Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 305032, 12 pages
http://dx.doi.org/10.1155/2013/305032

Research Article

Group Classification of a Generalized Lane-Emden System

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Received 11 September 2012; Accepted 25 November 2012

Academic Editor: Mehmet Pakdemirli

We perform the group classification of the generalized Lane-Emden system $xu'' + nu' + xH(v) = 0$, $xv'' + nv' + xg(u) = 0$, which occurs in many applications of physical phenomena such as pattern formation, population evolution, and chemical reactions. We obtain four cases depending on the values of n.

1. Introduction

The celebrated Lane-Emden equation

$$\frac{d^2 y}{dx^2} + \frac{n}{x}\frac{dy}{dx} + f(y) = 0, \tag{1}$$

where n is a real constant and $f(y)$ is a real-valued function of the variable y, has many applications in mathematical physics and astrophysics. Equation (1), for certain fixed values of n and $f(y)$, models several phenomena such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gaseous sphere, and the theory of thermionic currents [1–3]. Several methods for the solution and many applications of the Lane-Emden Equation (1) can be found in the literature. The interested reader is referred to [4] and the references therein. It is worth mentioning that Wong [5], in his review paper of 1975, presented more than 140 references on this topic.

A natural extension of (1), called the generalized Lane-Emden system [6], is given by

$$\begin{aligned} \frac{d^2 u}{dx^2} + \frac{n}{x}\frac{du}{dx} + H(v) &= 0, \\ \frac{d^2 v}{dx^2} + \frac{n}{x}\frac{dv}{dx} + G(u) &= 0. \end{aligned} \tag{2}$$

Such systems arise in the modeling of several physical phenomena, such as pattern formation, population evolution, chemical reactions, and so on [7], and in the past few years have attracted much attention. Various researchers have worked on existence and uniqueness results for the Lane-Emden systems [8, 9] and other related systems [10–12].

In [6] the authors studied Noether operators with respect to the standard Lagrangian of the generalized coupled Lane-Emden system (2). They obtained seven cases out of which six cases resulted in Noether point symmetries. The first integrals corresponding to the Noether operators in each case were also constructed.

The objective of this paper is to perform the Lie group classification of the generalized Lane-Emden system (2). The paper is organized as follows. In Section 2, we calculate the equivalence transformations of the Lane-Emden system (2). We determine the principal Lie algebra and perform the group classification of system (2) in Section 3. Finally, concluding remarks are presented in Section 4.

2. Equivalence Transformations

An equivalence transformation (see, e.g., [13]) of the system (2) is an invertible transformation involving the variables x, u, and v that map system (2) into itself, with possibly the form of the transformed functions being different from that of the original functions $H(v)$ and $G(u)$. We write system (2) as

$$\frac{d^2u}{dx^2} + \frac{n}{x}\frac{du}{dx} + H(v) = 0,$$
$$\frac{d^2v}{dx^2} + \frac{n}{x}\frac{dv}{dx} + G(u) = 0, \quad (3)$$
$$H_x = 0, \qquad H_u = 0, \qquad G_x = 0, \qquad G_v = 0,$$

where u and v are differential variables with independent variable x, and H is a differential function of the independent variables x and v, whereas G is a differential function of the independent variables x and u. We obtain the generators of the group of equivalence transformations as

$$Y = \xi(x,u,v)\frac{\partial}{\partial x} + \eta^1(x,u,v)\frac{\partial}{\partial u} + \eta^2(x,u,v)\frac{\partial}{\partial v} + \mu^1(x,u,v,H,G)\frac{\partial}{\partial H} + \mu^2(x,u,v,H,G)\frac{\partial}{\partial G}. \quad (4)$$

We apply Lie's infinitesimal approach by using the prolongation of Y to involve the derivatives in system (3) as, for example, in [14].

We summarize our results below.

Case 1 ($n \neq -1, 1, 3$). In this case system (3) has the nine-dimensional equivalence Lie algebra spanned by the equivalence generators

$$X_1 = x\frac{\partial}{\partial x} - 2H\frac{\partial}{\partial H} - 2G\frac{\partial}{\partial G},$$
$$X_2 = u\frac{\partial}{\partial u} + H\frac{\partial}{\partial H},$$
$$X_3 = v\frac{\partial}{\partial v} + G\frac{\partial}{\partial G},$$
$$X_4 = \frac{\partial}{\partial u},$$
$$X_5 = \frac{\partial}{\partial v}, \quad (5)$$
$$X_6 = x^{1-n}\frac{\partial}{\partial u},$$
$$X_7 = x^{1-n}\frac{\partial}{\partial v},$$
$$X_8 = x^2\frac{\partial}{\partial u} - 2(1+n)\frac{\partial}{\partial H},$$
$$X_9 = x^2\frac{\partial}{\partial v} - 2(1+n)\frac{\partial}{\partial G}$$

and hence the nine-parameter equivalence group is given by

$$X_1: \bar{x} = e^{a_1}x, \quad \bar{u} = u, \quad \bar{v} = v, \quad \bar{H} = e^{-2a_1}H, \quad \bar{G} = e^{-2a_1}G,$$
$$X_2: \bar{x} = x, \quad \bar{u} = e^{a_2}u, \quad \bar{v} = v, \quad \bar{H} = e^{a_2}H, \quad \bar{G} = G,$$
$$X_3: \bar{x} = x, \quad \bar{u} = u, \quad \bar{v} = e^{a_3}v, \quad \bar{H} = H, \quad \bar{G} = e^{a_3}G,$$
$$X_4: \bar{x} = x, \quad \bar{u} = u + a_4, \quad \bar{v} = v, \quad \bar{H} = H, \quad \bar{G} = G,$$
$$X_5: \bar{x} = x, \quad \bar{u} = u, \quad \bar{v} = v + a_5, \quad \bar{H} = H, \quad \bar{G} = G,$$
$$X_6: \bar{x} = x, \quad \bar{u} = u + a_6x^{1-n}, \quad \bar{v} = v, \quad \bar{H} = H, \quad \bar{G} = G,$$
$$X_7: \bar{x} = x, \quad \bar{u} = u, \quad \bar{v} = v + a_7x^{1-n}, \quad \bar{H} = H, \quad \bar{G} = G,$$
$$X_8: \bar{x} = x, \quad \bar{u} = u + a_8x^2, \quad \bar{v} = v, \quad \bar{H} = H - 2(1+n)a_8, \quad \bar{G} = G,$$
$$X_9: \bar{x} = x, \quad \bar{u} = u, \quad \bar{v} = v + a_9x^2, \quad \bar{H} = H, \quad \bar{G} = G - 2(1+n)a_9. \quad (6)$$

Thus the composition of these transformations gives

$$\bar{x} = e^{a_1}x,$$
$$\bar{u} = e^{a_2}\left(u + a_8x^2 + a_6x^{1-n} + a_4\right),$$
$$\bar{v} = e^{a_3}\left(v + a_9x^2 + a_7x^{1-n} + a_5\right), \quad (7)$$
$$\bar{H} = e^{a_2-2a_1}\left(H - 2(1+n)a_8\right),$$
$$\bar{G} = e^{a_3-2a_1}\left(G - 2(1+n)a_9\right).$$

