the n-dimensional Euclidean space, and $\mathscr{R}^{n\times m}$ refers to the set of all $n \times m$ real matrices and $\mathscr{N} = \{1, 2, \ldots, N\}$. The identity matrix of order n is denoted as I_n. $*$ represents the elements below the main diagonal of a symmetric matrix. The superscripts T and -1 stand for matrix transposition and matrix inverse, respectively.

2. Preliminaries

In this paper, we consider the following Markov jump system described by

$$\begin{aligned} \dot{x}(t) &= A_{r_t}x(t) + A_{\tau r_t}x\left(t - \tau_{s_t}(t)\right) + D_{r_t}\omega(t), \\ y(t) &= C_{yr_t}x(t) + C_{y\tau r_t}x\left(t - \tau_{s_t}(t)\right) + D_{yr_t}\omega(t), \\ z(t) &= C_{zr_t}x(t) + C_{z\tau r_t}x\left(t - \tau_{s_t}(t)\right) + D_{zr_t}\omega(t), \\ x(t) &= \varphi(t), \quad t = [-h, 0], \end{aligned} \tag{1}$$

where $x(t) \in \mathscr{R}^n$ is the state vector of the system, $y(t) \in \mathscr{R}^l$ is the measured output, $z(t) \in \mathscr{R}^q$ is the controlled output, $\varphi(t)$, $t \in [-h, 0]$ are initial conditions of continuous state, and $r_0 \in \mathscr{N}$, $s_0 \in \mathscr{M} = \{1, 2, \ldots, M\}$ are initial conditions of mode. $\omega(t) \in \mathscr{R}^q$ is the disturbance input, satisfying the following condition:

$$\int_0^\infty \omega^{\mathsf{T}}(t)\,\omega(t)\,dt \leq d. \tag{2}$$

Let the random form processes r_t, s_t be the Markov stochastic processes taking values on finite sets $\mathscr{N} = \{1, 2, \ldots, N\}$ and $\mathscr{M} = \{1, 2, \ldots, M\}$ with probability transition rate matrices $\Lambda = \{\lambda_{ij}\}$, $i, j \in \mathscr{N}$, and $\Pi = \{\pi_{m,n}\}$, $m, n \in \mathscr{M}$. The transition probabilities from mode j to mode j for Markov process r_t and from mode m to mode n for the Markov process s_t in time h are described as

$$\begin{aligned} \Pr\left(r_{t+\Delta} = j \mid r_t = i\right) &= \varrho_{ij} + \lambda_{ij}\Delta + o(\Delta), \\ \Pr\left(s_{t+\Delta} = n \mid s_t = m\right) &= \zeta_{mn} + \pi_{mn}\Delta + o(\Delta), \end{aligned} \tag{3}$$

where

$$\varrho_{ij} = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j, \end{cases} \qquad \zeta_{mn} = \begin{cases} 0, & \text{if } m \neq n, \\ 1, & \text{if } m = n, \end{cases} \tag{4}$$

and $\Delta > 0$, $\lambda_{ij} \geq 0$, for $i \neq j$, is the transition rate from mode i at time t to mode j at time $t + \Delta$ and

$$-\lambda_{ii} = \sum_{j=1, j\neq i}^{N} \lambda_{ij}, \tag{5}$$

for each mode $i \in \mathscr{N}$, $\lim_{\Delta \to 0_+}(o(\Delta)/\Delta) = 0$. $\pi_{mn} \geq 0$ for $m \neq n$ is the transition rate from mode m to mode n at time $t + \Delta$ and

$$-\pi_{mm} = \sum_{n=1, m\neq n}^{M} \pi_{mn}, \tag{6}$$

for each mode $i \in \mathscr{M}$, $\lim_{\Delta \to 0_+}(o(\Delta)/\Delta) = 0$. For convenience, we denote the Markov process r_t and s_t by i and m indices, respectively. $\tau_m(t)$ denotes the mode-dependent time-varying state delay in the system and satisfies the following condition:

$$\begin{aligned} 0 < \tau_m(t) &\leq h_m < \infty, \\ \dot{\tau}_m(t) &\leq \mu_m, \quad \forall m \in \mathscr{M}, \end{aligned} \tag{7}$$

where $h = \max\{h_i, i \in \mathscr{M}\}$ is prescribed integer representing the upper bounds of time-varying delay $\tau_m(t)$. Similarly, $\mu = \max\{\mu_m, m \in \mathscr{M}\}$ is prescribed integer representing the upper bounds of time-varying delay $\dot{\tau}_m(t)$. A_{r_t}, $A_{\tau r_t}$, D_{r_t}, C_{yr_t}, $C_{y\tau r_t}$, D_{yr_t}, C_{zr_t}, $C_{z\tau r_t}$, and D_{zr_t} are known mode-dependent

matrices with appropriate dimensions functions of the random jumping process $\{r_t\}$ and represent the nominal systems for each $r_t \in \mathcal{N}$. For notation simplicity, when the system operates in the i-th mode ($r_t = i$), A_{r_t}, $A_{\tau r_t}$, D_{r_t}, C_{yr_t}, $C_{y\tau r_t}$, D_{yr_t}, C_{zr_t}, $C_{z\tau r_t}$, and D_{zr_t} are denoted as A_i, $A_{\tau i}$, D_i, C_{yi}, $C_{y\tau i}$, D_{yi}, C_{zi}, $C_{z\tau i}$, and D_{zi}, respectively.

Here we are interested in designing a full-order filter described by

$$\begin{aligned} \dot{x}_f(t) &= A_{fr_t,s_t} x_f(t) + B_{fr_t,s_t} y(t), \\ z_f(t) &= C_{fr_t,s_t} x_f(t), \end{aligned} \tag{8}$$

where $x_f(t) \in \mathcal{R}^n$ is the filter state, $z_f(k) \in \mathcal{R}^q$, and the matrices A_{fr_t,s_t}, B_{fr_t,s_t}, and C_{fr_t,s_t} are unknown filter parameters to be designed.

Augmenting the model of (1) to include the filter (8), we obtain the following filtering error system:

$$\begin{aligned} \dot{\eta}(t) &= \overline{A}_{r_t}\eta(t) + \overline{B}_{r_t}\eta\left(t-\tau_m(t)\right) + \overline{D}_{r_t}\omega(t), \\ e(t) &= \overline{C}_{r_t}\eta(t) + \overline{E}_{r_t}\eta\left(t-\tau_m(t)\right), \end{aligned} \tag{9}$$

where

$$\begin{aligned} \eta(t) &= \begin{bmatrix} x(t) \\ x_f(t) \end{bmatrix}, \qquad e(t) = z(t) - z_f(t), \\ \overline{A}_{r_t} &= \begin{bmatrix} A_{r_t} & 0 \\ B_{fr_t,s_t}C_{yr_t} & A_{fr_t,s_t} \end{bmatrix}, \qquad \overline{B}_{r_t} = \begin{bmatrix} A_{\tau r_t} \\ B_{fr_t,s_t}C_{y\tau r_t} \end{bmatrix}, \\ \overline{D}_{r_t} &= \begin{bmatrix} D_{r_t} \\ B_{fr_t,s_t}D_{yr_t} \end{bmatrix}, \\ \overline{C}_{r_t} &= \begin{bmatrix} C_{zr_t}, & -C_{fr_t,s_t} \end{bmatrix}, \qquad \overline{E}_{r_t} = \begin{bmatrix} C_{z\tau r_t}, & 0 \end{bmatrix}. \end{aligned} \tag{10}$$

In order to more precisely describe the main objective, we introduce the following definitions and Lemmas for the underlying system.

Definition 1. System (1) is said to be finite-time bounded with respect to (c_1, c_2, T, R, d), if condition (2) and the following inequality hold:

$$\begin{aligned} &\sup_{-h\le\upsilon\le 0} \mathbb{E}\left\{\eta^{\mathsf{T}}(\upsilon) R\eta(\upsilon), \dot{\eta}^{\mathsf{T}}(\upsilon) R\dot{\eta}(\upsilon)\right\} \\ &\quad \le c_1 \Longrightarrow \mathbb{E}\left\{\eta^{\mathsf{T}}(t) R\eta(t)\right\} < c_2, \\ &\qquad\qquad \forall t \in [0, T], \end{aligned} \tag{11}$$

where $c_2 > c_1 \ge 0$ and $R > 0$.

Definition 2. Consider $V(\eta_t, r_t, s_t, t > 0)$ as the stochastic positive Lyapunov function; its weak infinitesimal operator is defined as

$$\begin{aligned} &\pounds V\left(\eta_t, r_t, s_t\right) \\ &\quad = \lim_{\Delta\to 0}\frac{1}{\Delta}\left[\mathbb{E}\left\{V\left(\eta_{t+\Delta}, r_{t+\Delta}, s_{t+\Delta}\right) \mid \eta_t, r_t, s_t\right\}\right. \\ &\qquad \left. -V\left(\eta_t, r_t, s_t\right)\right]. \end{aligned} \tag{12}$$

Definition 3. Given a constant $T > 0$, for all admissible $\omega(t)$ subject to condition (2), under zero initial conditions, if the closed-loop Markovian jump system (1) is finite-time bounded and the control outputs satisfy condition (8) with attenuation $\gamma > 0$,

$$\mathbb{E}\left\{\int_0^T e^{\mathsf{T}}(t) e(t)\, dt\right\} \le \gamma^2 e^{\eta T}\mathbb{E}\left\{\int_0^T \omega^{\mathsf{T}}(t)\omega(t)\, dt\right\}. \tag{13}$$

Then, the controller system (1) finite-time bounded with disturbance attenuation γ.

