

Research Article

Positive Interpolation Operators with Exponential-Type Weights

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We consider positive operators on the real line \mathbb{R} with property of interpolation, and we show the weighted L_p -convergence of the operators. We will construct an analogical operator of one which is studied by Knopfmacher (1986). Furthermore, we treat the Shepard-type interpolatory operator (cf. Xie et al. (1998)).

1. Introduction

In this paper, we consider two interpolatory positive operators. For $\gamma > 1$ and $-\infty < x_{n,n} < \dots < x_{1,n} < \infty$, we construct an operator

$$\begin{aligned} \mathcal{F}_{n,\gamma}[f](x) &= \frac{\sum_{k=1}^n h(x_{k,n}) w^{\gamma-2}(x_{k,n}) \lambda_{kn} f(x_{k,n}) |K_n(x, x_{k,n})|^\gamma}{\sum_{k=1}^n h(x_{k,n}) w^{\gamma-2}(x_{k,n}) \lambda_{kn} |K_n(x, x_{k,n})|^\gamma}. \end{aligned} \quad (1)$$

The details will be stated later, and the result is written in Section 2. Knopfmacher [1] studied the positive operator

$$F_{n,\gamma}[f](x) = \frac{\sum_{k=1}^n \lambda_{kn} f(x_{k,n}) |K_n(x, x_{k,n})|^\gamma}{\sum_{k=1}^n \lambda_{kn} |K_n(x, x_{k,n})|^\gamma}, \quad (2)$$

and for $1 < \gamma \leq 2$, he obtained a certain weighted-convergence theorem on the compact interval $I \subset \mathbb{R} = (-\infty, \infty)$. The operators (1) and (2) have the property of Hermite-Fejér interpolation, that is,

$$\begin{aligned} H_n[f](x_{k,n}) &= f(x_{k,n}), \\ H_n[f]'(x_{k,n}) &= 0, \quad k = 1, 2, \dots, n. \end{aligned} \quad (3)$$

We also treat the interpolatory positive operator of Shepard-type. Let us define $S_{n,\lambda}(f; x)$ for $f \in C(\mathbb{R})$ by

$$S_{n,\lambda}(f; x) := \frac{\sum_{j=1}^n f(x_{j,n}) \Phi_n^{(\lambda-1)/2}(x_{j,n}) |x - x_{j,n}|^{-\lambda}}{\sum_{j=1}^n \Phi_n^{(\lambda-1)/2}(x_{j,n}) |x - x_{j,n}|^{-\lambda}}, \quad (4)$$

$\lambda \geq 1, x \in \mathbb{R}.$

The operator $S_{n,\lambda}(f; x)$ is linear and positive, furthermore it interpolates $f(x)$ at the zeros $\{x_{i,n}\}_{i=1}^n$. In fact, we see that

$$\begin{aligned} S_{n,\lambda}(f; x_{k,n}) &= \lim_{\substack{x \neq x_{k,n}, \\ x \rightarrow x_{k,n}}} \left(f(x_{k,n}) \Phi_n^{(\lambda-1)/2}(x_{k,n}) \right. \\ &\quad \left. + \sum_{j \neq k} f(x_{j,n}) \Phi_n^{(\lambda-1)/2}(x_{j,n}) \right. \\ &\quad \left. \times |x - x_{j,n}|^{-\lambda} |x - x_{k,n}|^\lambda \right) \end{aligned}$$

$$\begin{aligned} & \times \left(\Phi_n^{(\lambda-1)/2}(x_{k,n}) + \sum_{j \neq k} \Phi_n^{(\lambda-1)/2}(x_{j,n}) \right. \\ & \left. \times |x - x_{j,n}|^{-\lambda} |x - x_{k,n}|^{\lambda} \right)^{-1} \\ & = f(x_{k,n}), \quad k = 1, 2, \dots, n. \end{aligned} \tag{5}$$

The related theorem is written in Section 4.

First we need the following definition from [2]. We say that $f : \mathbb{R} \rightarrow [0, \infty)$ is quasi-increasing (quasi-decreasing) if there exists $C > 0$ such that $f(x) \leq Cf(y)$ ($f(x) \geq Cf(y)$), $0 < x < y$.

Definition 1. Let $Q : \mathbb{R} \rightarrow [0, \infty)$ be an even function and satisfying the following properties.

- (a) $Q'(x)$ is continuous in \mathbb{R} , with $Q(0) = 0$.
- (b) $Q''(x)$ exists and is positive in $\mathbb{R} \setminus \{0\}$.
- (c) $\lim_{x \rightarrow \infty} Q(x) = \infty$.
- (d) The function

$$T(x) := \frac{xQ'(x)}{Q(x)}, \quad x \neq 0 \tag{6}$$

is quasi-increasing in $(0, \infty)$, with

$$T(x) \geq \Lambda > 1, \quad x \in \mathbb{R}^+ \setminus \{0\}. \tag{7}$$

- (e) There exists $C_1 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \leq C_1 \frac{|Q'(x)|}{Q(x)}, \quad \text{a.e. } x \in \mathbb{R} \setminus \{0\}. \tag{8}$$

Then, we write $w(x) = \exp(-Q(x)) \in \mathcal{F}(C^2)$. If there also exist a compact subinterval $J(\ni 0)$ of \mathbb{R} and $C_2 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \geq C_2 \frac{|Q'(x)|}{Q(x)}, \quad \text{a.e. } x \in \mathbb{R} \setminus J, \tag{9}$$

then we write $w(x) = \exp(-Q(x)) \in \mathcal{F}(C^2+)$.

Example 2. There are some typical examples of $Q(x)$ satisfying $w = \exp(-Q) \in \mathcal{F}(C^2+)$.

- (1) If $T(x)$ is bounded, then the weight $w = \exp(-Q)$ is the so-called the Freud-type weight. Then the typical Freud-type example would be

$$Q(x) = |x|^\alpha, \quad \alpha > 1. \tag{10}$$

- (2) If $T(x)$ is unbounded, then the weight $w = \exp(-Q)$ is called the Erdős-type weight. Erdős-type examples $w = \exp(-Q) \in \mathcal{F}(C^2+)$ are as follows.