Case 2 ($n = -1$). In this case system (3) has the nine-dimensional equivalence Lie algebra spanned by the equivalence generators

$$X_1 = x\frac{\partial}{\partial x} - 2H\frac{\partial}{\partial H} - 2G\frac{\partial}{\partial G},$$
$$X_2 = u\frac{\partial}{\partial u} + H\frac{\partial}{\partial H},$$

$$
\begin{aligned}
X_3 &= v\frac{\partial}{\partial v} + G\frac{\partial}{\partial G},\\
X_4 &= \frac{\partial}{\partial u},\\
X_5 &= \frac{\partial}{\partial v},\\
X_6 &= x^2 \ln x\frac{\partial}{\partial u} - 2\frac{\partial}{\partial H},\\
X_7 &= x^2 \ln x\frac{\partial}{\partial v} - 2\frac{\partial}{\partial G},\\
X_8 &= x^2\frac{\partial}{\partial u},\\
X_9 &= x^2\frac{\partial}{\partial v}
\end{aligned} \tag{8}
$$

and hence the nine-parameter equivalence group is given by

$$
\begin{aligned}
X_1 &: \overline{x} = e^{a_1}x, \quad \overline{u} = u, \quad \overline{v} = v,\\
&\quad \overline{H} = e^{-2a_1}H, \quad \overline{G} = e^{-2a_1}G,\\
X_2 &: \overline{x} = x, \quad \overline{u} = e^{a_2}u, \quad \overline{v} = v,\\
&\quad \overline{H} = e^{a_2}H, \quad \overline{G} = G,\\
X_3 &: \overline{x} = x, \quad \overline{u} = u, \quad \overline{v} = e^{a_3}v,\\
&\quad \overline{H} = H, \quad \overline{G} = e^{a_3}G,\\
X_4 &: \overline{x} = x, \quad \overline{u} = u + a_4, \quad \overline{v} = v,\\
&\quad \overline{H} = H, \quad \overline{G} = G,\\
X_5 &: \overline{x} = x, \quad \overline{u} = u, \quad \overline{v} = v + a_5,\\
&\quad \overline{H} = H, \quad \overline{G} = G,\\
X_6 &: \overline{x} = x, \quad \overline{u} = u + a_6x^2 \ln x, \quad \overline{v} = v,\\
&\quad \overline{H} = H - 2a_6, \overline{G} = G,\\
X_7 &: \overline{x} = x, \quad \overline{u} = u, \quad \overline{v} = v + a_7x^2 \ln x,\\
&\quad \overline{H} = H, \quad \overline{G} = G - 2a_7,\\
X_8 &: \overline{x} = x, \quad \overline{u} = u + a_8x^2, \quad \overline{v} = v,\\
&\quad \overline{H} = H, \quad \overline{G} = G,\\
X_9 &: \overline{x} = x, \quad \overline{u} = u, \quad \overline{v} = v + a_9x^2,\\
&\quad \overline{H} = H, \quad \overline{G} = G.
\end{aligned} \tag{9}
$$

Hence the composition of these transformations gives

$$
\begin{aligned}
\overline{x} &= e^{a_1}x,\\
\overline{u} &= e^{a_2}\left(u + a_6x^2 \ln x + x^2a_8 + a_4\right),\\
\overline{v} &= e^{a_3}\left(v + a_7x^2 \ln x + x^2a_9 + a_5\right),\\
\overline{H} &= e^{a_2-2a_1}\left(H - 2a_6\right),\\
\overline{G} &= e^{a_3-2a_1}\left(G - 2a_7\right).
\end{aligned} \tag{10}
$$

Case 3 ($n = 1$). In this case system (3) has the nine-dimensional equivalence Lie algebra spanned by the equivalence generators

$$
\begin{aligned}
X_1 &= x\frac{\partial}{\partial x} - 2H\frac{\partial}{\partial H} - 2G\frac{\partial}{\partial G},\\
X_2 &= u\frac{\partial}{\partial u} + H\frac{\partial}{\partial H},\\
X_3 &= v\frac{\partial}{\partial v} + G\frac{\partial}{\partial G},\\
X_4 &= \frac{\partial}{\partial u},\\
X_5 &= \frac{\partial}{\partial v},\\
X_6 &= \ln x\frac{\partial}{\partial u},\\
X_7 &= \ln x\frac{\partial}{\partial v},\\
X_8 &= x^2\frac{\partial}{\partial u} - 4\frac{\partial}{\partial H},\\
X_9 &= x^2\frac{\partial}{\partial v} - 4\frac{\partial}{\partial G}
\end{aligned} \tag{11}
$$

and hence the nine-parameter equivalence group is given by

$$
\begin{aligned}
X_1 &: \overline{x} = e^{a_1}x, \quad \overline{u} = u, \quad \overline{v} = v,\\
&\quad \overline{H} = e^{-2a_1}H, \quad \overline{G} = e^{-2a_1}G,\\
X_2 &: \overline{x} = x, \quad \overline{u} = e^{a_2}u, \quad \overline{v} = v,\\
&\quad \overline{H} = e^{a_2}H, \quad \overline{G} = G,\\
X_3 &: \overline{x} = x, \quad \overline{u} = u, \quad \overline{v} = e^{a_3}v,\\
&\quad \overline{H} = H, \quad \overline{G} = e^{a_3}G,\\
X_4 &: \overline{x} = x, \quad \overline{u} = u + a_4, \quad \overline{v} = v,\\
&\quad \overline{H} = H, \quad \overline{G} = G,
\end{aligned}
$$

Table 1: Lie symmetries for $n \neq -1, 1, 3$, for various functions $H(v)$ and $G(u)$.

$H(v)$ arbitrary, $G(u) = c$, c a constant	$X_1 = \frac{\partial}{\partial u}$, $X_2 = x^{(1-n)}\frac{\partial}{\partial u}$, $X_3 = (cx^2 + 2nv + 2v)\frac{\partial}{\partial u}$
$H(v) = d$, d a constant, $G(u)$ arbitrary	$X_1 = \frac{\partial}{\partial v}$, $X_2 = x^{(1-n)}\frac{\partial}{\partial v}$, $X_3 = (dx^2 + 2nu + 2u)\frac{\partial}{\partial v}$
$H(v) = d$, d a constant, $G(u) = \alpha + \beta u$, α and β constants $(\beta \neq 0)$	$X_1 = \frac{\partial}{\partial v}$, $X_2 = x^{(1-n)}\frac{\partial}{\partial v}$, $X_3 = (dx^2 + 2nu + 2u)\frac{\partial}{\partial v}$, $X_4 = \beta x^{(3-n)}\frac{\partial}{\partial v} + (2n-6)x^{(1-n)}\frac{\partial}{\partial u}$, $X_5 = (2n+2)\frac{\partial}{\partial u} - \beta x^2\frac{\partial}{\partial v}$, $X_6 = (2n^2 + 8n + 6)\,x\frac{\partial}{\partial x} - 4\left(n^2u + 3dx^2 + 4nu + 3u + ndx^2\right)\frac{\partial}{\partial u} + (\beta dx^2 - 2\alpha n - 6\alpha)x^2\frac{\partial}{\partial v}$, $X_7 = (16nx + 4n^2x + 12x)\frac{\partial}{\partial x} - (4n+12)dx^2\frac{\partial}{\partial u} + (8n^2v + 32nv + \beta dx^4 + 24v)\frac{\partial}{\partial v}$
$H(v) = d$, d a constant, $G(u) = \alpha + \beta u^{-p}$, α, p, and β constants $(\beta, p \neq 0)$	$X_1 = \frac{\partial}{\partial v}$, $X_2 = x^{(1-n)}\frac{\partial}{\partial v}$, $X_3 = (nx + x)\frac{\partial}{\partial x} + (2nu + 2u)\frac{\partial}{\partial u} - (2npv - 2nv + \alpha px^2 + 2pv - 2v)\frac{\partial}{\partial v}$, $X_4 = (2u + 2nu + dx^2)\frac{\partial}{\partial v}$
$H(v) = d$, d a constant, $G(u) = \alpha + \beta e^{-ku}$, α, k, and β constants $(\beta, k \neq 0)$	$X_1 = \frac{\partial}{\partial v}$, $X_2 = x^{(1-n)}\frac{\partial}{\partial v}$, $X_3 = (dx^2 + 2nu + 2u)\frac{\partial}{\partial v}$, $X_4 = (2n+2)\frac{\partial}{\partial u} - (\alpha kx^2 + 2knv + 2kv)\frac{\partial}{\partial v}$
$H(v) = d$, d a constant, $G(u) = \alpha + \beta \ln u$, α and β constants $(\beta \neq 0)$	$X_1 = \frac{\partial}{\partial v}$, $X_2 = x^{(1-n)}\frac{\partial}{\partial v}$, $X_3 = (dx^2 + 2nu + 2u)\frac{\partial}{\partial v}$, $X_4 = (xn + x)\frac{\partial}{\partial x} + (2u + 2nu)\frac{\partial}{\partial u} + (2nv - \beta x^2 + 2v)\frac{\partial}{\partial v}$
$H(v) = a + bv$, a and b constants $(b \neq 0)$, $G(u) = c$, c constant	$X_1 = \frac{\partial}{\partial u}$, $X_2 = x^{(1-n)}\frac{\partial}{\partial u}$, $X_3 = (cx^2 + 2nv + 2v)\frac{\partial}{\partial u}$, $X_4 = bx^{(3-n)}\frac{\partial}{\partial u} + (2n-6)x^{(1-n)}\frac{\partial}{\partial v}$, $X_5 = bx^2\frac{\partial}{\partial u} - (2n+2)\frac{\partial}{\partial v}$, $X_6 = (bcx^4 - 32nu - 12ax^2 - 24u - 8n^2u - 4anx^2)\frac{\partial}{\partial u} - (4cnx^2 + 24v + 8n^2v + 32nv + 12x^2)\frac{\partial}{\partial v}$, $X_7 = (16nx + 4n^2x + 12x)\frac{\partial}{\partial x} + (8n^2u + 24u + 32nu + bcx^4)\frac{\partial}{\partial u} - (12cx^2 + 4cnx^2)\frac{\partial}{\partial v}$
$H(v) = a + bv$, a and b constants $(b \neq 0)$, $G(u) = \alpha + \beta u$, α and β constants $(\beta \neq 0)$	$X_1 = F(x)\frac{\partial}{\partial v}$, $X_2 = W(x)\frac{\partial}{\partial u}$, $X_3 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}$, $X_4 = bv\frac{\partial}{\partial u} + \beta u\frac{\partial}{\partial v}$
$H(v) = a + bv$, a and b constants $(b \neq 0)$, $G(u) = \beta u^{-p}$, β and p constants $(\beta, p \neq 0)$	$X_1 = (bx + bpx)\frac{\partial}{\partial x} + 4bu\frac{\partial}{\partial u} + (2a + 2bv - 2ap - 2bpv)\frac{\partial}{\partial v}$
$H(v) = a + bv$, a and b constants $(b \neq 0)$, $G(u) = \beta e^{-ku}$, β and k constants $(\beta, k \neq 0)$	$X_1 = bkx\frac{\partial}{\partial x} + 4b\frac{\partial}{\partial u} - (2ak + 2bkv)\frac{\partial}{\partial v}$