Remark 4. It should be pointed out that the assumption of zero initial condition in system (1) is only for the purpose of technical simplification in the derivation, and it does not cause loss of generality. In fact, if this assumption is lost, the same control result can be obtained along the same line, except for adding extra manipulations in the derivation and extra terms in the control presentation. However, in real-world applications, the initial condition of the underlying system is generally not zero.

Lemma 5 (see [32]). *Let $f_i : \mathcal{R}^m \to \mathcal{R}$ $(i = 1, 2, \ldots, N)$ have positive values in an open subset $\mathcal{D}$ of $\mathcal{R}^m$. Then, the reciprocally convex combination of f_i over $\mathcal{D}$ satisfies*

$$\begin{aligned} &\min_{\{\beta_i|\beta_i>0, \sum_i \beta_i = 1\}} \sum_i \frac{1}{\beta_i} f_i(t) = \sum_i f_i(t) + \max_{g_{i,j}(t)} \sum_{i\ne j} g_{i,j}(t) \\ &\text{subject to} \quad \left\{g_{i,j} : \mathcal{R}^m \longrightarrow \mathcal{R}, g_{j,i}(t) = g_{i,j}(t), \right. \\ &\qquad \left. \begin{bmatrix} f_i(t) & g_{i,j}(t) \\ g_{i,j}(t) & f_j(t) \end{bmatrix} \ge 0 \right\}. \end{aligned} \tag{14}$$

Lemma 6 (Schur Complement [17]). *Given constant matrices X, Y, Z, where $X = X^{\mathsf{T}}$ and $0 < Y = Y^{\mathsf{T}}$, then $X + Z^{\mathsf{T}}Y^{-1}Z < 0$ if and only if*

$$\begin{bmatrix} X & Z^{\mathsf{T}} \\ * & -Y \end{bmatrix} < 0 \quad \text{or} \quad \begin{bmatrix} -Y & Z \\ * & X \end{bmatrix} < 0. \tag{15}$$

3. Finite-Time H_∞ Performance Analysis

Theorem 7. *System (9) is finite-time bounded with respect to (c_1, c_2, d, R, T), if there exist matrices $P_{im} > 0$, $Q_{li} > 0$ $(l = 1, 2)$, $Q > 0$, $X_{im} > 0$, $X > 0$, $Y_{im} > 0$, $Y > 0$, $H > 0$, S_{im}, scalars $c_1 < c_2$, $T > 0\lambda_s > 0$, $(s = 1, 2, \ldots, 10)$, $\eta > 0$, and $\Lambda > 0$, such that for all $i, j \in \mathcal{N}$ and $m, n \in \mathcal{M}$, the following inequalities hold:*

$$\sum_{j\in\mathcal{N}, j\ne i} \lambda_{ij}\left(Q_{1j} + Q_{2j}\right) + \sum_{n\in\mathcal{M}, m\ne n} \pi_{mn} Q_{2i} - Q < 0, \tag{16}$$

$$\sum_{n\in\mathcal{M}, m\ne n} \pi_{mn} X_{in} + \sum_{j\in\mathcal{N}, i\ne j} \lambda_{ij} X_{jm} - X < 0, \tag{17}$$

$$\sum_{n\in\mathcal{M},m\neq n}\pi_{mn}Y_{in}+\sum_{j\in\mathcal{N},i\neq j}\lambda_{ij}Y_{jm}-Y<0, \tag{18}$$

$$\begin{bmatrix}\frac{Y_{im}}{h} & S_{im}\\ * & \frac{Y_{im}}{h}\end{bmatrix}>0, \tag{19}$$

$$\Xi_{1im}=\begin{bmatrix}\Xi_{11im} & \Xi_{12im} & -S_{im} & \overline{A}_i^{\tau}V_{im}^{\tau} & h\overline{A}_i^{\tau}M_{im}^{\tau} & P_{im}\overline{D}_i\\ * & \Xi_{22im} & S_{im}-\frac{Y_{im}}{h} & \overline{B}_i^{\tau}V_{im}^{\tau} & h\overline{B}_i^{\tau}M_{im}^{\tau} & 0\\ * & * & -Q_{1i}+\frac{Y_{im}}{h} & 0 & 0 & 0\\ * & * & * & \Xi_{44im} & -hM_{im}^{\tau} & V_{im}\overline{D}_i\\ * & * & * & * & -hX_{im} & hM_{im}\overline{D}_i\\ * & * & * & * & * & -\delta H\end{bmatrix}<0, \tag{20}$$

$$\Xi_{2im}=\begin{bmatrix}\Xi_{11im} & \Xi_{12im} & -S_{im} & \overline{A}_i^{\tau}V_{im}^{\tau} & h\overline{A}_i^{\tau}M_{im}^{\tau} & P_{im}\overline{D}_i\\ * & \Xi_{22im} & S_{im}-\frac{Y_{im}}{h} & \overline{B}_i^{\tau}V_{im}^{\tau} & h\overline{B}_i^{\tau}N_{im}^{\tau} & 0\\ * & * & -Q_{1i}+\frac{Y_{im}}{h} & 0 & 0 & 0\\ * & * & * & \Xi_{44im} & -hN_{im}^{\tau} & V_{im}\overline{D}_i\\ * & * & * & * & -hX_{im} & hN_{im}\overline{D}_i\\ * & * & * & * & * & -\delta H\end{bmatrix}<0, \tag{21}$$

$$c_1\Lambda+d\delta\lambda_{10}\frac{1}{\eta}\left(1-e^{-\eta T}\right)<\lambda_1e^{-\eta T}c_2, \tag{22}$$

where

$$\begin{aligned}\Xi_{11im}&=\sum_{n\in\mathcal{M}}\pi_{mn}P_{in}+\sum_{j\in\mathcal{N}}\lambda_{ij}P_{jm}\\&\quad+\delta P_{im}+P_{im}\overline{A}_i+\overline{A}_i^{\tau}P_{im}+e^{\delta h}Q_{1i}\\&\quad+e^{\delta h}Q_{2i}+hQ\frac{e^{\delta h}-1}{\delta}X_{im}\\&\quad+\frac{e^{\delta h}-\delta he^{\delta h}-1}{\delta^2}X+\frac{Y_{im}}{h},\\ \Xi_{12im}&=P_{im}\overline{B}_i-\frac{Y_{im}}{h}+S_{im},\\ \Xi_{22im}&=r\left(\mu_m\right)Q_{2i}+\frac{2Y_{im}}{h}-S_{im}-S_{im}^{\tau},\\ \Xi_{44im}&=-V_{im}-V_{im}^{\tau}+\frac{e^{\delta h}-1}{\delta}Y_{im}+\frac{e^{\delta h}-\delta he^{\delta h}-1}{\delta^2}Y,\\ r\left(\tau_m\right)&=\begin{cases}-\left(1-\mu_m\right)e^{\delta h_i}, & \text{if } \mu_m>1,\\ -\left(1-\mu_m\right), & \text{if } \mu_m\le 1,\end{cases}\\ \Lambda&=\lambda_2+he^{\delta h}\left(\lambda_3+\lambda_4\right)+h^2e^{\delta h}\left(\lambda_5+\lambda_6+\lambda_8\right)\\&\quad+\frac{1}{2}h^3e^{\delta h}\left(\lambda_7+\lambda_9\right),\\ \lambda_1&=\min_{i\in\mathcal{N},m\in\mathcal{M}}\lambda_{\min}\left(\widetilde{P}_{im}\right),\qquad \lambda_2=\max_{i\in\mathcal{N},m\in\mathcal{M}}\lambda_{\max}\left(\widetilde{P}_{im}\right),\\ \lambda_3&=\max_{i\in\mathcal{N}}\lambda_{\max}\left(\widetilde{Q}_{1i}\right),\\ \lambda_4&=\max_{i\in\mathcal{N}}\lambda_{\max}\left(\widetilde{Q}_{2i}\right),\qquad \lambda_5=\lambda_{\max}\left(\widetilde{Q}\right),\\ \lambda_6&=\max_{i\in\mathcal{N},m\in\mathcal{M}}\lambda_{\max}\left(\widetilde{X}_{im}\right),\qquad \lambda_7=\lambda_{\max}\left(\widetilde{X}\right),\\ \lambda_8&=\max_{i\in\mathcal{N},m\in\mathcal{M}}\lambda_{\max}\left(\widetilde{Y}_{im}\right),\qquad \lambda_9=\lambda_{\max}\left(\widetilde{Y}\right),\\ \lambda_{10}&=\lambda_{\max}\left(H\right),\\ \widetilde{P}_{im}&=R^{-(1/2)}P_{im}R^{-(1/2)},\\ \widetilde{Q}_{li}&=R^{-(1/2)}Q_{li}R^{-(1/2)}\ (l=1,2),\\ \widetilde{Q}&=R^{-(1/2)}QR^{-(1/2)},\qquad \widetilde{X}_{im}=R^{-(1/2)}X_{im}R^{-(1/2)},\\ \widetilde{X}&=R^{-(1/2)}XR^{-(1/2)},\qquad \widetilde{Y}_{im}=R^{-(1/2)}Y_{im}R^{-(1/2)},\\ \widetilde{Y}&=R^{-(1/2)}YR^{-(1/2)}.\end{aligned} \tag{23}$$