- (a) (see [2, Example 1.2], [3, Theorem 3.1]) For $\alpha > 1$, $l = 1, 2, 3, \dots$

$$Q(x) = Q_{l,\alpha}(x) = \exp_l(|x|^\alpha) - \exp_l(0), \tag{11}$$

where

$$\exp_l(x) = \exp(\exp(\exp \cdots \exp x) \cdots) \quad (l\text{-times}). \tag{12}$$

More precisely, we define for $\alpha + m > 1$, $m \geq 0$, $l \geq 1$ and $\alpha \geq 0$,

$$Q_{l,\alpha,m}(x) := |x|^m (\exp_l(|x|^\alpha) - \alpha^* \exp_l(0)), \tag{13}$$

where $\alpha^* = 0$ if $\alpha = 0$, otherwise $\alpha^* = 1$ (but, note that $Q_{l,0,m}$ gives a Freud-type weight).

- (b) (see [3, Theorem 3.5]) For $\alpha > 1$, put $Q_\alpha(x) := (1 + |x|)^{|x|^\alpha} - 1$, $\alpha > 1$.

We construct the orthonormal polynomials $p_n(x) = p_n(w^2, x)$ of degree n for $w^2(x)$, that is,

$$\begin{aligned} & \int_{-\infty}^{\infty} p_n(w^2, x) p_m(w^2, x) w^2(x) dx \\ & = \delta_{mn} \quad (\text{Kronecker delta}). \end{aligned} \tag{14}$$

Let $f w \in L_p(\mathbb{R})$. The Fourier-type series of f is defined by

$$\tilde{f}(x) := \sum_{k=0}^{\infty} a_k(w^2, f) p_k(w^2, x), \tag{15}$$

$$a_k(w^2, f) := \int_{-\infty}^{\infty} f(t) p_k(w^2, t) w^2(t) dt.$$

We denote the partial sum of $\tilde{f}(x)$ by

$$s_n(f, x) := s_n(w^2, f, x) := \sum_{k=0}^{n-1} a_k(w^2, f) p_k(w^2, x). \tag{16}$$

If we use the Christoffel-Darboux formula, then we obtain

$$s_n(f, x) = \int_{-\infty}^{\infty} K_n(x, t) f(t) w^2(t) dt. \tag{17}$$

Here,

$$\begin{aligned} K_n(x, t) & := \sum_{k=0}^{n-1} p_k(x) p_k(t) \\ & = \frac{\gamma_{n-1} p_n(x) p_{n-1}(t) - p_n(t) p_{n-1}(x)}{\gamma_n (x - t)}, \end{aligned} \tag{18}$$

where $p_n(x) = \gamma_n x^n + \dots$. The polynomials of degree $\leq n$ are denoted by \mathcal{P}_n . We define the Christoffel numbers $\lambda_n(w; x)$ by

$$\lambda_n(w; x) := \inf_{P \in \mathcal{P}_{n-1}} \int_{-\infty}^{\infty} \frac{|Pw|^2(t) dt}{|P(x)|^2}, \quad (19)$$

then we have

$$\lambda_n(w; x) = \frac{1}{K_n(x, x)} = \frac{1}{\sum_{k=0}^{n-1} p_k^2(w^2, x)}. \quad (20)$$

We denote the zeros of the orthonormal polynomial $p_n(w^2, x)$ by $x_{n,n} < x_{n-1,n} < \dots < x_{1,n}$. Then we define the Christoffel numbers $\lambda_{k,n}$, $k = 1, 2, \dots, n$ such as $\lambda_{k,n} := \lambda_n(w, x_{k,n})$.

2. Preliminaries and Theorems

We need the Mhaskar-Rakhmanov-Saff number a_x ;

$$x = \frac{2}{\pi} \int_0^1 \frac{a_x t Q'(a_x t)}{(1-t^2)^{1/2}} dt, \quad x > 0. \quad (21)$$

We define

$$\varphi_u(x) = \begin{cases} \frac{a_u}{u} \frac{1 - (|x|/a_{2u})}{\sqrt{1 - (|x|/a_u) + \delta_u}}, & |x| \leq a_u; \\ \varphi_u(a_u), & a_u < |x|, \end{cases} \quad (22)$$

$$\Phi_n(x) = \begin{cases} 1 - \frac{|x|}{a_n} + \delta_n, & |x| \leq a_n; \\ \delta_n, & a_n < |x|, \end{cases} \quad (23)$$

where

$$\delta_u = \{uT(a_u)\}^{-2/3} \quad u > 0. \quad (24)$$

Moreover, we define a function $\psi_n(x)$ for $\gamma > 1$ and $x \in \mathbb{R}$

$$\psi_n(x) = \begin{cases} a_n^{2-\gamma} \varphi_n^{\gamma-1}(x), & 1 < \gamma < 2; \\ \varphi_n(x) \log a_n, & \gamma = 2; \\ \varphi_n(x), & \gamma > 2, \end{cases} \quad (25)$$

$$\psi_n^* := \begin{cases} a_n^{2-\gamma} \phi_n^{\gamma-1}, & 1 < \gamma < 2; \\ \phi_n \log n, & \gamma = 2; \\ \phi_n, & 2 < \gamma, \end{cases} \quad (26)$$

where $\phi_n := \max\{a_n/n, a_n \delta_n\}$. For the Freud-type weight w we suppose to hold $\psi_n^* \rightarrow 0$ as $n \rightarrow \infty$. If $w \in \mathcal{F}(C^2+)$ is the Erdős-type weight, then it always holds. So for the Freud-type weight we need to limit slightly the weights.

To state our main result, we assume some conditions for $h(x)$ as follows.

- (1) $h(x)$ is even, positive, and quasi-decreasing on $[0, \infty)$.
- (2) $h(x_{k,n}) \sim h(x_{k+1,n})$ for $k = 1, 2, \dots, n$ and $n = 1, 2, \dots$
- (3) $h(x)\Phi_n^{-\gamma/4}(x)$ is bounded on \mathbb{R} for $n = 1, 2, \dots$

Let $\{x_{j,n}\}_{j=1}^n$ be the zeros of the orthonormal polynomial $p_n(w^2, x)$. Then we define the operator $\mathcal{F}_{n,\gamma}[f](x)$ by (1) with $\gamma > 1, h(x)$, and for each $f \in C(\mathbb{R})$ we define a pointwise modulus of continuity $\omega_x(f; t) = \sup_{\{y: |x-y| \leq t, y \in \mathbb{R}\}} |f(x) - f(y)|$. When $f \in C(\mathbb{R})$ is uniformly continuous on \mathbb{R} , we set

$$\Omega(f; t) = \sup_{x \in \mathbb{R}} \omega_x(f; t). \quad (27)$$

Then our first theorem is as follows.