Table 1: Continued.

$H(v) = a + bv^{-m}$, a, m, and b constants $(b, m \neq 0)$, $G(u) = c$, c a constant $X_1 = \frac{\partial}{\partial u}$, $X_2 = x^{(1-n)}\frac{\partial}{\partial u}$, $X_3 = (cx^2 + 2nv + 2v)\frac{\partial}{\partial u}$, $X_4 = (nx + x)\frac{\partial}{\partial x} + (2u - 2nmu + 2nu - amx^2 - 2mu)\frac{\partial}{\partial u} + (2nv + 2v)\frac{\partial}{\partial v}$
$H(v) = bv^{-m}$, m and b constants $(b, m \neq 0)$, $G(u) = \alpha + \beta u$, α and β constants $(\beta \neq 0)$ $X_1 = (\beta x + \beta mx)\frac{\partial}{\partial x} + (2\alpha - 2\alpha m + 2\beta u - 2\beta mu)\frac{\partial}{\partial u} + 4\beta v\frac{\partial}{\partial v}$
$H(v) = bv^{-m}$, m and b constants $(b, m \neq 0)$, $G(u) = \beta u^{-p}$, β and p constants $(\beta, p \neq 0)$ $X_1 = (pmx - x)\frac{\partial}{\partial x} + (2mu - 2u)\frac{\partial}{\partial u} + (2pv - 2v)\frac{\partial}{\partial v}$
$H(v) = bv^{-m}$, m and b constants $(b, m \neq 0)$, $G(u) = \beta e^{-ku}$, β and k constants $(\beta, k \neq 0)$ $X_1 = kmx\frac{\partial}{\partial x} + (2m - 2)\frac{\partial}{\partial u} + 2kv\frac{\partial}{\partial v}$
$H(v) = a + be^{-mv}$, a, m and b constants $(b, m \neq 0)$, $G(u) = c$, c constant $X_1 = \frac{\partial}{\partial u}$, $X_2 = x^{(1-n)}\frac{\partial}{\partial u}$, $X_3 = (cx^2 + 2nv + 2v)\frac{\partial}{\partial u}$, $X_4 = (ax^2 + 2u + 2nu)\, m\frac{\partial}{\partial u} - 2(n + 1)\frac{\partial}{\partial v}$
$H(v) = be^{-mv}$, m and b constants $(b, m \neq 0)$, $G(u) = \alpha + \beta u$ $(\beta \neq 0)$ $X_1 = \beta mx\frac{\partial}{\partial x} - (2\alpha m + 2\beta mu)\frac{\partial}{\partial u} + 4\beta\frac{\partial}{\partial v}$
$H(v) = be^{-mv}$, m and b constants $(b, m \neq 0)$, $G(u) = \beta u^{-p}$, β and p constants $(\beta, p \neq 0)$ $X_1 = mpx\frac{\partial}{\partial x} + 2mu\frac{\partial}{\partial u} + (2p - 2)\frac{\partial}{\partial v}$
$H(v) = be^{-mv}$, m and b constants $(b, m \neq 0)$, $G(u) = \beta e^{-ku}$, β and k constants $(\beta, k \neq 0)$ $X_1 = mkx\frac{\partial}{\partial x} + 2m\frac{\partial}{\partial u} + 2k\frac{\partial}{\partial v}$
$H(v) = a + b\ln v$, a, m and b constants $(b \neq 0)$, $G(u) = c$, c constant $X_1 = \frac{\partial}{\partial u}$, $X_2 = x^{(1-n)}\frac{\partial}{\partial u}$, $X_3 = (cx^2 + 2v + 2nv)\frac{\partial}{\partial u}$, $X_4 = (n + 1)x\frac{\partial}{\partial x} + (2nu - bx^2 + 2u)\frac{\partial}{\partial u} + (2nv + 2v)\frac{\partial}{\partial v}$

$$\begin{aligned}
X_5 &: \bar{x} = x, \quad \bar{u} = u, \quad \bar{v} = v + a_5, \\
&\quad \bar{H} = H, \quad \bar{G} = G, \\
X_6 &: \bar{x} = x, \quad \bar{u} = u + a_6 \ln x, \quad \bar{v} = v, \\
&\quad \bar{H} = H, \quad \bar{G} = G, \\
X_7 &: \bar{x} = x, \quad \bar{u} = u, \quad \bar{v} = v + a_7 \ln x, \\
&\quad \bar{H} = H, \quad \bar{G} = G, \\
X_8 &: \bar{x} = x, \quad \bar{u} = u + a_8 x^2, \quad \bar{v} = v, \\
&\quad \bar{H} = H - 4a_8, \quad \bar{G} = G, \\
X_9 &: \bar{x} = x, \quad \bar{u} = u, \quad \bar{v} = v + a_9 x^2, \\
&\quad \bar{H} = H, \quad \bar{G} = G - 4a_9
\end{aligned} \tag{12}$$

and so the composition of these transformations gives

$$\begin{aligned}
\bar{x} &= e^{a_1} x, \\
\bar{u} &= e^{a_2}\left(u + a_6 \ln x + a_8 x^2 + a_4\right), \\
\bar{v} &= e^{a_3}\left(v + a_7 \ln x + a_9 x^2 + a_5\right), \\
\bar{H} &= e^{a_2 - 2a_1}\left(H - 4a_8\right), \\
\bar{G} &= e^{a_3 - 2a_1}\left(G - 4a_9\right).
\end{aligned} \tag{13}$$

Case 4 ($n = 3$). In this case system (3) has the ten-dimensional equivalence Lie algebra spanned by the equivalence generators

$$X_1 = x\frac{\partial}{\partial x} - 2H\frac{\partial}{\partial H} - 2G\frac{\partial}{\partial G},$$

$$X_2 = x^{-1}\frac{\partial}{\partial x} - 2x^{-2}u\frac{\partial}{\partial u} - 2x^{-2}v\frac{\partial}{\partial v},$$

Table 2: Lie symmetries for $n = -1$, for various functions $H(v)$ and $G(u)$.