Proof. First, in order to cast our model into the framework of the Markov processes, we define a new process $\{(\eta_t,r_t,s_t),t\ge 0\}$ by

$$\eta_t(s)=\eta(t+s),\quad s\in[-h,0]. \tag{24}$$

Now, we consider the following Lyapunov-Krasovskii functional:

$$V\left(\eta_t,r_t,s_t\right)=\sum_{l=1}^{4}V_l\left(\eta_t,r_t,t\right), \tag{25}$$

where

$$\begin{aligned}V_1\left(\eta_t,r_t,s_t\right)&=\eta(t)^{\tau}e^{\delta t}P_{r_t,s_t}\eta(t),\\ V_2\left(\eta_t,r_t,s_t\right)&=\int_{t-h}^{t}e^{\delta(s+h)}\eta^{\tau}(s)Q_{1r_t}\eta(s)\,ds\\&\quad+\int_{t-\tau_{s_t}(t)}^{t}e^{\delta(s+h)}\eta^{\tau}(s)Q_{2r_t}\eta(s)\,ds\\&\quad+\int_{-h}^{0}\int_{t+\theta}^{t}e^{\delta(s+h)}\eta^{\tau}(s)Q\eta(s)\,ds\,d\theta,\\ V_3\left(\eta_t,r_t,s_t\right)&=\int_{-h}^{0}\int_{t+\theta}^{t}e^{\delta(s-\theta)}\eta^{\tau}(s)X_{r_t,s_t}\eta(s)\,ds\,d\theta\\&\quad+\int_{-h}^{0}\int_{\theta}^{0}\int_{t+\upsilon}^{t}e^{\delta(s-\theta)}\eta^{\tau}(s)X\eta(s)\,ds\,d\upsilon\,d\theta,\\ V_4\left(\eta_t,r_t,s_t\right)&=\int_{-h}^{0}\int_{t+\theta}^{t}e^{\delta(s-\theta)}\dot{\eta}^{\tau}(s)Y_{r_t,s_t}\dot{\eta}(s)\,ds\,d\theta\\&\quad+\int_{-h}^{0}\int_{\theta}^{0}\int_{t+\theta}^{t}e^{\delta(s-\theta)}\dot{\eta}^{\tau}(s)Y\dot{\eta}(s)\,ds\,d\upsilon\,d\theta.\end{aligned} \tag{26}$$

Then, for each $r_t = i$, $s_t = m$, we have

$$
\begin{aligned}
&\pounds V_1(\eta_t, i, m) \\
&= \lim_{\Delta\to 0^+}\frac{1}{\Delta}\mathbb{E}\left\{\eta^{\mathsf{T}}(t+\Delta)\, e^{\delta(t+\Delta)} P_{r_{t+\Delta}, s_{t+\Delta}}\eta(t+\Delta)\right. \\
&\qquad \left. -\eta^{\mathsf{T}}(t)\, e^{\delta t} P_{im}\eta(t)\right\} \\
&= \lim_{\Delta\to 0^+}\frac{1}{\Delta}\mathbb{E}\left\{\eta^{\mathsf{T}}(t+\Delta)\left[\left(1+\lambda_{ii}\Delta+o(\Delta)\right)\right.\right. \\
&\qquad \times\left(1+\pi_{mm}\Delta+o(\Delta)\right) e^{\delta(t+\Delta)} P_{im} \\
&\qquad +\left(1+\lambda_{ii}\Delta+o(\Delta)\right) \\
&\qquad \times\left(\sum_{n\in\mathcal{M}}\left(\pi_{mn}\Delta+o(\Delta)\right)\right) e^{\delta(t+\Delta)} P_{in} \\
&\qquad +\left(1+\pi_{mm}\Delta+o(\Delta)\right) \\
&\qquad \times\left(\sum_{j\in\mathcal{N}}\left(\lambda_{ij}\Delta+o(\Delta)\right)\right) e^{\delta(t+\Delta)} P_{jm} \\
&\qquad +\left(\sum_{n\in\mathcal{M}}\left(\pi_{mn}\Delta+o(\Delta)\right)\right) \\
&\qquad \left.\left.\times\left(\sum_{j\in\mathcal{N}}\left(\lambda_{ij}\Delta+o(\Delta)\right)\right) e^{\delta(t+\Delta)} P_{jn}\right] \right. \\
&\qquad \times\eta(t+\Delta) \\
&\qquad \left. -\eta^{\mathsf{T}}(t)\, e^{\delta t} P_{im}\eta(t)\right\} \\
&= \delta e^{\delta t}\eta^{\mathsf{T}}(t) P_{im}\eta(t) + 2e^{\delta t}\eta^{\mathsf{T}}(t) P_{im}\dot{\eta}(t) \\
&\quad + e^{\delta t}\eta^{\mathsf{T}}(t)\left(\sum_{n\in\mathcal{M}}\pi_{mn}P_{in}+\sum_{j\in\mathcal{N}}\lambda_{ij}P_{jm}\right)\eta(t) \\
&= e^{\delta t}\eta^{\mathsf{T}}(t)\left(\sum_{n\in\mathcal{M}}\pi_{mn}P_{in}+\sum_{j\in\mathcal{N}}\lambda_{ij}P_{jm}+\delta P_{im}\right. \\
&\qquad \left. + P_{im}\overline{A}_i + \overline{A}_i^{\mathsf{T}} P_{im}\right)\eta(t) \\
&\quad + 2e^{\delta t}\eta^{\mathsf{T}}(t) P_{im}\overline{B}_i\eta\left(t-\tau_m(t)\right) + 2e^{\delta t}\eta^{\mathsf{T}}(t) P_{im}\overline{D}_i\omega(t),
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
&\pounds V_2(\eta_t, i, m) \\
&= \lim_{\Delta\to 0^+}\frac{1}{\Delta}\mathbb{E}\left\{\int_{t+\Delta-h}^{t+\Delta}\eta^{\mathsf{T}}(s)\, e^{\delta(s+h)} Q_{1r_{t+\Delta}}\eta(s)\, ds\right. \\
&\qquad \left. -\int_{t-h}^{t}\eta^{\mathsf{T}}(s)\, e^{\delta(s+h)} Q_{1i}\eta(s)\, ds\right\} \\
&\quad + \lim_{\Delta\to 0^+}\frac{1}{\Delta}\mathbb{E}\left\{\int_{t+\Delta-\tau_{s_{t+\Delta}}(t)}^{t+\Delta}\eta^{\mathsf{T}}(s)\, e^{\delta(s+h)} Q_{2r_{t+\Delta}}\eta(s)\, ds\right. \\
&\qquad \left. -\int_{t-\tau_m(t)}^{t}\eta^{\mathsf{T}}(s)\, e^{\delta(s+h)} Q_{2i}\eta(s)\, ds\right\} \\
&\quad + \lim_{\Delta\to 0^+}\frac{1}{\Delta}\mathbb{E}\left\{\int_{-h}^{0}\int_{t+\Delta+\theta}^{t+\Delta} e^{\delta(s+h)}\eta^{\mathsf{T}}(s) Q\eta(s)\, ds\, d\theta\right. \\
&\qquad \left. -\int_{-h}^{0}\int_{t+\theta}^{t} e^{\delta(s+h)}\eta^{\mathsf{T}}(s) Q\eta(s)\, ds\, d\theta\right\} \\
&= \lim_{\Delta\to 0^+}\frac{1}{\Delta}\mathbb{E}\left\{\int_{t+\Delta-h}^{t+\Delta}\eta^{\mathsf{T}}(s)\, e^{\delta(s+h)} Q_{1i}\eta(s)\, ds\right. \\
&\qquad +\int_{t+\Delta-h}^{t+\Delta}\eta^{\mathsf{T}}(s) \\
&\qquad\quad \times\sum_{j\in\mathcal{N}}\left(\lambda_{ij}\Delta+o(\Delta)\right) \\
&\qquad\quad \times e^{\delta(s+h)} Q_{1i}\eta(s)\, ds \\
&\qquad \left. -\int_{t-h}^{t}\eta^{\mathsf{T}}(s) Q_{1i}\eta(s)\, ds\right\} \\
&\quad + \lim_{\Delta\to 0^+}\frac{1}{\Delta}\mathbb{E}\left[\int_{t+\Delta-\tau_j(t+\Delta)}^{t+\Delta}\eta^{\mathsf{T}}(s)\, e^{\delta(s+h)} Q_{2i}\eta(s)\, ds\right. \\
&\qquad +\sum_{n\in\mathcal{M}}\left(\pi_{mn}\Delta+o(\Delta)\right) \\
&\qquad \times\int_{t+\Delta-\tau_n(t+\Delta)}^{t}\eta^{\mathsf{T}}(s)\, e^{\delta(s+h)} Q_{2i}\eta(s)\, ds \\
&\qquad +\int_{t+\Delta}^{t+\Delta}-\tau_j(t+\Delta)^{t+\Delta}\eta^{\mathsf{T}}(s) \\
&\qquad\quad \times\sum_{j\in\mathcal{N}}\left(\lambda_{ij}\Delta+o(\Delta)\right) \\
&\qquad\quad \times e^{\delta(s+h)} Q_{2j}\eta(s)\, ds \\
&\qquad \left. -\int_{t-\tau_m(t)}^{t}\eta^{\mathsf{T}}(s)\, e^{\delta(s+h)} Q_{2i}\eta(s)\, ds\right] \\
&\quad + e^{\delta t}\eta^{\mathsf{T}}(t)\, hQ\eta(t) - \int_{t-h} e^{\delta(s+h)}\eta^{\mathsf{T}}(s) Q\eta(s)\, ds \\
&\le e^{\delta t}\eta^{\mathsf{T}}(t)\left(e^{\delta h}Q_{1i}+e^{\delta h}Q_{2i}+hQ\right)\eta(t) \\
&\quad - e^{\delta t}\eta^{\mathsf{T}}(t-h) Q_{1i}\eta(t-h) \\
&\quad - \left(1-\tau_m(t)\right) e^{\delta t}\eta^{\mathsf{T}}\left(t-\tau_m(t)\right) Q_{2i}\eta\left(t-\tau_m(t)\right) \\
&\quad + \int_{t-h}^{t} e^{\sigma(s+h)}\eta^{\mathsf{T}}(s)\left(\sum_{j\in\mathcal{N}, j\ne i}\lambda_{ij}\left(Q_{1j}+Q_{2j}\right)\right.
\end{aligned}
$$