Theorem 3. Let $w \in \mathcal{F}(C^2+)$, and let $\psi_n^* \rightarrow 0$ as $n \rightarrow \infty$. Let $\gamma > 1$. Then we have the following.

- (a) For $x_{n,n} \leq x \leq x_{1,n}$,

$$|\mathcal{F}_{n,\gamma}[f](x) - f(x)| \leq C\omega_x(f; \psi_n(x)) h^{-1}(x) \Phi_n^{-\gamma/4}(x), \quad (28)$$

and for $|x| > x_{1,n}$

$$|\mathcal{F}_{n,\gamma}[f](x) - f(x)| \leq C\omega_x(f; \psi_n(x)) |x| \psi_n^{-1}(x). \quad (29)$$

- (b) Let $0 < p \leq \infty$ and w^* be an integrable function satisfying the following condition:

$$\begin{aligned} & \|w^*(x) h^{-1}(x) \Phi_n^{-\gamma/4}(x)\|_{L_p([x_{n,n}, x_{1,n}])} \\ & + \|x \psi_n^{-1}(x) w^*(x)\|_{L_p(|x| \geq x_{1,n})} < \infty. \end{aligned} \quad (30)$$

Then one has for $f(x)$ being uniformly continuous and bounded on \mathbb{R}

$$\|w^* \{\mathcal{F}_{n,\gamma}[f] - f\}\|_{L_p(\mathbb{R})} = O(1) \Omega(f; \psi_n^*), \quad (31)$$

where ψ_n^* are defined in (26).

We prepare some lemmas for the proof of the theorem.

Lemma 4. Let $w = \exp(-Q) \in \mathcal{F}(C^2)$.

- (1) (see [2, Lemma 3.5 (3.27)–(3.29)]) For fixed $L > 0$ and uniformly for $t > 0$,

$$\begin{aligned} a_{Lt} & \sim a_t, & T(a_{Lt}) & \sim T(a_t), \\ Q^{(j)}(a_{Lt}) & \sim Q^{(j)}(a_t), & j & = 0, 1. \end{aligned} \quad (32)$$

Moreover,

$$T(a_{Lt}) \sim T(a_t). \quad (33)$$

- (2) (see [2, Lemma 3.4 (3.18), (3.17), Lemma 3.8 (3.42)])

$$Q(a_t) \sim \frac{t}{\sqrt{T(a_t)}}, \quad Q'(a_t) \sim \frac{t\sqrt{T(a_t)}}{a_t}, \quad (34)$$

and for $x \in [0, a_n/2]$,

$$Q'(x) \sim \frac{n}{a_n} \left(\frac{x}{a_n}\right)^{\Lambda-1}, \tag{35}$$

where $\Lambda > 1$ is defined in Definition 1(d).

(3) (see [2, Lemma 3.11 (a), (b)]) Given fixed $0 < \alpha$, one has uniformly for $t > 0$,

$$\left|1 - \frac{a_{\alpha t}}{a_t}\right| \sim \frac{1}{T(a_t)}. \tag{36}$$

(4) (see [2, Lemma 3.7 (3.38)]) For some $0 < \varepsilon \leq 2$, and for large enough t ,

$$T(a_t) \leq t^{2-\varepsilon}. \tag{37}$$

(5) (see [2, Lemma 3.8 (a)]) For $x \in [0, a_t]$,

$$Q'(x) \leq C \frac{t}{a_t} \frac{1}{\sqrt{1 - (x/a_t)}}. \tag{38}$$

Lemma 5 ([4, Theorem 2.7]). *There exists $C > 0$ such that*

$$\sup_{x \in \mathbb{R}} |p_n(x) w(x) \Phi_n^{1/4}(x)| \leq C a_n^{-1/2}. \tag{39}$$

Lemma 6. *Let $w(x) = \exp(-Q(x)) \in \mathcal{F}(C^2+)$.*

(1) *Let $x_{j,n}$ be the zero of $p_n(x)$. Then for $n \geq 1$ and $1 \leq j \leq n-1$,*

$$x_{j,n} - x_{j+1,n} \sim \varphi_n(x_{j,n}), \tag{40}$$

$$\varphi_n(x_{j,n}) \sim \varphi_n(x_{j+1,n}). \tag{41}$$

(2) *For $n \geq 1$ and $1 \leq j \leq n-1$,*

$$\Phi_n(x_{j,n}) \sim \Phi_n(x_{j+1,n}). \tag{42}$$

Proof. (1) This follows from [2, Corollary 13.4, Theorem 5.7 (b)].

(2) Recall the definition of $\Phi_n(x)$ in (21). We have

$$\begin{aligned} \Phi_n(x_{j,n}) &= 1 - \frac{|x_{j,n}|}{a_n} + \delta_n \\ &= 1 - \frac{|x_{j+1,n}|}{a_n} \\ &\quad + \delta_n - \frac{x_{j,n} - x_{j+1,n}}{a_n} \\ &\sim 1 - \frac{|x_{j+1,n}|}{a_n} + \delta_n - \frac{\varphi_n(x_{j,n})}{a_n} \\ &= 1 - \frac{|x_{j+1,n}|}{a_n} + \delta_n \\ &\quad - \frac{1}{n} \frac{1 - |x_{j,n}/a_{2n}|}{\sqrt{1 - |x_{j,n}|/a_n}}. \end{aligned} \tag{43}$$

Hence, if $|x_{k,n}|, |x_{k+1,n}| \leq a_n/2$, then we see

$$\Phi_n(x_{j,n}) \sim 1 - \frac{|x_{j+1,n}|}{a_n} + \delta_n - \frac{1}{n} \sqrt{\frac{1 - |x_{j,n}|}{a_n} + \delta_n}. \tag{44}$$