$H(v)$ arbitrary, $G(u) = c$, c a constant $X_1 = \frac{\partial}{\partial u}$, $X_2 = x^2 \frac{\partial}{\partial u}$, $X_3 = (cx^2 \ln x + 2v)\frac{\partial}{\partial u}$
$H(v) = d$, d a constant, $G(u)$ arbitrary $X_1 = \frac{\partial}{\partial v}$, $X_2 = x^2 \frac{\partial}{\partial v}$, $X_3 = (dx^2 \ln x + 2u)\frac{\partial}{\partial v}$
$H(v) = d$, d a constant, $G(u) = \alpha + \beta u$, α and β constants $(\beta \neq 0)$ $X_1 = \frac{\partial}{\partial v}$, $X_2 = x^2 \frac{\partial}{\partial v}$, $X_3 = (dx^2 \ln x + 2u)\frac{\partial}{\partial v}$, $X_4 = \beta x^4 \frac{\partial}{\partial v} - 8x^2 \frac{\partial}{\partial u}$, $X_5 = \beta x^2 \ln x \frac{\partial}{\partial v} - 2\frac{\partial}{\partial u}$, $X_6 = (4d\beta x^4 \ln x - 32\alpha x^2 \ln x - 3\beta d x^4 - 64v)\frac{\partial}{\partial v} - (32x^2 \ln x + 64u)\frac{\partial}{\partial u}$, $X_7 = 16x\frac{\partial}{\partial x} + (4d\beta x^4 \ln x - 16\alpha x^2 \ln x - 3\beta d x^4)\frac{\partial}{\partial v} - (32dx^2 \ln x + 32u)\frac{\partial}{\partial u}$
$H(v) = d$, d a constant, $G(u) = \alpha + \beta u^{-p}$, α, p, and β constants $(\beta, p \neq 0)$ $X_1 = \frac{\partial}{\partial v}$, $X_2 = x^2 \frac{\partial}{\partial v}$, $X_3 = (dx^2 \ln x + 2u)\frac{\partial}{\partial v}$, $X_4 = x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u} + (2v - 2pv - \alpha p x^2 \ln x)\frac{\partial}{\partial v}$
$H(v) = d$, d a constant, $G(u) = \alpha + \beta e^{-ku}$, α, k, and β constants $(\beta, k \neq 0)$ $X_1 = \frac{\partial}{\partial v}$, $X_2 = x^2 \frac{\partial}{\partial v}$, $X_3 = (dx^2 \ln x + 2u)\frac{\partial}{\partial v}$, $X_4 = 2\frac{\partial}{\partial u} - (\alpha k x^2 \ln x + 2kv)\frac{\partial}{\partial v}$
$H(v) = d$, d a constant, $G(u) = \alpha + \beta \ln u$, α and β constants $(\beta \neq 0)$ $X_1 = \frac{\partial}{\partial v}$, $X_2 = x^2 \frac{\partial}{\partial v}$, $X_3 = (dx^2 \ln x + 2u)\frac{\partial}{\partial v}$, $X_4 = x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u} + (2v - \beta x^2 \ln x)\frac{\partial}{\partial v}$
$H(v) = a + bv$, a and b constants $(b \neq 0)$, $G(u) = c$, c constant $X_1 = \frac{\partial}{\partial u}$, $X_2 = x^2 \frac{\partial}{\partial u}$, $X_3 = (cx^2 \ln x + 2v)\frac{\partial}{\partial u}$, $X_4 = cx^2 \ln x \frac{\partial}{\partial u} - 2\frac{\partial}{\partial v}$, $X_5 = bx^4 \ln x \frac{\partial}{\partial u} - 8x^2 \frac{\partial}{\partial v}$, $X_6 = (4bcx^4 \ln x - 32ax^2 \ln x - 3bcx^4 - 64u)\frac{\partial}{\partial u} - (32cx^2 \ln x + 64v)\frac{\partial}{\partial v}$, $X_7 = 16x\frac{\partial}{\partial x} + (4cdx^4 \ln x - 16ax^2 \ln x - 3bcx^4)\frac{\partial}{\partial u} - (32cx^2 \ln x + 32v)\frac{\partial}{\partial v}$
$H(v) = a + bv$, a and b constants $(b \neq 0)$, $G(u) = \alpha + \beta u$, α and β constants $(\beta \neq 0)$ $X_1 = F(x)\frac{\partial}{\partial v}$, $X_2 = W(x)\frac{\partial}{\partial u}$, $X_3 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}$, $X_4 = bv\frac{\partial}{\partial u} + \beta u\frac{\partial}{\partial v}$
$H(v) = a + bv$, a and b constants $(b \neq 0)$, $G(u) = \beta u^{-p}$, β and p constants $(\beta, p \neq 0)$ $X_1 = (bx + bpx)\frac{\partial}{\partial x} + 4bu\frac{\partial}{\partial u} + (2a + 2bv - 2ap - 2bpv)\frac{\partial}{\partial v}$
$H(v) = a + bv$, a and b constants $(b \neq 0)$, $G(u) = \beta e^{-ku}$, β and k constants $(\beta, k \neq 0)$ $X_1 = bkx\frac{\partial}{\partial x} + 4b\frac{\partial}{\partial u} - (2ak + 2bkv)\frac{\partial}{\partial v}$
$H(v) = a + bv^{-m}$, a, m, and b constants $(b, m \neq 0)$, $G(u) = c$, c a constant $X_1 = \frac{\partial}{\partial u}$, $X_2 = x^2 \frac{\partial}{\partial u}$, $X_3 = (cx^2 \ln x + 2v)\frac{\partial}{\partial u}$, $X_4 = x\frac{\partial}{\partial x} + (2u - amx^2 \ln x - 2mu)\frac{\partial}{\partial u} + 2v\frac{\partial}{\partial v}$

TABLE 2: Continued.

$H(v) = bv^{-m}, m$ and b constants $(b, m \neq 0), G(u) = \alpha + \beta u, \alpha$ and β constants $(\beta \neq 0)$ $X_1 = (\beta x + \beta m x)\frac{\partial}{\partial x} + (2\alpha - 2\alpha m + 2\beta u - 2\beta m u)\frac{\partial}{\partial u} + 4\beta v\frac{\partial}{\partial v}$
$H(v) = bv^{-m}, m$ and b constants $(b, m \neq 0), G(u) = \beta u^{-p}, p$ and β constants $(\beta, p \neq 0)$ $X_1 = (pmx - x)\frac{\partial}{\partial x} + (2mu - 2u)\frac{\partial}{\partial u} + (2pv - 2v)\frac{\partial}{\partial v}$
$H(v) = bv^{-m}, m$ and b constants $(b, m \neq 0), G(u) = \beta e^{-ku}, k$ and β constants $(\beta, k \neq 0)$ $X_1 = kmx\frac{\partial}{\partial x} + (2m - 2)\frac{\partial}{\partial u} + 2kv\frac{\partial}{\partial v}$
$H(v) = a + be^{-mv}, a, m,$ and b constants $(b, m \neq 0), G(u) = c, c$ constant $X_1 = \frac{\partial}{\partial u}, X_2 = x^2\frac{\partial}{\partial u}, X_3 = (cx^2 \ln x + 2v)\frac{\partial}{\partial u}, X_4 = (\alpha m x^2 \ln x + 2mu)\frac{\partial}{\partial u} - 2\frac{\partial}{\partial v}$
$H(v) = be^{-mv}, m$ and b constants $(b, m \neq 0), G(u) = \alpha + \beta u, \alpha$ and β constants $(\beta \neq 0)$ $X_1 = \beta m x\frac{\partial}{\partial x} - (2\alpha m + 2\beta m u)\frac{\partial}{\partial u} + 4\beta\frac{\partial}{\partial v}$
$H(v) = be^{-mv}, m$ and b constants $(b, m \neq 0), G(u) = \beta u^{-p}, \beta$ and p constants $(\beta, p \neq 0)$ $X_1 = mpx\frac{\partial}{\partial x} + 2mu\frac{\partial}{\partial u} + (2p - 2)\frac{\partial}{\partial v}$
$H(v) = be^{-mv}, m$ and b constants $(b, m \neq 0), G(u) = \beta e^{-ku}, \beta$ and k constants $(\beta, k \neq 0)$ $X_1 = mkx\frac{\partial}{\partial x} + 2m\frac{\partial}{\partial u} + 2k\frac{\partial}{\partial v}$
$H(v) = a + b\ln v, a, m,$ and b constants $(b \neq 0), G(u) = c, c$ constant $X_1 = \frac{\partial}{\partial u}, X_2 = x^2\frac{\partial}{\partial u}, X_3 = (cx^2 \ln x + 2v)\frac{\partial}{\partial u}, X_4 = x\frac{\partial}{\partial x} + (2u - bx^2 \ln x)\frac{\partial}{\partial u} + 2v\frac{\partial}{\partial v}$