$$+\sum_{n\in\mathcal{M},m\neq n}\pi_{mn}Q_{2i}-Q\Bigg)\times\eta(s)\,ds. \tag{28}$$

Since

$$0\le\tau_m(t)\le h_m, \tag{29}$$

we define

$$r(\tau_m)=\begin{cases}-(1-\mu_m)e^{\delta h_m}, & \text{if } \mu_m>1,\\ -(1-\mu_m), & \text{if } \mu_m\le 1.\end{cases} \tag{30}$$

Then,

$$\begin{aligned}\pounds V_2(\eta_t,i,m)\le{}& e^{\delta t}\eta^{\mathsf T}(t)\left(e^{\delta h}Q_{1i}+e^{\delta h}Q_{2i}+hQ\right)\eta(t)\\ &-e^{\delta t}\eta^{\mathsf T}(t-h)Q_{1i}\eta(t-h)\\ &+e^{\delta t}\eta^{\mathsf T}(t-\tau_m(t))\,r(\tau_m)Q_{2i}\eta(t-\tau_m(t))\\ &+\int_{t-h}^{t}e^{\sigma(s+h)}\eta^{\mathsf T}(s)\\ &\quad\times\Bigg(\sum_{j\in\mathcal{N},j\neq i}\lambda_{ij}\left(Q_{1j}+Q_{2j}\right)\\ &\qquad+\sum_{n\in\mathcal{M},m\neq n}\pi_{mn}Q_{2i}-Q\Bigg)\\ &\quad\times\eta(s)\,ds.\end{aligned} \tag{31}$$

Similar to the previous process, we can obtain

$$\begin{aligned}\pounds V_3(\eta_t,i,m)\le{}&\int_{-h}^{0}\int_{t+\theta}^{t}e^{\delta(s-\theta)}\eta^{\mathsf T}(s)\\ &\quad\times\Bigg(\sum_{n\in\mathcal{M},m\neq n}\pi_{mn}X_{in}\\ &\qquad+\sum_{j\in\mathcal{N},i\neq j}\lambda_{ij}X_{jm}-X\Bigg)\\ &\quad\times\eta(s)\,ds\,d\theta\\ &+e^{\delta t}\eta^{\mathsf T}(t)X_{im}\eta(t)\int_{-h}^{0}e^{-\delta\upsilon}d\upsilon\\ &-e^{\delta t}\int_{t-h}^{t}\eta^{\mathsf T}(s)X_{im}\eta(s)\,ds\\ &+e^{\delta t}\eta^{\mathsf T}(t)X\eta(t)\int_{-h}^{0}\int_{\upsilon}^{0}e^{-\delta\upsilon}d\theta d\upsilon.\end{aligned} \tag{32}$$

By using Lemma 5, it yields that

$$\begin{aligned}-\int_{t-h}^{t}\eta^{\mathsf T}(s)X_{im}\eta(s)\,ds={}&-\int_{t-\tau_m(t)}^{t}\eta^{\mathsf T}(s)X_{im}\eta(s)\,ds\\ &-\int_{t-h}^{t-\tau_m(t)}\eta^{\mathsf T}(s)X_{im}\eta(s)\,ds\\ \le{}&-\tau_m(t)U_1^{\mathsf T}X_{im}U_1\\ &-(h-\tau_m(t))U_2^{\mathsf T}X_{im}U_2,\end{aligned} \tag{33}$$

where

$$\begin{aligned}U_1&=\frac{1}{\tau_m(t)}\int_{t-\tau_m(t)}^{t}\eta(s)\,ds,\\ U_2&=\frac{1}{h-\tau_m(t)}\int_{t-h}^{t-\tau_m(t)}\eta(s)\,ds,\\ \lim_{\tau_m(t)\to 0}&\frac{1}{\tau_m(t)}\int_{t-\tau_m(t)}^{t}\eta(s)\,ds=\eta(t),\\ \lim_{\tau_m(t)\to h}&\frac{1}{h-\tau_m(t)}\int_{t-h}^{t-\tau_m(t)}\eta(s)\,ds=\eta(t-h).\end{aligned} \tag{34}$$

From the Newton-Leibniz formula, the following equation is true for any matrices M_{im}, N_{im}, and V_{im} with appropriate dimensions:

$$\begin{aligned}&\left(2\tau_m(t)U_1^{\mathsf T}M_{im}+2(h-\tau_m(t))U_2^{\mathsf T}N_{im}+2\dot\eta^{\mathsf T}(t)V_{im}\right)\\ &\times\left[-\dot\eta(t)+\overline{A}_i\eta(t)+\overline{B}_{\tau i}\eta(t-\tau_m(t))+\overline{D}_i\omega(t)\right]=0,\end{aligned} \tag{35}$$

$$\begin{aligned}&\pounds V_4(\eta_t,i,m)\\ &\le\int_{-h}^{0}\int_{t+\theta}^{t}e^{\delta(s-\theta)}\dot\eta^{\mathsf T}(s)\\ &\quad\times\Bigg(\sum_{n\in\mathcal{M},m\neq n}\pi_{mn}Y_{in}\\ &\qquad+\sum_{j\in\mathcal{N},i\neq j}\lambda_{ij}Y_{jm}-Y\Bigg)\\ &\quad\times\dot\eta(s)\,ds\,d\theta\\ &+e^{\delta t}\dot\eta^{\mathsf T}(t)Y_{im}\dot\eta(t)\int_{-h}^{0}e^{-\delta\upsilon}d\upsilon\\ &-e^{\delta t}\int_{t-h}^{t}\dot\eta^{\mathsf T}(s)Y_{im}\dot\eta(s)\,ds\\ &+e^{\delta t}\dot\eta^{\mathsf T}(t)Y\dot\eta(t)\int_{-h}^{0}\int_{\upsilon}^{0}e^{-\delta\upsilon}d\theta\,d\upsilon.\end{aligned} \tag{36}$$