Here we see

$$\begin{aligned} \Phi_n(x_{j,n}) + \frac{C}{n} \sqrt{\Phi_n(x_{j,n})} \\ = \sqrt{\Phi_n(x_{j,n})} \left\{ \sqrt{\Phi_n(x_{j,n})} + \frac{C}{n} \right\} \sim \Phi_n(x_{j,n}), \end{aligned} \tag{45}$$

because of $\sqrt{\Phi_n(x_{j,n})} \geq C/\sqrt{T(a_n)} > 1/n^{1-\varepsilon} > (1/n)(\varepsilon > 0)$. (see Lemma 4 (3), (4)). Therefore, we have

$$\Phi_n(x_{j,n}) \sim \Phi_n(x_{j+1,n}). \tag{46}$$

Let $a_n/2 < |x_{j,n}|$. Then we see

$$\frac{1}{n} \frac{1 - |x_{j,n}/a_{2n}|}{\sqrt{1 - |x_{j,n}|/a_n + \delta_n}} \leq C \frac{1}{n} \frac{(nT(a_n))^{1/3}}{T(a_n)} \sim \delta_n. \tag{47}$$

Therefore we see

$$\Phi_n(x_{j,n}) \sim 1 - \frac{|x_{j+1,n}|}{a_n} + \delta_n = \Phi_n(x_{j+1,n}). \tag{48}$$

□

Lemma 7 ([2, Theorem 13.3 (13.9)]). *If $x \in [x_{k+1,n}, x_{k,n}]$, then*

$$(l_{k,n}w)(x) w^{-1}(x_{k,n}) + (l_{k+1,n}w)(x) w^{-1}(x_{k+1,n}) \sim 1. \tag{49}$$

Lemma 8. *Let $w = \exp(-Q) \in \mathcal{F}(C^2+)$. Then the following results hold.*

(a) For $|x| \leq a_n(1 + \delta_n)$,

$$K_n(x, x) = \sum_{k=0}^{n-1} p_k^2(w^2, x) = \lambda_{n,2}^{-1}(w; x) \sim \varphi_n^{-1}(x) w^{-2}(x). \tag{50}$$

(b) For $x \in \mathbb{R}$

$$K_n(x, x) \leq C \varphi_n^{-1}(x) w^{-2}(x). \tag{51}$$

Proof. From [2, Theorem 9.3], we have the following.

(1) Uniformly for $n \geq 1$ and $|x| \leq a_n(1 + \eta_n)$, we have

$$\lambda_n(w; x) \sim \varphi_n(x) w^2(x). \tag{52}$$

(2) Moreover, uniformly for $n \geq 1$ and $x \in \mathbb{R}$,

$$\lambda_n(w; x) \geq C \varphi_n(x) w^2(x). \tag{53}$$

Since $K_n(x, x) = 1/\lambda_n(w; x)$, we have the following results. \square

3. Proof of Theorem 3

To estimate the difference $|\mathcal{F}_{n,\gamma}[f](x) - f(x)|$, we split $\sum_{k=1}^n$ into two parts.

To prove the theorem we start the estimation of the denominator for the operator $\mathcal{F}_{n,\gamma}$. We will need it in Step 4.

Step 1. Let

$$H_{n,\gamma}(x) := \sum_{k=1}^n h(x_{k,n}) w^{\gamma-2}(x_{k,n}) \lambda_{k,n} |K_n(x, x_{k,n})|^\gamma. \tag{54}$$

Then we have the following.

Lemma 9. *There exists $C > 0$ such that uniformly, for $x \in [x_{n,n}, x_{1,n}]$,*

$$H_{n,\gamma}(x) \geq Ch(x) \varphi_n^{1-\gamma}(x) w^{-\gamma}(x). \tag{55}$$

Proof. By Lemma 7, if $x \in [x_{k+1,n}, x_{k,n}]$, $x_{k+1,n} \geq 0$, $k \geq 1$, then

$$(l_{k,n}w)(x) w^{-1}(x_{k,n}) + (l_{k+1,n}w)(x) w^{-1}(x_{k+1,n}) \sim 1. \tag{56}$$

Since $l_{k,n}(x) = \lambda_{k,n}K_n(x, x_{k,n})$, we see

$$\begin{aligned} & \lambda_{k,n} \frac{w(x)}{w(x_{k,n})} K_n(x, x_{k,n}) \\ & + \lambda_{k+1,n} \frac{w(x)}{w(x_{k+1,n})} K_n(x, x_{k+1,n}) \sim 1, \end{aligned} \tag{57}$$

and this implies that

$$\begin{aligned} 0 < C & \leq \left(\lambda_{k,n} \frac{w(x)}{w(x_{k,n})} |K_n(x, x_{k,n})| \right. \\ & \left. + \lambda_{k+1,n} \frac{w(x)}{w(x_{k+1,n})} |K_n(x, x_{k+1,n})| \right) \\ & = \left(\frac{\lambda_{k,n} w(x)}{w^2(x_{k,n})} w(x_{k,n}) |K_n(x, x_{k,n})| \right. \\ & \left. + \frac{\lambda_{k+1,n} w(x)}{w^2(x_{k+1,n})} w(x_{k+1,n}) |K_n(x, x_{k+1,n})| \right) \\ & \sim \frac{\lambda_{k,n} w(x)}{w^2(x_{k,n})} (w(x_{k,n}) |K_n(x, x_{k,n})| \\ & \quad + w(x_{k+1,n}) |K_n(x, x_{k+1,n})|) \\ & \leq C \frac{\lambda_{k,n} w(x)}{w^2(x_{k,n})} (|K_n(x, x_{k,n}) w(x_{k,n})|^\gamma \\ & \quad + |K_n(x, x_{k+1,n}) w(x_{k+1,n})|^\gamma)^{1/\gamma}. \end{aligned} \tag{58}$$

Therefore, from (41) and (52) we can obtain

$$\begin{aligned} & \varphi_n^{1-\gamma}(x) w^{-\gamma}(x) \\ & \sim \left(\frac{\lambda_{k,n}}{w^2(x_{k,n})} \right)^{1-\gamma} w^{-\gamma}(x) \\ & \leq C \frac{\lambda_{k,n}}{w^2(x_{k,n})} (|K_n(x, x_{k,n}) w(x_{k,n})|^\gamma \\ & \quad + |K_n(x, x_{k+1,n}) w(x_{k+1,n})|^\gamma). \end{aligned} \tag{59}$$