$$\begin{aligned}
X_3 &= u\frac{\partial}{\partial u} + H\frac{\partial}{\partial H},\\
X_4 &= v\frac{\partial}{\partial v} + G\frac{\partial}{\partial G},\\
X_5 &= \frac{\partial}{\partial u},\\
X_6 &= \frac{\partial}{\partial v},\\
X_7 &= x^2\frac{\partial}{\partial u} - 8\frac{\partial}{\partial H},\\
X_8 &= x^2\frac{\partial}{\partial v} - 8\frac{\partial}{\partial G},\\
X_9 &= x^{-2}\frac{\partial}{\partial u},\\
X_{10} &= x^{-2}\frac{\partial}{\partial v}
\end{aligned} \tag{14}$$

and hence the ten-parameter equivalence group is given by

$$\begin{aligned}
X_1 &: \overline{x} = e^{a_1}x, \quad \overline{u} = u, \quad \overline{v} = v,\\
&\quad \overline{H} = e^{-2a_1}H, \quad \overline{G} = e^{-2a_1}G,\\
X_2 &: \overline{x} = \left(x^2 + 2a_2\right)^{1/2}, \quad \overline{u} = ux^2\left(x^2 + 2a_2\right)^{-1},\\
&\quad \overline{v} = vx^2\left(x^2 + 2a_2\right)^{-1}, \quad \overline{H} = H, \quad \overline{G} = G,\\
X_3 &: \overline{x} = x, \quad \overline{u} = e^{a_3}u, \quad \overline{v} = v,\\
&\quad \overline{H} = e^{a_3}H, \quad \overline{G} = G,\\
X_4 &: \overline{x} = x, \quad \overline{u} = u, \quad \overline{v} = e^{a_4}v,\\
&\quad \overline{H} = H, \quad \overline{G} = e^{a_4}G,\\
X_5 &: \overline{x} = x, \quad \overline{u} = u + a_5, \quad \overline{v} = v,\\
&\quad \overline{H} = H, \quad \overline{G} = G,\\
X_6 &: \overline{x} = x, \quad \overline{u} = u, \quad \overline{v} = v + a_6,\\
&\quad \overline{H} = H, \quad \overline{G} = G,\\
X_7 &: \overline{x} = x, \quad \overline{u} = u + a_7x^2, \quad \overline{v} = v,\\
&\quad \overline{H} = H - 8a_7, \quad \overline{G} = G,\\
X_8 &: \overline{x} = x, \quad \overline{u} = u, \quad \overline{v} = v + a_8x^2,\\
&\quad \overline{H} = H, \quad \overline{G} = G - 8a_8,\\
X_9 &: \overline{x} = x, \quad \overline{u} = u + a_9x^{-2}, \quad \overline{v} = v,\\
&\quad \overline{H} = H, \quad \overline{G} = G,\\
X_{10} &: \overline{x} = x, \quad \overline{u} = u, \quad \overline{v} = v + a_{10}x^{-2},\\
&\quad \overline{H} = H, \quad \overline{G} = G.
\end{aligned} \tag{15}$$

Table 3: Lie symmetries for $n = 1$, for various functions $H(v)$ and $G(u)$.

$H(v)$ arbitrary, $G(u) = c$, c a constant $X_1 = \frac{\partial}{\partial u}$, $X_2 = \ln x \frac{\partial}{\partial u}$, $X_3 = (cx^2 + 4v)\frac{\partial}{\partial u}$
$H(v) = d$, d a constant, $G(u)$ arbitrary $X_1 = \frac{\partial}{\partial v}$, $X_2 = \ln x \frac{\partial}{\partial v}$, $X_3 = (dx^2 \ln x + 4u)\frac{\partial}{\partial v}$
$H(v) = d$, d a constant, $G(u) = \alpha + \beta u$, α and β constants ($\beta \neq 0$) $X_1 = \frac{\partial}{\partial v}$, $X_2 = \ln x \frac{\partial}{\partial v}$, $X_3 = (dx^2 \ln x + 4u)\frac{\partial}{\partial v}$, $X_4 = \beta x^2 \frac{\partial}{\partial v} - 4\frac{\partial}{\partial u}$, $X_5 = (\beta x^2 \ln x - \beta x^2)\frac{\partial}{\partial v} - 4 \ln x \frac{\partial}{\partial u}$, $X_6 = (16dx^2 + 64u)\frac{\partial}{\partial u} + (16\alpha x^2 + 64v - \beta dx^4)\frac{\partial}{\partial v}$, $X_7 = 16x\frac{\partial}{\partial x} - (8\alpha x^2 - \beta dx^4)\frac{\partial}{\partial v} - (16dx^2 + 32u)\frac{\partial}{\partial u}$
$H(v) = d$, d a constant, $G(u) = \alpha + \beta u^{-p}$, α, p, and β constants ($\beta, p \neq 0$) $X_1 = \frac{\partial}{\partial v}$, $X_2 = \ln x \frac{\partial}{\partial v}$, $X_3 = (dx^2 \ln x + 4u)\frac{\partial}{\partial v}$, $X_4 = 2x\frac{\partial}{\partial x} + 4u\frac{\partial}{\partial u} - (4vp + \alpha p x^2 - 4v)\frac{\partial}{\partial v}$
$H(v) = d$, d a constant, $G(u) = \alpha + \beta e^{-ku}$, α, k, and β constants ($\beta, k \neq 0$) $X_1 = \frac{\partial}{\partial v}$, $X_2 = \ln x \frac{\partial}{\partial v}$, $X_3 = (dx^2 + 4u)\frac{\partial}{\partial v}$, $X_4 = 4\frac{\partial}{\partial u} - (\alpha k x^2 + 4kv)\frac{\partial}{\partial v}$
$H(v) = d$, d a constant, $G(u) = \alpha + \beta \ln u$, α and β constants ($\beta \neq 0$) $X_1 = \frac{\partial}{\partial v}$, $X_2 = \ln x \frac{\partial}{\partial v}$, $X_3 = (dx^2 + 4u)\frac{\partial}{\partial v}$, $X_4 = 2x\frac{\partial}{\partial x} + 4u\frac{\partial}{\partial u} + (4v - \beta x^2)\frac{\partial}{\partial v}$
$H(v) = a + bv$, a and b constants ($b \neq 0$), $G(u) = c, c$ constant $X_1 = \frac{\partial}{\partial u}$, $X_2 = \ln x \frac{\partial}{\partial u}$, $X_3 = (cx^2 + 4v)\frac{\partial}{\partial u}$, $X_4 = 4 \ln x \frac{\partial}{\partial v} - (bx^2 \ln x - bx^2)\frac{\partial}{\partial u}$, $X_5 = bx^2 \frac{\partial}{\partial u} - 4\frac{\partial}{\partial v}$, $X_6 = 32x\frac{\partial}{\partial x} - 16cx^2\frac{\partial}{\partial v} + (bcx^4 + 64u)\frac{\partial}{\partial u}$, $X_7 = 16x\frac{\partial}{\partial x} - (8ax^2 - bcx^4)\frac{\partial}{\partial u} - (16cx^2 + 32v)\frac{\partial}{\partial v}$
$H(v) = a + bv$, a and b constants ($b \neq 0$), $G(u) = \alpha + \beta u$, α and β constants ($\beta \neq 0$) $X_1 = F(x)\frac{\partial}{\partial v}$, $X_2 = W(x)\frac{\partial}{\partial u}$, $X_3 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}$, $X_4 = bv\frac{\partial}{\partial u} + \beta u\frac{\partial}{\partial v}$
$H(v) = a + bv$, a and b constants ($b \neq 0$), $G(u) = \beta u^{-p}$, β and p constants ($\beta, p \neq 0$) $X_1 = (bx + bpx)\frac{\partial}{\partial x} + 4bu\frac{\partial}{\partial u} + (2a + 2bv - 2ap - 2bpv)\frac{\partial}{\partial v}$
$H(v) = a + bv$, a and b constants ($b \neq 0$), $G(u) = \beta e^{-ku}$, β and k constants ($\beta, k \neq 0$) $X_1 = bkx\frac{\partial}{\partial x} + 4b\frac{\partial}{\partial u} - (2ak + 2bkv)\frac{\partial}{\partial v}$
$H(v) = a + bv^{-m}$, a, m, and b constants ($b, m \neq 0$), $G(u) = c, c$ a constant $X_1 = \frac{\partial}{\partial u}$, $X_2 = \ln x \frac{\partial}{\partial u}$, $X_3 = (cx^2 + 4v)\frac{\partial}{\partial u}$, $X_4 = 2x\frac{\partial}{\partial x} + (4u - amx^2 - 4mu)\frac{\partial}{\partial u} + 4v\frac{\partial}{\partial v}$