From Lemma 5, it yields that

$$
\begin{aligned}
&-\int_{t-h}^{t}\dot{\eta}^{\mathsf{T}}(s)\,Y_{im}\dot{\eta}(s)\,ds\\
&=-\int_{t-\tau_m(t)}^{t}\dot{\eta}^{\mathsf{T}}(s)\,Y_{im}\dot{\eta}(s)\,ds\\
&\quad-\int_{t-h}^{t-\tau_m(t)}\dot{\eta}^{\mathsf{T}}(s)\,Y_{im}\dot{\eta}(s)\,ds\\
&\le-\frac{h}{\tau_m(t)}\left[\int_{t-\tau_m(t)}^{t}\dot{\eta}(s)\,ds\right]^{\mathsf{T}}\frac{Y_{im}}{h}\left[\int_{t-\tau_m(t)}^{t}\dot{\eta}(s)\,ds\right]\\
&\quad-\frac{h}{h-\tau_m(t)}\left[\int_{t-h}^{t-\tau_m(t)}\dot{\eta}(s)\,ds\right]^{\mathsf{T}}\frac{Y_{im}}{h}\left[\int_{t-h}^{t-\tau_m(t)}\dot{\eta}(s)\,ds\right]\\
&\le-\begin{bmatrix}\displaystyle\int_{t-\tau_m(t)}^{t}\dot{\eta}(s)\,ds\\ \displaystyle\int_{t-h}^{t-\tau_m(t)}\dot{\eta}(s)\,ds\end{bmatrix}^{\mathsf{T}}\begin{bmatrix}\frac{Y_{im}}{h} & S_{im}\\ * & \frac{Y_{im}}{h}\end{bmatrix}\begin{bmatrix}\displaystyle\int_{t-\tau_m(t)}^{t}\dot{\eta}(s)\,ds\\ \displaystyle\int_{t-h}^{t-\tau_m(t)}\dot{\eta}(s)\,ds\end{bmatrix}.
\end{aligned}
\tag{37}
$$

From (25)–(37), we can eventually obtain

$$
\pounds V(\eta_t,r_t,s_t)-\delta\omega^{\mathsf{T}}(t)\,H\omega(t)\le e^{\delta t}\xi^{\mathsf{T}}(t)\,\overline{\Xi}_{im}\xi(t),
\tag{38}
$$

where

$$
\xi^{\mathsf{T}}(t)=\left[\eta^{\mathsf{T}}(t),\,\eta^{\mathsf{T}}(t-\tau_m(t)),\,\eta^{\mathsf{T}}(t-h),\,\dot{\eta}^{\mathsf{T}}(t),\,U_1^{\mathsf{T}},\,U_2^{\mathsf{T}},\,\omega^{\mathsf{T}}(t)\right],
$$

$$
\overline{\Xi}_{im}=\begin{bmatrix}
\Xi_{11im} & \Xi_{12im} & -S_{im} & A_i^{\mathsf{T}}V_{im}^{\mathsf{T}} & \tau_m(t)\,A_i^{\mathsf{T}}M_{im}^{\mathsf{T}} & (h-\tau_m(t))\,A_i^{\mathsf{T}}N_{im}^{\mathsf{T}} & P_{im}D_i\\
* & \Xi_{22im} & S_{im}-\frac{Y_{im}}{h} & A_{\tau i}^{\mathsf{T}}V_{im}^{\mathsf{T}} & \tau_m(t)\,A_{\tau i}^{\mathsf{T}}M_{im}^{\mathsf{T}} & (h-\tau_m(t))\,A_{\tau i}^{\mathsf{T}}N_{im}^{\mathsf{T}} & 0\\
* & * & -Q_{1i}+\frac{Y_{im}}{h} & 0 & 0 & 0 & 0\\
* & * & * & \Xi_{44im} & -\tau_m(t)\,M_{im}^{\mathsf{T}} & -(h-\tau_m(t))\,N_{im}^{\mathsf{T}} & V_{im}D_i\\
* & * & * & * & -\tau_m(t)\,X_{im} & 0 & \tau_m(t)\,M_{im}D_i\\
* & * & * & * & * & -(h-\tau_m(t))\,X_{im} & (h-\tau_m(t))\,N_{im}D_i\\
* & * & * & * & * & * & -\delta H
\end{bmatrix}.
\tag{39}
$$

The LMIs (20) and (21) lead to $\tau_m(t)\to h$ and $\tau_m(t)\to 0$, respectively. It is easy to see that Ξ_{1im} results from $\overline{\Xi}_{im}|\tau_m(t)=h$ and $\overline{\Xi}_{im}|\tau_m(t)=0$. Thus, we obtain

$$
\mathbb{E}\left\{\pounds V(\eta_t,r_t,s_t)\right\}\le\eta\mathbb{E}\left[V(\eta_t,r_t,s_t)\right]+\delta\omega^{\mathsf{T}}(t)\,H\omega(t).
\tag{40}
$$

Multiplying the aforementioned inequality by $e^{-\eta t}$, we can get

$$
\mathbb{E}\left\{\pounds\left[e^{-\eta t}V(\eta_t,r_t,s_t)\right]\right\}\le e^{-\eta t}\delta\omega^{\mathsf{T}}(t)\,H\omega(t).
\tag{41}
$$

By integrating the aforementioned inequality between 0 and t, it follows that

$$
\begin{aligned}
&e^{-\eta t}\mathbb{E}\left[V(\eta_t,r_t,s_t)\right]-\mathbb{E}\left[V(\eta_0,r_0,s_0)\right]\\
&\quad\le\delta\int_0^t e^{-\eta s}\omega^{\mathsf{T}}(s)\,H\omega(s)\,ds.
\end{aligned}
\tag{42}
$$

Denote that $\widetilde{P}_{im}=R^{-(1/2)}P_{im}R^{-(1/2)}$, $\widetilde{Q}_{si}=R^{-(1/2)}Q_{si}R^{-(1/2)}$ $(s=1,2)$, $\widetilde{Q}=R^{-(1/2)}QR^{-(1/2)}$, $\widetilde{X}_{im}=R^{-(1/2)}X_{im}R^{-(1/2)}$, $\widetilde{X}=R^{-(1/2)}XR^{-(1/2)}$, $\widetilde{Y}_{im}=R^{-(1/2)}Y_{im}R^{-(1/2)}$, and $\widetilde{Y}=R^{-(1/2)}YR^{-(1/2)}$; it yields that

$$
\begin{aligned}
&\mathbb{E}\left[V(\eta_0,r_0,s_0)\right]\\
&\le\max_{i\in\mathcal{N},m\in\mathcal{M}}\lambda_{\max}\left(\widetilde{P}_{im}\right)\eta^{\mathsf{T}}(0)\,R\eta(0)\\
&\quad+\left(\max_{i\in\mathcal{N}}\lambda_{\max}\left(\widetilde{Q}_{1i}\right)+\max_{i\in\mathcal{N}}\lambda_{\max}\left(\widetilde{Q}_{2i}\right)\right)e^{\delta h}\\
&\quad\times\int_{-h}^{0}e^{\delta s}\eta^{\mathsf{T}}(s)\,R\eta(s)\,ds\\
&\quad+e^{\delta h}\lambda_{\max}\left(\widetilde{Q}\right)\int_{-h}^{0}\int_{\theta}^{0}e^{\delta s}\eta^{\mathsf{T}}(s)\,R\eta(s)\,ds\\
&\quad+e^{\delta h}\max_{i\in\mathcal{N},m\in\mathcal{M}}\lambda_{\max}\left(\widetilde{X}_{im}\right)\\
&\quad\times\int_{-h}^{0}\int_{\theta}^{0}e^{-\delta\theta}\eta^{\mathsf{T}}(s)\,R\eta(s)\,ds\,d\theta\\
&\quad+e^{\delta h}\lambda_{\max}\left(\widetilde{X}\right)\\
&\quad\times\int_{-h}^{0}\int_{\theta}^{0}\int_{\upsilon}^{0}e^{-\delta\theta}\eta^{\mathsf{T}}(s)\,R\eta(s)\,ds\,d\theta\,d\upsilon\\
&\quad+e^{\delta h}\max_{i\in\mathcal{N},m\in\mathcal{M}}\lambda_{\max}\left(\widetilde{Y}_{im}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\times \int_{-h}^{0}\int_{\theta}^{0} e^{-\delta\theta}\dot{\eta}^{\mathsf{T}}(s)\,R\dot{\eta}(s)\,ds\,d\theta \\
&+ e^{\delta h}\lambda_{\max}\left(\widetilde{Y}\right) \\
&\times \int_{-h}^{0}\int_{\theta}^{0}\int_{\upsilon}^{0} e^{-\delta\theta}\dot{\eta}^{\mathsf{T}}(s)\,R\dot{\eta}(s)\,ds\,d\theta\,d\upsilon \\
\le\ &\Bigg\{\max_{i\in\mathcal{N},m\in\mathcal{M}}\lambda_{\max}\left(\widetilde{P}_{im}\right) \\
&+ he^{\delta h}\left(\max_{i\in\mathcal{N}}\lambda_{\max}\left(Q_{1i}+\max_{i\in\mathcal{N}}\lambda_{\max}\left(Q_{2i}\right)\right)\right) \\
&+ h^{2}e^{\delta h}\Bigg(\max_{i\in\mathcal{N},m\in\mathcal{M}}\lambda_{\max}\left(\widetilde{X}_{im}\right)+\lambda_{\max}\left(\widetilde{Q}\right) \\
&\qquad + \max_{i\in\mathcal{N},m\in\mathcal{M}}\lambda_{\max}\left(\widetilde{Y}_{im}\right)\Bigg) \\
&+\frac{1}{2}h^{3}e^{\delta h}\left(\lambda_{\max}\left(\widetilde{X}\right)+\lambda_{\max}\left(\widetilde{Y}\right)\right)\Bigg\} \\
&\times \sup_{-h\le s\le 0}\left\{\eta^{\mathsf{T}}(s)\,R\eta(s),\dot{\eta}^{\mathsf{T}}(s)\,R\dot{\eta}(s)\right\} \\
=\ &c_{1}\Lambda.
\end{aligned}
\tag{43}
$$