Using the fact $h(x_{k,n}) \sim h(x) \sim h(x_{k+1,n})$ (see the definition of $h(x)$), we have by (41) and (52)

$$\begin{aligned} & H_{n,\gamma}(x) \\ & \geq h(x_{k,n}) \lambda_{k,n} w^{\gamma-2}(x_{k,n}) |K_n(x, x_{k,n})|^\gamma \\ & \quad + h(x_{k+1,n}) \lambda_{k+1,n} w^{\gamma-2}(x_{k+1,n}) |K_n(x, x_{k+1,n})|^\gamma \\ & \geq Ch(x) \frac{\lambda_{k,n}}{w^2(x_{k,n})} (|K_n(x, x_{k,n}) w(x_{k,n})|^\gamma \\ & \quad + |K_n(x, x_{k+1,n}) w(x_{k+1,n})|^\gamma) \\ & \geq Ch(x) \varphi_n^{1-\gamma}(x) w^{-\gamma}(x). \end{aligned} \tag{60}$$

In another case, that is, when $x_{k+1,n} < 0$, we also have the same result. \square

Step 2. Let $|x - x_{k,n}| \leq \varphi_n(x)$. Let $f(x)$ be uniformly continuous and bounded on \mathbb{R} , and let $\gamma > 1$. Then we have

$$|f(x) - f(x_{k,n})| \leq \omega_x(f; \varphi_n(x)). \tag{61}$$

Now, let

$$\sum_1 := \frac{1}{H_{n,\gamma}(x)} \sum_{|x-x_{k,n}| \leq \varphi_n(x)} h(x_{k,n}) w^{\gamma-2}(x_{k,n}) \lambda_{kn} \times |f(x) - f(x_{k,n})| |K_n(x, x_{k,n})|^\gamma. \quad (62)$$

We have the following estimation.

Lemma 10. For $x \in \mathbb{R}$,

$$\sum_1 \leq \omega_x(f; \varphi_n(x)). \quad (63)$$

Proof. By (61),

$$\begin{aligned} \sum_1 &\leq \omega_x(f; \varphi_n(x)) \frac{1}{H_{n,\gamma}(x)} \\ &\quad \times \sum_{|x-x_{k,n}| \leq \varphi_n(x)} h(x_{k,n}) w^{\gamma-2}(x_{k,n}) \lambda_{kn} |K_n(x, x_{k,n})|^\gamma \\ &\leq \omega_x(f; \varphi_n(x)), \end{aligned} \quad (64)$$

because we know from the definition of $H_{n,\gamma}(x)$ in (54) that

$$\sum_{|x-x_{k,n}| \leq \varphi_n(x)} h(x_{k,n}) w^{\gamma-2}(x_{k,n}) \lambda_{kn} |K_n(x, x_{k,n})|^\gamma \leq H_{n,\gamma}(x). \quad (65)$$

□

Step 3. Next, we estimate $\sum_{|x-x_{k,n}| > \varphi_n(x)}$. Let $|x - x_{k,n}| > \varphi_n(x)$ and let $|x - x_{m,n}| = \min\{|x - x_{k,n}|, k = 1, 2, \dots, n\}$. To do so, we prepare the following. By Lemma 6,

$$\begin{aligned} |K_n(x, x_{k,n})| &\leq C a_n \frac{|p_n(x) p_{n-1}(x_{k,n})|}{|x - x_{k,n}|} \\ &\leq C w^{-1}(x) w^{-1}(x_{k,n}) \Phi_n^{-1/4}(x) \\ &\quad \times \Phi_n^{-1/4}(x_{k,n}) \frac{1}{|x - x_{k,n}|}. \end{aligned} \quad (66)$$

From the property of the modulus of continuity we have, for $|x - x_{k,n}| > \varphi_n(x_{k,n})$,

$$|f(x) - f(x_{k,n})| \leq C (|x - x_{k,n}| \psi_n^{-1}(x) + 1) \omega_x(f; \psi_n(x)), \quad (67)$$

where $\psi_n(x)$ is defined in (25) as $\psi_n(x) \rightarrow 0$ uniformly in \mathbb{R} as $n \rightarrow \infty$.

We have the following estimate.

Lemma 11. For any $x \in \mathbb{R}$,

$$B_{n,k}(x) := \frac{1}{H_{n,\gamma}(x)} h(x_{k,n}) w^{\gamma-2}(x_{k,n}) \lambda_{kn} |K_n(x, x_{k,n})|^\gamma. \quad (68)$$

Then

$$\sum_{|x-x_{k,n}| > \varphi_n(x)} B_{n,k}(x) \leq 1. \quad (69)$$

Proof.

$$\begin{aligned} &\sum_{|x-x_{k,n}| > \varphi_n(x)} B_{n,k}(x) \\ &= \frac{1}{H_{n,\gamma}(x)} \sum_{|x-x_{k,n}| > \varphi_n(x)} h(x_{k,n}) w^{\gamma-2}(x_{k,n}) \lambda_{kn} \\ &\quad \times |K_n(x, x_{k,n})|^\gamma \\ &\leq 1, \end{aligned} \quad (70)$$

because we know from the definition of $H_{n,\gamma}(x)$ in (54) that

$$\sum_{|x-x_{k,n}| > \varphi_n(x)} h(x_{k,n}) w^{\gamma-2}(x_{k,n}) \lambda_{kn} |K_n(x, x_{k,n})|^\gamma \leq H_{n,\gamma}(x). \quad (71)$$

□

Step 4. Let $|x - x_{k,n}| > \varphi_n(x)$. Using the result of Step 1, we have the following estimate.

Lemma 12. For any $x \in \mathbb{R}$, one sets

$$C_{n,k}(x) := \frac{1}{H_{n,\gamma}(x)} h(x_{k,n}) w^{\gamma-2}(x_{k,n}) \lambda_{kn} \times |x - x_{k,n}| |K_n(x, x_{k,n})|^\gamma. \quad (72)$$

Then for $x \in [x_{n,n}, x_{1,n}]$,

$$\sum_{|x-x_{k,n}| > \varphi_n(x)} C_{n,k}(x) \leq Ch^{-1}(x) \Phi_n^{-\gamma/4}(x) \psi_n(x), \quad (73)$$

and for $|x| > x_{1,n}$,

$$\sum_{|x-x_{k,n}| > \varphi_n(x)} C_{n,k}(x) \leq 2|x|. \quad (74)$$

Proof. First, let $x \in [x_{n,n}, x_{1,n}]$. Then using (52), (66), and Lemma 9, we have

$$\begin{aligned} C_{n,k}(x) &\leq Ch(x_{k,n}) \varphi_n(x_{k,n}) \Phi_n^{-\gamma/4} \\ &\quad \times (x) \Phi_n^{-\gamma/4}(x_{k,n}) \frac{1}{|x - x_{k,n}|^{\gamma-1}} \\ &\quad \times h^{-1}(x) \varphi_n^{\gamma-1}(x). \end{aligned} \quad (75)$$