Table 3: Continued.

$H(v)=bv^{-m}$, m and b constants $(b,m\neq 0)$, $G(u)=\alpha+\beta u$, α and β constants $(\beta\neq 0)$ $X_1=(\beta x+\beta m x)\frac{\partial}{\partial x}+(2\alpha-2\alpha m+2\beta u-2\beta m u)\frac{\partial}{\partial u}+4\beta v\frac{\partial}{\partial v}$
$H(v)=bv^{-m}$, m and b constants $(b,m\neq 0)$, $G(u)=\beta u^{-p}$, p and β constants $(\beta,p\neq 0)$ $X_1=(pmx-x)\frac{\partial}{\partial x}+(2mu-2u)\frac{\partial}{\partial u}+(2pv-2v)\frac{\partial}{\partial v}$
$H(v)=bv^{-m}$, m and b constants $(b,m\neq 0)$, $G(u)=\beta e^{-ku}$, k and β constants $(\beta,k\neq 0)$ $X_1=kmx\frac{\partial}{\partial x}+(2m-2)\frac{\partial}{\partial u}+2kv\frac{\partial}{\partial v}$
$H(v)=a+be^{-mv}$, $a,m,$ and b constants $(b,m\neq 0)$, $G(u)=c$, c constant $X_1=\frac{\partial}{\partial u}$, $X_2=\ln x\frac{\partial}{\partial u}$, $X_3=(cx^2+4v)\frac{\partial}{\partial u}$, $X_4=(amx^2+4mu)\frac{\partial}{\partial u}-4\frac{\partial}{\partial v}$
$H(v)=be^{-mv}$, m and b constants $(b,m\neq 0)$, $G(u)=\alpha+\beta u$, α and β constants $(\beta\neq 0)$ $X_1=\beta mx\frac{\partial}{\partial x}-(2\alpha m+2\beta m u)\frac{\partial}{\partial u}+4\beta\frac{\partial}{\partial v}$
$H(v)=be^{-mv}$, m and b constants $(b,m\neq 0)$, $G(u)=\beta u^{-p}$, β and p constants $(\beta,p\neq 0)$ $X_1=mpx\frac{\partial}{\partial x}+2mu\frac{\partial}{\partial u}+(2p-2)\frac{\partial}{\partial v}$
$H(v)=be^{-mv}$, m and b constants $(b,m\neq 0)$, $G(u)=\beta e^{-ku}$, β and k constants $(\beta,k\neq 0)$ $X_1=mkx\frac{\partial}{\partial x}+2m\frac{\partial}{\partial u}+2k\frac{\partial}{\partial v}$, $X_2=(mkx\ln x-kmx)\frac{\partial}{\partial x}+2m\ln x\frac{\partial}{\partial u}+2k\ln x\frac{\partial}{\partial v}$
$H(v)=a+b\ln v$, $a,m,$ and b constants $(b\neq 0)$, $G(u)=c$, c constant $X_1=\frac{\partial}{\partial u}$, $X_2=\ln x\frac{\partial}{\partial u}$, $X_3=(cx^2+4v)\frac{\partial}{\partial u}$, $X_4=2x\frac{\partial}{\partial x}+(4u-bx^2)\frac{\partial}{\partial u}+4v\frac{\partial}{\partial v}$

Therefore the composition of these transformations gives

$$\begin{gathered}\overline{x}=e^{a_1}\left(x^2+2a_2\right)^{1/2},\\ \overline{u}=e^{a_3}\left(x^2+2a_2\right)^{-1}\left(x^2u+a_9+a_7x^4+a_5x^2\right),\\ \overline{v}=e^{a_4}\left(x^2+2a_2\right)^{-1}\left(x^2v+a_{10}+a_8x^4+a_6x^2\right),\\ \overline{H}=e^{a_3-2a_1}\left(H-8a_7\right),\\ \overline{G}=e^{a_4-2a_1}\left(G-8a_8\right).\end{gathered}\tag{16}$$

3. Principal Lie Algebra and Lie Group Classification

The generalized Lane-Emden system (2) admits a Lie point symmetry

$$X=\xi(x,u,v)\frac{\partial}{\partial x}+\eta^1(x,u,v)\frac{\partial}{\partial u}+\eta^2(x,u,v)\frac{\partial}{\partial v}\tag{17}$$

if

$$\begin{gathered}X^{[2]}\left(\frac{d^2u}{dx^2}+\frac{n}{x}\frac{du}{dx}+H(v)\right)\bigg|_{(2)}=0,\\ X^{[2]}\left(\frac{d^2v}{dx^2}+\frac{n}{x}\frac{dv}{dx}+G(u)\right)\bigg|_{(2)}=0.\end{gathered}\tag{18}$$

After some albeit tedious and lengthy calculations, the above determining equation gives

$$\begin{gathered}\xi=e(x),\\ \eta^1=k(x)u+l(x),\\ \eta^2=c(x)v+d(x),\\ -e''+\frac{n}{x}e'-\frac{n}{x^2}e+2k'=0,\\ -e''+\frac{n}{x}e'-\frac{n}{x^2}e+2c'=0,\\ (ku+l)G'(u)+\left(2e'-c\right)G(u)+\frac{n}{x}d'+d''=0,\\ (cv+d)H'(v)+\left(2e'-k\right)H(v)+\frac{n}{x}l'+l''=0.\end{gathered}\tag{19}$$

Consequently, we conclude that the principal Lie algebra of (2) is trivial and the classifying relations are

$$\begin{gathered}(\alpha u+\beta)G'(u)+\gamma G(u)+\delta=0,\\ (\theta v+\lambda)H'(v)+\varphi H(v)+\omega=0,\end{gathered}\tag{20}$$

where $\alpha,\beta,\gamma,\delta,\theta,\lambda,\psi,$ and ω are constants.

Table 4: Lie symmetries for $n = 3$, for various functions $H(v)$ and $G(u)$.