For given $\eta > 0$ and $0 \le t \le T$, we have

$$
\begin{aligned}
\mathbb{E}\left[V\left(\eta_{t},r_{t},s_{t}\right)\right] &\le \mathbb{E}\left[e^{\eta t}V\left(\eta_{0},r_{0},s_{0}\right)\right] \\
&\quad + e^{\eta t}\delta\int_{0}^{t} e^{-\eta s}\omega^{\mathsf{T}}(s)\,H\omega(s)\,ds \\
&\le e^{\eta T}c_{1}\Lambda + d\delta e^{\eta T}\lambda_{\max}(H)\int_{0}^{T} e^{-\eta s}ds \\
&\le e^{\eta T}\left\{c_{1}\Lambda + d\delta\lambda_{10}\frac{1}{\eta}\left(1-e^{-\eta T}\right)\right\}.
\end{aligned}
\tag{44}
$$

On the other hand, it follows from (25) that

$$
\begin{aligned}
\mathbb{E}\left[V\left(\eta_{t},r_{t},s_{t}\right)\right] &\ge \mathbb{E}\left[\eta^{\mathsf{T}}(t)\,e^{\lambda t}P_{im}\eta(t)\right] \\
&\ge \max_{i\in\mathcal{N}}\lambda_{\min}\left(P_{im}\right)\mathbb{E}\left[\eta^{\mathsf{T}}(t)\,R\eta(t)\right] \\
&= \lambda_{1}\mathbb{E}\left[\eta^{\mathsf{T}}(t)\,R\eta(t)\right].
\end{aligned}
\tag{45}
$$

It can be derived from (43)–(45) that

$$
\mathbb{E}\left[\eta^{\mathsf{T}}(t)\,R\eta(t)\right] \le \frac{e^{\eta T}}{\lambda_{1}}\left\{c_{1}\Lambda + d\delta\lambda_{10}\frac{1}{\eta}\left(1-e^{-\eta T}\right)\right\}. \tag{46}
$$

From (22) and (46), we have

$$
\mathbb{E}\left[\eta^{\mathsf{T}}(t)\,R\eta(t)\right] < c_{2}. \tag{47}
$$

Then, the system is finite-time bounded with respect to (c_1, c_2, d, R, T). □

Remark 8. In this paper, $\tau_m(t)$ and $\dot{\tau}_m(t)$ may have different upper bounds in various delay intervals satisfying (7), respectively. However, in previous work such as [20, 21], $\tau_m(t)$ and $\dot{\tau}_m(t)$ are enlarged to $\tau_m(t) \le h = \max\{h_i, m \in \mathcal{M}\}$ and $\dot{\tau}_m(t) \le \mu = \max\{\mu_m, m \in \mathcal{M}\}$, respectively, which may lead to conservativeness inevitably. However, the previouse case can be taken fully into account by employing the Lyapunov-Krasovskii functional (25).

Remark 9. When dealing with term $-\int_{t-h}^{t}\dot{\eta}^{\mathsf{T}}(s)Y_{im}\dot{\eta}(s)ds$, the convex combination is not employed, Lemma 5 is used in this paper, and then the free-weighting matrices-dependent null add items are necessary to be introduced in our proof, which lead to the decrease of the number of LMIs and LMIs scalar decision variables.

Remark 10. The feature of this paper is the way to deal with the integral term. Many researchers have enlarged the derivative of the Lyapunov functional in order to deal with the integral term in mathematical operations. In this paper, we propose a novel delay-dependent sufficient criterion, which ensures that the Markovian jump system with different mode systems is finite-time stable.

Remark 11. It should be pointed out that the novelty of the Lyapunov functional (25) lies in distinct Lyapunov matrices (P_{im}, X_{im}, Y_{im}) which is chosen for different system modes i $(i = 1, 2, \ldots, N)$ and m $(m = 1, 2, \ldots, M)$.

Theorem 12. *System* (9) *is finite-time bounded with respect to* (c_1, c_2, d, R, T), *if there exist matrices* $P_i > 0$, $Q_{li} > 0$ $(l = 1, 2)$, $Q > 0$, $X_{im} > 0$, $X > 0$, $Y_{im} > 0$, $Y > 0$, S_{im}, *scalars* $c_1 < c_2$, $T > 0$, $\lambda_s > 0$ $(s = 1, 2, \ldots, 9)$, $\eta > 0$, $\gamma > 0$, *and* $\Lambda > 0$, *such that for all* $i, j \in \mathcal{N}$ *and* $m, n \in \mathcal{M}$ *and* (16)–(19), *the following inequalities hold:*

$$
\Sigma_{1im} = \begin{bmatrix}
\Xi_{11im} & \Xi_{12im} & -S_{im} & \overline{A}_i^{\mathsf{T}}V_{im}^{\mathsf{T}} & h\overline{A}_i^{\mathsf{T}}M_{im}^{\mathsf{T}} & P_{im}\overline{D}_i & \overline{C}_i^{\mathsf{T}} \\
* & \Xi_{22im} & S_{im}-\frac{Y_{im}}{h} & \overline{B}_i^{\mathsf{T}}V_{im}^{\mathsf{T}} & h\overline{B}_i^{\mathsf{T}}M_{im}^{\mathsf{T}} & 0 & \overline{E}_i^{\mathsf{T}} \\
* & * & -Q_{1i}+\frac{Y_{im}}{h} & 0 & 0 & 0 & 0 \\
* & * & * & \Xi_{44im} & -hM_{im}^{\mathsf{T}} & V_{im}\overline{D}_i & 0 \\
* & * & * & * & -hX_{im} & hM_{im}\overline{D}_i & 0 \\
* & * & * & * & * & -\gamma^{2}I & 0 \\
* & * & * & * & * & * & -I
\end{bmatrix} < 0,
\tag{48}
$$

$$
\Sigma_{2im} = \begin{bmatrix}
\Xi_{11im} & \Xi_{12im} & -S_{im} & \overline{A}_i^{\mathsf{T}}V_{im}^{\mathsf{T}} & h\overline{A}_i^{\mathsf{T}}N_{im}^{\mathsf{T}} & P_{im}\overline{D}_i & \overline{C}_i^{\mathsf{T}} \\
* & \Xi_{22im} & S_{im}-\frac{Y_{im}}{h} & \overline{B}_i^{\mathsf{T}}V_{im}^{\mathsf{T}} & h\overline{B}_i^{\mathsf{T}}N_{im}^{\mathsf{T}} & 0 & \overline{E}_i^{\mathsf{T}} \\
* & * & -Q_{1i}+\frac{Y_{im}}{h} & 0 & 0 & 0 & 0 \\
* & * & * & \Xi_{44im} & -hN_{im}^{\mathsf{T}} & V_{im}\overline{D}_i & 0 \\
* & * & * & * & -hX_{im} & hN_{im}\overline{D}_i & 0 \\
* & * & * & * & * & -\gamma^{2}I & 0 \\
* & * & * & * & * & * & -I
\end{bmatrix} < 0,
\tag{49}
$$

$$
c_{1}\Lambda + d\delta\gamma^{2}\frac{1}{\eta}\left(1-e^{-\eta T}\right) < \lambda_{1}e^{-\eta T}c_{2}. \tag{50}
$$

Proof. We now consider the H_∞ performance of system (9). Select the same Lyapunov-Krasovskii functional as Theorem 7; it yields that

$$\pounds V(\eta_t, r_t, s_t) + e^\intercal(t)\, e(t) - \gamma^2 \omega^\intercal(t)\, \omega(t) \le \xi^\intercal(t)\, \Sigma_{lim} \xi(t) \quad (l = 1, 2). \tag{51}$$

It follows from (49)-(50) that

$$\mathbb{E}\{\pounds V(\eta_t, r_t, s_t)\} \le \mathbb{E}\left[\eta V(\eta_t, r_t, s_t)\right] + \gamma^2 \omega^\intercal(t)\, \omega(t) - \mathbb{E}\left[e^\intercal(t)\, e(t)\right]. \tag{52}$$

Multiplying the aforementioned inequality by $e^{-\eta t}$, one has

$$\mathbb{E}\left\{\pounds\left[e^{-\eta t} V(\eta_t, r_t, s_t)\right]\right\} \le e^{-\eta t}\left[\gamma^2 \omega^\intercal(t)\, \omega(t) - e^\intercal(t)\, e(t)\right]. \tag{53}$$

In zero initial condition and $\mathbb{E}[V(\eta_t, r_t, s_t)] > 0$, by integrating the aforementioned inequality between 0 and T, we can get

$$\int_0^T e^{-\eta t}\left[\gamma^2 \omega^\intercal(t)\, \omega(t) - e^\intercal(t)\, e(t)\right] dt \le \mathbb{E}\left\{\int_0^T \pounds\left[e^{-\eta t} V(\eta_t, r_t, s_t)\right] dt\right\} \le V(\eta_0, r_0, s_0) = 0. \tag{54}$$

Using Dynkins formula, it results that

$$\mathbb{E}\left[\int_0^T e^{-\eta \upsilon} e^\intercal(\upsilon)\, e(\upsilon)\, d\upsilon\right] \le \gamma^2 \mathbb{E}\left[\int_0^T e^{-\eta \upsilon} \omega^\intercal(\upsilon)\, \omega(\upsilon)\, d\upsilon\right]. \tag{55}$$

Then, it yields that

$$\mathbb{E}\left[\int_0^T e^\intercal(\upsilon)\, e(\upsilon)\, d\upsilon\right] \le \gamma^2 e^{\eta T} \mathbb{E}\left[\int_0^T \omega^\intercal(\upsilon)\, \omega(\upsilon)\, d\upsilon\right]. \tag{56}$$

Thus, it is concluded by Definition 3 that system (9) is finite-time bounded with an H_∞ performance γ. This completes the proof. □

Remark 13. From the proof process of Theorems 7 and 12, it is easy to see that neither bounding technique for cross-terms nor model transformation is involved. In other words, the obtained result is expected to be less conservative.