From the fact that $h(x)\Phi_n^{-\gamma/4}(x)$ is bounded (recall the definition of $h(x)$), we can continue as

$$C_{n,k}(x) \leq Ch^{-1}(x) \Phi_n^{-\gamma/4}(x) \varphi_n^{\gamma-1}(x) \frac{\varphi_n(x_{k,n})}{|x - x_{k,n}|^{\gamma-1}}. \quad (76)$$

Then by (25) and (40),

$$\begin{aligned} & \sum_{|x-x_{k,n}|>\varphi_n(x)} C_{n,k}(x) \\ & \leq C \sum_{|x-x_{k,n}|>\varphi_n(x)} \frac{\varphi_n(x_{k,n})}{|x-x_{k,n}|^{\gamma-1}} h^{-1}(x) \Phi_n^{-\gamma/4}(x) \varphi_n^{\gamma-1}(x) \\ & \leq Ch^{-1}(x) \Phi_n^{-\gamma/4}(x) \varphi_n^{\gamma-1}(x) \\ & \quad \times \begin{cases} a_n^{2-\gamma}, & 1 < \gamma < 2; \\ \log a_n, & \gamma = 2; \\ \varphi_n^{2-\gamma}(x), & \gamma > 2, \end{cases} \\ & \leq Ch^{-1}(x) \Phi_n^{-\gamma/4}(x) \psi_n(x). \end{aligned} \tag{77}$$

Next, suppose $x > x_{1,n}$. Then since

$$\begin{aligned} C_{n,k}(x) & \leq 2|x| \frac{1}{H_{n,\gamma}(x)} h(x_{k,n}) \omega^{\gamma-2}(x_{k,n}) \lambda_{k,n} \\ & \quad \times |K_n(x, x_{k,n})|^\gamma, \end{aligned} \tag{78}$$

we have from Lemma 11,

$$\sum_{|x-x_{k,n}|>\varphi_n(x)} C_{n,k}(x) \leq 2|x|. \tag{79}$$

□

Step 5. Using (67) and Lemmas 11 and 12, we can estimate the part $\sum_{|x-x_{k,n}|>\varphi_n(x)}$ as follows:

$$\begin{aligned} \sum_2 & := \left(\sum_{|x-x_{k,n}|>\varphi_n(x)} h(x_{k,n}) \omega^{\gamma-2}(x_{k,n}) \lambda_{k,n} \right. \\ & \quad \left. \times |f(x) - f(x_{k,n})| |K_n(x, x_{k,n})|^\gamma \right) \\ & \quad \times \left(\sum_{k=1}^n h(x_{k,n}) \omega^{\gamma-2}(x_{k,n}) \lambda_{k,n} |K_n(x, x_{k,n})|^\gamma \right)^{-1}. \end{aligned} \tag{80}$$

Then for $x \in [x_{n,n}, x_{1,n}]$

$$\begin{aligned} \sum_2 & \leq C\omega_x(f; \psi_n(x)) \\ & \quad \times \left(\sum_{|x-x_{k,n}|>\varphi_n(x)} B_{n,k}(x) + \psi_n^{-1}(x) \right. \\ & \quad \left. \times \sum_{|x-x_{k,n}|>\varphi_n(x)} C_{n,k}(x) \right) \\ & \leq C\omega_x(f; \psi_n(x)) (1 + h^{-1}(x) \Phi_n^{-\gamma/4}(x)) \\ & \leq C\omega_x(f; \psi_n(x)) h^{-1}(x) \Phi_n^{-\gamma/4}(x), \end{aligned} \tag{81}$$

and for $|x| \geq x_{1,n}$,

$$\begin{aligned} \sum_2 & \leq C\omega_x(f; \psi_n(x)) \\ & \quad \times \left(\sum_{|x-x_{k,n}|>\varphi_n(x)} B_{n,k}(x) + \psi_n^{-1}(x) \right. \\ & \quad \left. \times \sum_{|x-x_{k,n}|>\varphi_n(x)} C_{n,k}(x) \right) \\ & \leq C\omega_x(f; \psi_n(x)) (1 + |x| \psi_n^{-1}(x)) \\ & \leq C\omega_x(f; \psi_n(x)) |x| \psi_n^{-1}(x). \end{aligned} \tag{82}$$

Therefore, with Lemma 10 we have the following result.

Lemma 13. For $x_{n,n} \leq x \leq x_{1,n}$,

$$|\mathcal{F}_{n,\gamma}[f](x) - f(x)| \leq C\omega_x(f; \psi_n(x)) h^{-1}(x) \Phi_n^{-\gamma/4}(x) \tag{83}$$

and for $|x| > x_{1,n}$

$$|\mathcal{F}_{n,\gamma}[f](x) - f(x)| \leq C\omega_x(f; \psi_n(x)) |x| \psi_n^{-1}(x). \tag{84}$$

Proof of Theorem 3. (a) follows from Lemma 13. We will show (b). Let $0 < p \leq \infty$. Then since we know that $\varphi(x) \leq C\phi_n$ and so $\psi_n(x) \leq C\psi_n^*$ for all $x \in \mathbb{R}$, we have

$$\begin{aligned} \|w^*(\mathcal{F}_{n,\gamma}[f] - f)\|_{L_p(\mathbb{R})} & = O(1) \left\| w^* \left(\sum_1 + \sum_2 \right) \right\|_{L_p(\mathbb{R})} \\ & = O(1) \Omega(f; \psi_n^*). \end{aligned} \tag{85}$$

□

Example 14. Let $h(x) = \Phi^{\gamma/4}(x)$ and

$$w^*(x) = \frac{\Phi^{\gamma/2}(x)}{(1 + |x|)^{\beta+1}}, \quad \beta p > 1, \tag{86}$$

where

$$\Phi(x) := \frac{1}{(1 + Q(x))^{2/3} T(x)}. \tag{87}$$

Then the condition (30) is satisfied.

4. Shepard-Type Operator

Let us define the positive interpolatory operator (4) for $f \in C(\mathbb{R})$ and the zeros $\{x_{j,n}\}_{j=1}^n$ of the orthonormal polynomial $p_n(w^2, x)$.