$H(v)$ arbitrary, $G(u) = c$, c a constant $X_1 = \frac{\partial}{\partial u}$, $X_2 = x^{-2}\frac{\partial}{\partial u}$, $X_3 = (cx^2 + 8v)\frac{\partial}{\partial u}$
$H(v) = d$, d a constant, $G(u)$ arbitrary $X_1 = \frac{\partial}{\partial v}$, $X_2 = x^{-2}\frac{\partial}{\partial v}$, $X_3 = (dx^2 + 8u)\frac{\partial}{\partial v}$
$H(v) = d$, d a constant, $G(u) = \alpha + \beta u$, α and β constants ($\beta \neq 0$) $X_1 = \frac{\partial}{\partial v}$, $X_2 = x^{-2}\frac{\partial}{\partial v}$, $X_3 = (dx^2 + 8u)\frac{\partial}{\partial v}$, $X_4 = \beta \ln x\frac{\partial}{\partial v} - 2x^{-2}\frac{\partial}{\partial u}$, $X_5 = \beta x^2\frac{\partial}{\partial v} - 8\frac{\partial}{\partial u}$, $X_6 = 96x\frac{\partial}{\partial x} - 24dx^2\frac{\partial}{\partial u} + (\beta dx^4 + 192v)\frac{\partial}{\partial v}$, $X_7 = (192u + 24dx^4)\frac{\partial}{\partial u} + (192v - \beta dx^2 + 24\alpha x^2)\frac{\partial}{\partial v}$
$H(v) = d$, d a constant, $G(u) = \alpha + \beta u^{-p}$, α, p, and β constants ($\beta, p \neq 0$) $X_1 = \frac{\partial}{\partial v}$, $X_2 = x^{-2}\frac{\partial}{\partial v}$, $X_3 = (dx^2 + 8u)\frac{\partial}{\partial v}$, $X_4 = 4x\frac{\partial}{\partial x} + 8u\frac{\partial}{\partial u} - (\alpha px^2 - 8v + 8pv)\frac{\partial}{\partial v}$
$H(v) = d$, d a constant, $G(u) = \alpha + \beta e^{-ku}$, α, k, and β constants ($\beta, k \neq 0$) $X_1 = \frac{\partial}{\partial v}$, $X_2 = x^{-2}\frac{\partial}{\partial v}$, $X_3 = (dx^2 + 8u)\frac{\partial}{\partial v}$, $X_4 = 8\frac{\partial}{\partial u} - (\alpha kx^2 + 8kv)\frac{\partial}{\partial v}$
$H(v) = d$, d a constant, $G(u) = \alpha + \beta \ln u$, α and β constants ($\beta \neq 0$) $X_1 = \frac{\partial}{\partial v}$, $X_2 = x^{-2}\frac{\partial}{\partial v}$, $X_3 = (dx^2 + 8u)\frac{\partial}{\partial v}$, $X_4 = 4x\frac{\partial}{\partial x} + 8u\frac{\partial}{\partial u} + (8v - \beta x^2)\frac{\partial}{\partial v}$, $X_5 = x^{-1}\frac{\partial}{\partial x} - 2x^{-2}u\frac{\partial}{\partial u} - (2x^{-2}v - \beta \ln x)\frac{\partial}{\partial v}$
$H(v) = a + bv$, a and b constants ($b \neq 0$), $G(u) = c$, c constant $X_1 = \frac{\partial}{\partial u}$, $X_2 = x^{-2}\frac{\partial}{\partial u}$, $X_3 = (cx^2 + 8v)\frac{\partial}{\partial u}$, $X_4 = b\ln x\frac{\partial}{\partial u} - 2x^{-2}\frac{\partial}{\partial v}$, $X_5 = bx^2\frac{\partial}{\partial u} - 8\frac{\partial}{\partial v}$, $X_6 = 96x\frac{\partial}{\partial x} + (bcx^4 + 192u)\frac{\partial}{\partial u} - 24cx^2\frac{\partial}{\partial v}$, $X_7 = (24ax^2 + 192u - bcx^4)\frac{\partial}{\partial u} + (24cx^2 + 192v)\frac{\partial}{\partial v}$
$H(v) = a + bv$, a and b constants ($b \neq 0$), $G(u) = \alpha + \beta u$, α and β constants ($\beta \neq 0$) $X_1 = F(x)\frac{\partial}{\partial v}$, $X_2 = W(x)\frac{\partial}{\partial u}$, $X_3 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}$, $X_4 = bv\frac{\partial}{\partial u} + \beta u\frac{\partial}{\partial v}$
$H(v) = a + bv$, a and b constants ($b \neq 0$), $G(u) = \beta u^{-p}$, β and p constants ($\beta, p \neq 0$) $X_1 = (bx + bpx)\frac{\partial}{\partial x} + 4bu\frac{\partial}{\partial u} + (2a + 2bv - 2ap - 2bpv)\frac{\partial}{\partial v}$
$H(v) = a + bv$, a and b constants ($b \neq 0$) $G(u) = \beta e^{-ku}$, β and k constants ($\beta, k \neq 0$) $X_1 = bkx\frac{\partial}{\partial x} + 4b\frac{\partial}{\partial u} - (2ak + 2bkv)\frac{\partial}{\partial v}$

Table 4: Continued.

$H(v)=a+bv^{-m}$, a, m, and b constants $(b, m\neq 0)$, $G(u)=c$, c a constant $X_1=\frac{\partial}{\partial u}$, $X_2=x^{-2}\frac{\partial}{\partial u}$, $X_3=(cx^2+8v)\frac{\partial}{\partial u}$, $X_4=4x\frac{\partial}{\partial x}+(8u-amx^2-8mu)\frac{\partial}{\partial u}+8v\frac{\partial}{\partial v}$
$H(v)=bv^{-m}$, m and b constants $(b, m\neq 0)$, $G(u)=\alpha+\beta u$, α and β constants $(\beta\neq 0)$ $X_1=(\beta x+\beta mx)\frac{\partial}{\partial x}+(2\alpha-2\alpha m+2\beta u-2\beta mu)\frac{\partial}{\partial u}+4\beta v\frac{\partial}{\partial v}$
$H(v)=bv^{-m}$, m and b constants $(b, m\neq 0)$, $G(u)=\beta u^{-p}$, β and p constants $(\beta, p\neq 0)$ $X_1=(pmx-x)\frac{\partial}{\partial x}+(2mu-2u)\frac{\partial}{\partial u}+(2pv-2v)\frac{\partial}{\partial v}$
$H(v)=bv^{-m}$, m and b constants $(b, m\neq 0)$, $G(u)=\beta e^{-ku}$, β and k constants $(\beta, k\neq 0)$ $X_1=kmx\frac{\partial}{\partial x}+(2m-2)\frac{\partial}{\partial u}+2kv\frac{\partial}{\partial v}$
$H(v)=a+be^{-mv}$, a, m, and b constants $(b, m\neq 0)$, $G(u)=c$, c constant $X_1=\frac{\partial}{\partial u}$, $X_2=x^{-2}\frac{\partial}{\partial u}$, $X_3=(cx^2+8v)\frac{\partial}{\partial u}$, $X_4=(amx^2+8mu)\frac{\partial}{\partial u}-8\frac{\partial}{\partial v}$
$H(v)=be^{-mv}$, m and b constants $(b, m\neq 0)$, $G(u)=\alpha+\beta u$, α and β constants $(\beta\neq 0)$ $X_1=\beta mx\frac{\partial}{\partial x}-(2\alpha m+2\beta mu)\frac{\partial}{\partial u}+4\beta\frac{\partial}{\partial v}$
$H(v)=be^{-mv}$, m and b constants $(b, m\neq 0)$, $G(u)=\beta u^{-p}$, β and p constants $(\beta, p\neq 0)$ $X_1=mpx\frac{\partial}{\partial x}+2mu\frac{\partial}{\partial u}+(2p-2)\frac{\partial}{\partial v}$
$H(v)=be^{-mv}$, m and b constants $(b, m\neq 0)$, $G(u)=\beta e^{-ku}$, β and k constants $(\beta, k\neq 0)$ $X_1=mkx\frac{\partial}{\partial x}+2m\frac{\partial}{\partial u}+2k\frac{\partial}{\partial v}$
$H(v)=a+b\ln v$, a, m, and b constants $(b\neq 0)$, $G(u)=c$, c constant $X_1=\frac{\partial}{\partial u}$, $X_2=x^{-2}\frac{\partial}{\partial u}$, $X_3=(cx^2+8v)\frac{\partial}{\partial u}$, $X_4=4x\frac{\partial}{\partial x}+(8u-bx^2)\frac{\partial}{\partial u}+8v\frac{\partial}{\partial v}$, $X_5=x^{-1}\frac{\partial}{\partial x}-(2x^{-2}u-b\ln x)\frac{\partial}{\partial u}-2x^{-2}v\frac{\partial}{\partial v}$