Remark 14. The Lyapunov asymptotic stability and finite-time stability of a class of system are independent concepts. A Lyapunov asymptotic stability system may not be finite-time stability. Moreover, finite-time stability system may also not be Lyapunov asymptotic stability. There exist some results on Lyapunov stability, while finite-time stability also needs our full investigation, which was neglected by most previous work.

4. Finite-Time H_∞ Filtering

Theorem 15. *System* (9) *is finite-time bounded with respect to* (c_1, c_2, d, R, T), *if there exist matrices* $P_{im} > 0$, M_{im}, N_{im}, V_{im}, $Q_{li} > 0$ $(l = 1, 2)$, $Q > 0$, $X_{im} > 0$, $X > 0$, $Y_{im} > 0$, $Y > 0$, $\overline{A}_{fim}$, $\overline{B}_{fim}$, $\overline{C}_{fim}$, $\overline{M}_{Aim}$, $\overline{M}_{Bim}$, $\overline{N}_{Aim}$, $\overline{N}_{Bim}$, $\overline{V}_{Aim}$, $\overline{V}_{Bim}$, S_{im} *scalars* $c_1 < c_2$, $T > 0$, $\sigma_s > 0$ $(s = 1, 2, \ldots, 9)$, $\delta > 0$, $\eta > 0$, *and* $\overline{\Lambda} > 0$, *such that for all* $i, j \in \mathcal{N}$ *and* $m, n \in \mathcal{M}$, *the following inequalities hold:*

$$P_{im} = \begin{bmatrix} P_{1im} & P_{2im} \\ P_{2im} & P_{2im} \end{bmatrix} > 0, \qquad M_{im} = \begin{bmatrix} M_{1im} & M_{2im} \\ M_{2im} & M_{2im} \end{bmatrix}, \tag{57}$$

$$N_{im} = \begin{bmatrix} N_{1im} & N_{2im} \\ N_{2im} & N_{2im} \end{bmatrix}, \qquad V_{im} = \begin{bmatrix} V_{1im} & V_{2im} \\ V_{2im} & V_{2im} \end{bmatrix}, \tag{58}$$

$$\Gamma_{1im} = \begin{bmatrix} \Gamma_{11im} & \Gamma_{12im} & -S_{im} & \Gamma_{14im} & \Gamma_{15im} & \Gamma_{16im} & \Gamma_{17im} \\ * & \Xi_{22im} & S_{im} - \frac{Y_{im}}{h} & \Gamma_{24im} & \Gamma_{25im} & 0 & \begin{bmatrix} C_{z\tau i}^\intercal \\ 0 \end{bmatrix} \\ * & * & -Q_{1i} + \frac{Y_{im}}{h} & 0 & 0 & 0 & 0 \\ * & * & * & \Xi_{44im} & -hM_{im}^\intercal & \Gamma_{46im} & 0 \\ * & * & * & * & -hX_{im} & \Gamma_{56im} & 0 \\ * & * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & * & -I \end{bmatrix} < 0, \tag{59}$$

$$\Gamma_{2im} = \begin{bmatrix} \Gamma_{11im} & \Gamma_{12im} & -S_{im} & \Gamma_{14im} & \Gamma'_{15im} & \Gamma_{16im} & \Gamma_{17im} \\ * & \Xi_{22im} & S_{im} - \frac{Y_{im}}{h} & \Gamma_{24im} & \Gamma'_{25im} & 0 & \begin{bmatrix} C_{z\tau i}^\intercal \\ 0 \end{bmatrix} \\ * & * & -Q_{1i} + \frac{Y_{im}}{h} & 0 & 0 & 0 & 0 \\ * & * & * & \Xi_{44im} & -hN_{im}^\intercal & \Gamma'_{46im} & 0 \\ * & * & * & * & -hX_{im} & \Gamma'_{56im} & 0 \\ * & * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & * & -I \end{bmatrix} < 0, \tag{60}$$

where

$$\Gamma_{11im} = \sum_{n \in \mathcal{M}} \pi_{mn} P_{in} + \sum_{j \in \mathcal{N}} \lambda_{ij} P_{jm} + e^{\delta h} Q_{1i} + e^{\delta h} Q_{2i} + hQ + \frac{e^{\delta h} - 1}{\delta} X_{im} + \frac{e^{\delta h} - \delta h e^{\delta h} - 1}{\delta^2} X + \frac{Y_{im}}{h} + \begin{bmatrix} \delta P_{1im} + P_{1im} A_i + A_i^\intercal P_{1im} + \overline{B}_{fim} C_{yi} + C_{yi}^\intercal \overline{B}_{fim}^\intercal & \delta P_{2im} + \overline{A}_{fim} + \overline{A}_{fim}^\intercal \\ \delta P_{2im} + P_{2im} A_i + A_i^\intercal P_{2im} + \overline{B}_{fim} C_{yi} + C_{yi}^\intercal \overline{B}_{fim}^\intercal & \delta P_{2im} + \overline{A}_{fim} + \overline{A}_{fim}^\intercal \end{bmatrix}, \tag{61}$$

$$\Gamma_{12im} = -\frac{Y_{im}}{h} + S_{im} + \begin{bmatrix} P_{1im}A_{\tau i} + \overline{B}_{fim}C_{y\tau i} \\ P_{2im}A_{\tau i} + \overline{B}_{fim}C_{y\tau i} \end{bmatrix},$$

$$\Gamma_{14im} = \begin{bmatrix} A_i^{\mathsf{T}}V_{1im}^{\mathsf{T}} + C_{yi}^{\mathsf{T}}\overline{V}_{Bim}^{\mathsf{T}} & \overline{V}_{Aim}^{\mathsf{T}} \\ A_i^{\mathsf{T}}V_{2im}^{\mathsf{T}} + C_{yi}^{\mathsf{T}}\overline{V}_{Bim}^{\mathsf{T}} & \overline{V}_{Aim}^{\mathsf{T}} \end{bmatrix},$$

$$\Gamma_{15im} = \begin{bmatrix} hA_i^{\mathsf{T}}M_{1im}^{\mathsf{T}} + hC_{yi}^{\mathsf{T}}\overline{M}_{Bim}^{\mathsf{T}} & h\overline{M}_{Aim}^{\mathsf{T}} \\ hA_i^{\mathsf{T}}M_{2im}^{\mathsf{T}} + hC_{yi}^{\mathsf{T}}\overline{M}_{Bim}^{\mathsf{T}} & h\overline{M}_{Aim}^{\mathsf{T}} \end{bmatrix},$$

$$\Gamma'_{15im} = \begin{bmatrix} hA_i^{\mathsf{T}}N_{1im}^{\mathsf{T}} + hC_{yi}^{\mathsf{T}}\overline{N}_{Bim}^{\mathsf{T}} & h\overline{N}_{Aim}^{\mathsf{T}} \\ hA_i^{\mathsf{T}}N_{2im}^{\mathsf{T}} + hC_{yi}^{\mathsf{T}}\overline{N}_{Bim}^{\mathsf{T}} & h\overline{N}_{Aim}^{\mathsf{T}} \end{bmatrix},$$

$$\Gamma_{16im} = \begin{bmatrix} P_{1im}D_i + \overline{B}_{fim}D_{yi} \\ P_{2im}D_i + \overline{B}_{fim}D_{yi} \end{bmatrix},$$

$$\Gamma_{17im} = \begin{bmatrix} C_{zi}^{\mathsf{T}} \\ -\overline{C}_{fim} \end{bmatrix},$$

$$\Gamma_{24im} = \begin{bmatrix} A_{\tau i}^{\mathsf{T}}V_{1im}^{\mathsf{T}} + C_{y\tau i}^{\mathsf{T}}\overline{V}_{Bim}^{\mathsf{T}} \\ A_{\tau i}^{\mathsf{T}}V_{2im}^{\mathsf{T}} + C_{y\tau i}^{\mathsf{T}}\overline{V}_{Bim}^{\mathsf{T}} \end{bmatrix},$$