Let

$$\Phi(x) := \frac{1}{(1 + Q(x))^{2/3} T(x)}. \tag{88}$$

Lemma 15 ([5, Lemma 3.3]). For $x \in \mathbb{R}$, one has

$$\Phi(x) \leq C\Phi_n(x), \quad n \geq 1. \tag{89}$$

Assumption 1. We suppose that, for each $\varepsilon > 0$,

$$T(a_n) \leq C(\varepsilon)n^\varepsilon, \quad n = 1, 2, 3, \dots, \tag{90}$$

where $C(\varepsilon)$ is a constant depending only on ε .

Remark 16. Let $w = \exp(-Q) \in \mathcal{F}(C^2+)$, and let us define

$$\begin{aligned} \nu &:= \limsup_{x \rightarrow \infty} \frac{Q''(x)/Q'(x)}{Q'(x)/Q(x)}, \\ \mu &:= \liminf_{x \rightarrow \infty} \frac{Q''(x)/Q'(x)}{Q'(x)/Q(x)}. \end{aligned} \tag{91}$$

If $\nu = \mu$, then we say that the weight w is regular. The regular weights satisfy the condition (90) (see [6, Corollry 5.5]). All weights in Example 2 are regular weights.

Lemma 17 ([3, Theorem 1.6]). Let $w = \exp(-Q) \in \mathcal{F}(C^2+)$, and let a_n be defined by (21). Then there exists $C > 0$ such that for every $n > 0$

$$a_n \leq Cn^{1/\Lambda}, \tag{92}$$

where $\Lambda > 1$ is defined in Definition 1 (d). In particular, for the weight w_α one has $\Lambda = \alpha$. Furthermore, if w is an Erdős-type, then for any $\eta > 0$, there exists $C(\eta) > 0$ such that, for every $n > 0$,

$$a_n \leq C(\eta)n^\eta. \tag{93}$$

For each $3/2 < \lambda < 3$ let us set

$$\mu_n = \begin{cases} \frac{a_n T^{\lambda/3}(a_n)}{n^{1-\lambda/3}}, & 2 < \lambda < 3; \\ \frac{a_n T^{\lambda/3}(a_n) \log n}{n^{1-\lambda/3}}, & \lambda = 2; \\ \frac{a_n T^{\lambda/3}(a_n)}{n^{(2\lambda-3)/3}}, & \frac{3}{2} < \lambda < 2. \end{cases} \tag{94}$$

Our second theorem is as follows.

Theorem 18. Let $f \in C(\mathbb{R})$ be uniformly continuous on \mathbb{R} and let $3/2 < \lambda < 3$. Assume $U(x)$ is a nonnegative and decreasing function with $U(x) \leq C\Phi^{(\lambda-1)/2}(x)$. Then one has for the Erdős-type weights,

$$\|U(x)(S_{n,\lambda}(f;x) - f(x))\|_{L^\infty(\mathbb{R})} \leq C\Omega(f; \mu_n), \tag{95}$$

where μ_n is defined in (94).

For the Freud weights we have the following. For $\Lambda > 3$, let us set $(3/2)(1 + (1/\Lambda)) < \lambda < 3(1 - (1/\Lambda))$ and

$$\mu_{n,\Lambda} = \begin{cases} \frac{1}{n^{1-\lambda/3-1/\Lambda}}, & 2 < \lambda < 3\left(1 - \frac{1}{\Lambda}\right); \\ \frac{1}{n^{1/3-1/\Lambda}}, & \lambda = 2; \\ \frac{1}{n^{(2\lambda-3)/3-1/\Lambda}}, & \frac{3}{2}\left(1 + \frac{1}{\Lambda}\right) < \lambda < 2 \end{cases} \tag{96}$$

(note (92) and (94)).

Corollary 19. Let $\Lambda > 3$, where Λ is defined in Definition 1 (d), and let $(3/2)(1 + (1/\Lambda)) < \lambda < 3(1 - (1/\Lambda))$. Then, for the Freud-type weights, (95) holds with $\mu_{n,\Lambda}$. In particular, when $w(x) = \exp(-|x|^\alpha)$, one can take $\Lambda = \alpha$.

Remark 20. For the Freud-type weights we see $\lim_{n \rightarrow \infty} \mu_{n,\Lambda} = 0$. If we assume (90), then for the Erdős-type weights, from Lemma 17 (93), we also have $\lim_{n \rightarrow \infty} \mu_n = 0$.

Proof of Theorem 18. Let $3/2 < \lambda < 3$. We see that

$$\begin{aligned} S_{n,\lambda}(f;x) - f(x) &= \frac{\sum_{j=1}^n \{f(x_{j,n}) - f(x)\} \Phi_n^{(\lambda-1)/2}(x_{j,n}) |x - x_{j,n}|^{-\lambda}}{\sum_{j=1}^n \Phi_n^{(\lambda-1)/2}(x_{j,n}) |x - x_{j,n}|^{-\lambda}}. \end{aligned} \tag{97}$$

Let $(x_{m+1,n} + x_{m,n})/2 < x \leq x_{m,n}$ or $(x_{m,n} + x_{m-1,n})/2 < x \leq x_{m,n}$. Then, we see

$$\begin{aligned} |f(x_{m,n}) - f(x)| &\leq \omega_x(f; |x_{m,n} - x|) \\ &\leq C\omega_x(f; \varphi_n(x)) \leq C\omega(f; \mu_n), \end{aligned} \tag{98}$$

where μ_n is defined in (94). If $j \neq m$, then we have

$$\begin{aligned} |f(x_{j,n}) - f(x)| &\leq \omega_x(f; |x - x_{j,n}|) \\ &\leq (|x - x_{j,n}| \mu_n^{-1} + 1) \Omega(f; \mu_n). \end{aligned} \tag{99}$$

Let

$$\begin{aligned} \sum_1 &:= \frac{\sum_{j \neq m} \Phi_n^{(\lambda-1)/2}(x_{j,n}) |x - x_{j,n}|^{-\lambda}}{\sum_{j=1}^n \Phi_n^{(\lambda-1)/2}(x_{j,n}) |x - x_{j,n}|^{-\lambda}}, \\ \sum_2 &:= \frac{\sum_{j \neq m} \Phi_n^{(\lambda-1)/2}(x_{j,n}) |x - x_{j,n}|^{-(\lambda-1)}}{\sum_{j=1}^n \Phi_n^{(\lambda-1)/2}(x_{j,n}) |x - x_{j,n}|^{-\lambda}}. \end{aligned} \tag{100}$$