These classifying relations are invariant under the equivalence transformation (7) if

$$\begin{aligned}\overline{\alpha}&=\alpha,\\ \overline{\beta}&=\alpha\left(a_8x^2+a_6x^{1-n}+a_4\right)+\beta e^{-a_2},\\ \overline{\gamma}&=\gamma,\\ \overline{\delta}&=\delta e^{2a_1-a_3}-2\gamma(1+n)a_9,\\ \overline{\theta}&=\theta,\\ \overline{\lambda}&=\theta\left(a_9x^2+a_7x^{1-n}+a_5\right)+\lambda e^{-a_3},\\ \overline{\varphi}&=\varphi,\\ \overline{\omega}&=\omega e^{2a_1-a_3}-2\varphi(1+n)a_8.\end{aligned}\tag{21}$$

The classifying relations (20) are also invariant under the equivalence transformation (10) if

$$\begin{aligned}\overline{\alpha}&=\alpha,\\ \overline{\beta}&=\alpha\left(a_8x^2+a_6x^2\ln x+a_4\right)+\beta e^{-a_2},\\ \overline{\gamma}&=\gamma,\\ \overline{\delta}&=\delta e^{2a_1-a_3}-2\gamma a_7,\\ \overline{\theta}&=\theta,\\ \overline{\lambda}&=\theta\left(a_9x^2+a_7x^2\ln x+a_5\right)+\lambda e^{-a_3},\\ \overline{\varphi}&=\varphi,\\ \overline{\omega}&=\omega e^{2a_1-a_2}-2\varphi a_6.\end{aligned}\tag{22}$$

It is also noted that the classifying relations (20) are invariant under the equivalence transformation (13) if

$$\begin{aligned}\overline{\alpha} &= \alpha,\\ \overline{\beta} &= \alpha\left(a_6 \ln x + a_8 x^2 + a_4\right) + \beta e^{-a_2},\\ \overline{\gamma} &= \gamma,\\ \overline{\delta} &= \delta e^{2a_1 - a_3} - 4\gamma a_9,\\ \overline{\theta} &= \theta,\\ \overline{\lambda} &= \theta\left(a_7 \ln x + a_9 x^2 + a_5\right) + \lambda e^{-a_3},\\ \overline{\varphi} &= \varphi,\\ \overline{\omega} &= \omega e^{2a_1 - a_2} - 4\varphi a_8.\end{aligned} \tag{23}$$

The classifying relations (20) are also invariant under the equivalence transformation (16) if

$$\begin{aligned}\overline{\alpha} &= \alpha,\\ \overline{\beta} &= \alpha\left(a_9 x^{-2} + a_7 x^2 + a_5\right) + \beta\left(1 + 2a_2 x^{-2}\right) e^{-a_3},\\ \overline{\gamma} &= \gamma,\\ \overline{\delta} &= \delta e^{2a_1 - a_4} - 8\gamma a_8,\\ \overline{\theta} &= \theta,\\ \overline{\lambda} &= \theta\left(a_{10} x^{-2} + a_8 x^2 + a_6\right) + \lambda\left(1 + 2a_2 x^{-2}\right) e^{-a_4},\\ \overline{\varphi} &= \varphi,\\ \overline{\omega} &= \omega e^{2a_1 - a_3} - 8\varphi a_7.\end{aligned} \tag{24}$$

The above relations are now used to find the nonequivalence forms of H and G and their corresponding Lie point symmetry. Several cases arise and are presented in Tables 1, 2, 3, and 4.

The Noether symmetries given in [6] from (25) to (44) always form a proper subalgebra of the Lie algebra that is obtained above. This can be seen from Tables 1, 2, 3, and 4. However, in [6] the first integrals were also presented.

4. Concluding Remarks

We have studied a generalized coupled Lane-Emden system from the algebraic viewpoint. A complete group classification of the underlying system was performed. We showed that the generalized coupled Lane-Emden system admits a nine- or ten-dimensional equivalence Lie algebra. The principal Lie algebra, which was found to be trivial, had several possible extensions. We deduced the results for all possible cases of the values of n. There were in fact four cases that arose.

Acknowledgment

Two of the authors, B. Muatjetjeja and C. M. Khalique, would like to thank the Organizing Committee of “Symmetries, Differential Equations, and Applications: Galois Bicentenary,” (SDEA2012) Conference for their kind hospitality during the conference.

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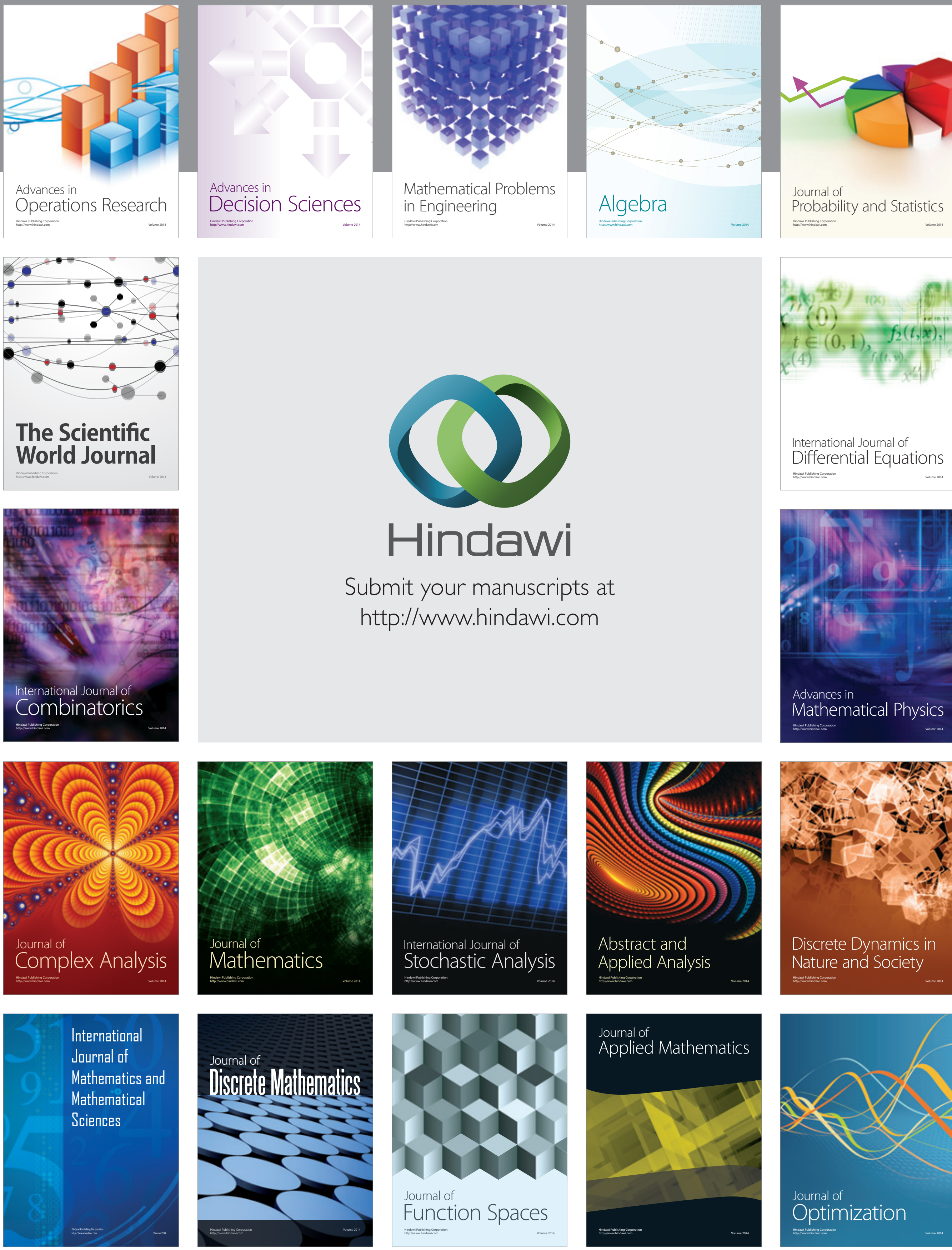
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