$$\Gamma_{25im} = \begin{bmatrix} hA_{\tau i}^{\mathsf{T}}M_{1im}^{\mathsf{T}} + hC_{y\tau i}^{\mathsf{T}}\overline{M}_{Bim}^{\mathsf{T}} \\ hA_{\tau i}^{\mathsf{T}}M_{2im}^{\mathsf{T}} + hC_{y\tau i}^{\mathsf{T}}\overline{M}_{Bim}^{\mathsf{T}} \end{bmatrix},$$

$$\Gamma'_{25im} = \begin{bmatrix} hA_{\tau i}^{\mathsf{T}}N_{1im}^{\mathsf{T}} + hC_{y\tau i}^{\mathsf{T}}\overline{N}_{Bim}^{\mathsf{T}} \\ hA_{\tau i}^{\mathsf{T}}N_{2im}^{\mathsf{T}} + hC_{y\tau i}^{\mathsf{T}}\overline{N}_{Bim}^{\mathsf{T}} \end{bmatrix},$$

$$\Gamma_{46im} = \begin{bmatrix} V_{1im}D_i + \overline{V}_{Bim}D_{yi} \\ V_{2im}D_i + \overline{V}_{Bim}D_{yi} \end{bmatrix},$$

$$\Gamma_{56im} = \begin{bmatrix} hM_{1im}D_i + h\overline{M}_{Bim}D_{yi} \\ hM_{2im}D_i + h\overline{M}_{Bim}D_{yi} \end{bmatrix},$$

$$\Gamma'_{56im} = \begin{bmatrix} hN_{1im}D_i + h\overline{N}_{Bim}D_{yi} \\ hN_{2im}D_i + h\overline{N}_{Bim}D_{yi} \end{bmatrix}. \tag{62}$$

Then, a desired filter can be chosen with parameters as

$$A_{fim} = P_{2im}^{-1}\overline{A}_{fim}, \qquad B_{fim} = P_{2im}^{-1}\overline{B}_{fim}, \qquad C_{fim} = \overline{C}_{fim}. \tag{63}$$

Proof. We denote that

$$P_{im} = \begin{bmatrix} P_{1im} & P_{2im} \\ P_{2im} & P_{2im} \end{bmatrix}. \tag{64}$$

The term $P_{im}\overline{A}_i$ can be rewritten as

$$P_{im}\overline{A}_i = \begin{bmatrix} P_{1im}A_i + P_{2im}B_{fim}C_{yi} & P_{2im}A_{fim} \\ P_{2im}A_i + P_{2im}B_{fim}C_{yi} & P_{2im}A_{fim} \end{bmatrix}. \tag{65}$$

Similarly, we have

$$P_{im}\overline{B}_i = \begin{bmatrix} P_{1im}A_{\tau i} + P_{2im}B_{fim}C_{y\tau i} \\ P_{2im}A_{\tau i} + P_{2im}B_{fim}C_{y\tau i} \end{bmatrix}, \qquad P_{im}\overline{D}_i = \begin{bmatrix} P_{1im}D_i + P_{2im}B_{fim}D_{yi} \\ P_{2im}D_i + P_{2im}B_{fim}D_{yi} \end{bmatrix} \tag{66}$$

Define $\overline{A}_{fim} = P_{2im}A_{fim}$, $\overline{B}_{fim} = P_{2im}B_{fim}$, $\overline{C}_{fim} = C_{fim}$, $\overline{M}_{Aim} = M_{2im}A_{fim}$, $\overline{M}_{Bim} = M_{2im}B_{fim}$, $\overline{N}_{Aim} = N_{2im}A_{fim}$, $\overline{N}_{Bim} = N_{2im}B_{fim}$, $\overline{V}_{Aim} = V_{2im}A_{fim}$, and $\overline{V}_{Bim} = V_{2im}B_{fim}$. Therefore, if (59) and (60) hold, system (9) is finite-time bounded with a prescribed H_∞ performance index γ. The proof is completed. □

Remark 16. In many actual applications, the minimum value of $\gamma_{\min}^2$ is of interest. In Theorem 12, with a fixed λ, $\gamma_{\min}$ can be obtained through the following optimization procedure:

$$\begin{aligned} &\min \quad \gamma^2 \\ &\text{s.t.} \quad (48)\text{–}(50). \end{aligned} \tag{67}$$

In Theorem 15, as for finite-time stability and boundedness, once the state bound c_2 is not ascertained, the minimum value $c_{2\min}$ is of interest. With a fixed λ, define $\lambda_1 = 1$; then the following optimization problem can be formulated to get minimum value $c_{2\min}$:

$$\begin{aligned} &\min \quad \varsigma\gamma^2 + (1-\varsigma)c_2 \\ &\text{s.t.} \quad (59)\text{-}(60) \text{ and } (50), \end{aligned} \tag{68}$$

where ς is weighted factor, and $\varsigma \in [0, 1]$.

5. Illustrative Example

Example 17. Consider that the Markovian jump system and the delay mode switching are governed by a Markov process with the following transition rates:

$$\Lambda = \begin{bmatrix} -0.7 & 0.7 \\ 0.9 & -0.9 \end{bmatrix}, \qquad \Pi = \begin{bmatrix} -1 & 1 \\ 1.2 & -1.2 \end{bmatrix}, \tag{69}$$

as well as with the following parameters:

$$\begin{aligned} &A_1 = \begin{bmatrix} -3.5 & 0.86 \\ -0.64 & -3.25 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -2.5 & 0.34 \\ 1.4 & -0.02 \end{bmatrix}, \\ &A_{\tau 1} = \begin{bmatrix} -0.8 & -1.3 \\ -0.7 & -2.2 \end{bmatrix}, \qquad A_{\tau 2} = \begin{bmatrix} -2.8 & 0.5 \\ -0.8 & -1.4 \end{bmatrix}, \\ &D_1 = D_2 = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, \qquad C_{y1} = [0.4, 0.7], \\ &C_{y2} = [0.5, 0.8], \qquad C_{y\tau 1} = [0.1, 0.2], \\ &C_{y\tau 2} = [0.1, 0.1], \\ &D_{y1} = D_{y2} = 0.1, \qquad C_{z1} = [0.1,\ 0.05], \\ &C_{z2} = [0.2,\ 0.09], \qquad C_{z\tau 1} = [0.3,\ 0.6], \\ &C_{z\tau 2} = [0.4,\ 0.8], \qquad D_{z1} = D_{z2} = 0.05. \end{aligned} \tag{70}$$

Then, we choose $R = I$, $T = 2$, $c_1 = 1$, and $d = 0.01$; through Theorem 15, it yields that the mode-dependent filters are as follows:

$$A_{f11} = \begin{bmatrix} -6.1254 & 1.5565 \\ -0.2347 & -0.3763 \end{bmatrix},$$

$$A_{f12} = \begin{bmatrix} -6.2682 & 1.4338 \\ -0.3552 & -0.5327 \end{bmatrix},$$

$$A_{f21} = \begin{bmatrix} -8.9214 & 2.4223 \\ -1.5476 & -0.6234 \end{bmatrix},$$

$$A_{f22} = \begin{bmatrix} -8.4638 & 2.8237 \\ -1.3545 & -0.4322 \end{bmatrix},$$

$$B_{f11} = \begin{bmatrix} 5.1465 & -2.8516 \\ -9.1216 & 9.5171 \end{bmatrix},$$

$$B_{f12} = \begin{bmatrix} 5.4203 & -2.3121 \\ -9.3156 & 9.6332 \end{bmatrix}, \quad (71)$$

$$B_{f21} = \begin{bmatrix} 4.6193 & -1.1984 \\ -16.2397 & 16.3006 \end{bmatrix},$$

$$B_{f22} = \begin{bmatrix} 4.6512 & -1.0311 \\ -16.1132 & 16.2312 \end{bmatrix},$$

$$C_{f11} = [0.0294, -0.0397],$$

$$C_{f12} = [0.0271, -0.03368],$$

$$C_{f21} = [0.1163, -0.1432],$$

$$C_{f22} = [0.1342, -0.3672].$$

This paper deals with the finite-time filter design problem for a class of Markovian jump systems; particularly, two different Markov processes are considered for modeling the randomness of system matrix and the state delay. Then, through the numerical example, we can see that results in this paper are feasible, which further verified the correctness of our theory. Therefore, the paper shorten this gap.

6. Conclusions

In this paper, we have examined the problems of finite-time H_∞ filtering for a class of Markovian jump systems with different system modes. Based on a novel approach, a sufficient condition is derived such that the closed-loop Markovian jump system is finite-time bounded and satisfies a prescribed level of H_∞ disturbance attenuation in a finite time interval. Finally, a numerical example is also given to illustrate the effectiveness of the proposed design approach. It should be noted that one of future research topics would be to investigate the problems of fault detection and fault tolerant control for time-varying Markovian jump systems with incomplete information over a finite-time horizon.

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