Then we see that $0 < \sum_1 \leq 1$. Now, we will estimate \sum_2 . We see that

$$\begin{aligned} \frac{1}{|x - x_{j,n}|} &\sim \left(\sum_{\substack{j \leq s \leq m+1 \\ m-1 \leq s \leq j}} \varphi_n(x_{s,n}) \right)^{-1} \\ &\sim \frac{n}{a_n} \left(\sum_{\substack{j \leq s \leq m+1 \\ m-1 \leq s \leq j}} \frac{1 - |x_{s,n}|/a_{2n}}{(1 - |x_{s,n}|/a_n + \delta_n)^{1/2}} \right)^{-1} \\ &\geq \frac{n}{a_n} (nT(a_n))^{-1/3} \left(\sum_{\substack{j \leq s \leq m+1 \\ m-1 \leq s \leq j}} (1 - |x_{s,n}|/a_{2n}) \right)^{-1} \\ &\geq \frac{n^{2/3}}{a_n T^{1/3}(a_n)} \frac{1}{|m - j|}. \end{aligned} \tag{101}$$

Hence we have

$$\begin{aligned} &\sum_{j=1}^n \Phi_n^{(\lambda-1)/2}(x_{j,n}) |x - x_{j,n}|^{-\lambda} \\ &\geq \left(\frac{n^{2/3}}{a_n T^{1/3}(a_n)} \right)^\lambda \sum_{j \neq m} \Phi_n^{(\lambda-1)/2}(x_{j,n}) \frac{1}{|m - j|^\lambda} \\ &\geq \left(\frac{n^{2/3}}{a_n T^{1/3}(a_n)} \right)^\lambda \sum_{\substack{|x_{j,n}| \leq a_n/2 \\ j \neq m}} \Phi_n^{(\lambda-1)/2}(x_{j,n}) \frac{1}{|m - j|^\lambda} \tag{102} \\ &\geq C \left(\frac{n^{2/3}}{a_n T^{1/3}(a_n)} \right)^\lambda \sum_{\substack{|x_{j,n}| \leq a_n/2 \\ j \neq m}} \frac{1}{|m - j|^\lambda} \\ &\geq C \left(\frac{n^{2/3}}{a_n T^{1/3}(a_n)} \right)^\lambda \begin{cases} 1, & \lambda > 1; \\ \log n, & \lambda = 1. \end{cases} \end{aligned}$$

Using for $1 \leq j \leq n$

$$1 - \frac{|x_{j,n}|}{a_{2n}} \geq C \left(1 - \frac{|x_{j,n}|}{a_n} + \delta_n \right), \tag{103}$$

we see that

$$\begin{aligned} \frac{1}{|x - x_{j,n}|} &\sim \left(\sum_{\substack{j \leq s \leq m+1 \\ m-1 \leq s \leq j}} \varphi_n(x_{s,n}) \right)^{-1} \\ &\sim \frac{n}{a_n} \left(\sum_{\substack{j \leq s \leq m+1 \\ m-1 \leq s \leq j}} \frac{1 - |x_{s,n}|/a_{2n}}{(1 - |x_{s,n}|/a_n + \delta_n)^{1/2}} \right)^{-1} \\ &\leq C \frac{n}{a_n} \left(\sum_{\substack{j \leq s \leq m+1 \\ m-1 \leq s \leq j}} (1 - |x_{s,n}|/a_n + \delta_n)^{1/2} \right)^{-1} \\ &\leq C \frac{n}{a_n} (\Phi_n^{-1/2}(x) + \Phi_n^{-1/2}(x_{j,n})) \frac{1}{|m - j|}. \end{aligned} \tag{104}$$

Therefore, we have

$$\begin{aligned} &\sum_{j \neq m} U(x) \Phi_n^{(\lambda-1)/2}(x_{j,n}) |x - x_{j,n}|^{-(\lambda-1)} \\ &\leq C \left(\frac{n}{a_n} \right)^{\lambda-1} \sum_{j \neq m} (U(x) \Phi_n^{-(\lambda-1)/2}(x) \\ &\quad \times \Phi_n^{(\lambda-1)/2}(x_{j,n}) + U(x)) \\ &\quad \times \frac{1}{|m - j|^{\lambda-1}} \tag{105} \\ &\leq C \left(\frac{n}{a_n} \right)^{\lambda-1} \sum_{j \neq m} \frac{1}{|m - j|^{\lambda-1}} \\ &\leq C \left(\frac{n}{a_n} \right)^{\lambda-1} \begin{cases} 1, & 2 < \lambda; \\ \log n, & \lambda = 2; \\ n^{2-\lambda}, & 1 \leq \lambda < 2. \end{cases} \end{aligned}$$

Then, with (102) we see

$$\begin{aligned} &\left| U(x) \sum_2(x) \right| \\ &= \left| \frac{\sum_{j \neq m} U(x) \Phi_n^{(\lambda-1)/2}(x_{j,n}) |x - x_{j,n}|^{-(\lambda-1)}}{\sum_{j=1}^n \Phi_n^{(\lambda-1)/2}(x_{j,n}) |x - x_{j,n}|^{-\lambda}} \right| \\ &\leq C \left(\frac{a_n T^{1/3}(a_n)}{n^{2/3}} \right)^\lambda \left(\frac{n}{a_n} \right)^{\lambda-1} \end{aligned}$$

$$\begin{aligned} & \times \begin{cases} 1, & 2 < \lambda; \\ \log n, & \lambda = 2; \\ n^{2-\lambda}, & 1 < \lambda < 2; \\ \frac{n^{2-\lambda}}{\log n}, & \lambda = 1, \end{cases} \\ & \leq C \frac{a_n T^{\lambda/3}(a_n)}{n^{1-\lambda/3}} \begin{cases} 1, & 2 < \lambda; \\ \log n, & \lambda = 2; \\ n^{2-\lambda}, & 1 < \lambda < 2; \\ \frac{n^{2-\lambda}}{\log n}, & \lambda = 1. \end{cases} \end{aligned} \quad (106)$$

Hence, using μ_n in (94), we have that, for $3/2 < \lambda < 3$,

$$\left| U(x) \sum_2(x) \right| \leq C \mu_n. \quad (107)$$

Consequently, with $0 < \sum_1 \leq 1$ we have

$$U(x) |S_{n,\lambda}(f; x) - f(x)| \leq C \Omega(f; \mu_n). \quad (108)$$

□

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