

Research Article

Multiple Solutions for Generalized Asymptotical Linear Hamiltonian Systems Satisfying Bolza Boundary Conditions

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This paper is devoted to multiple solutions of generalized asymptotical linear Hamiltonian systems satisfying Bolza boundary conditions. We classify the linear Hamiltonian systems by the index theory and obtain the existence and multiplicity of solutions for the Hamiltonian systems, based on an application of the classical symmetric mountain pass lemma.

1. Introduction

This paper is concerned with a classical problem on the existence of solutions for Hamiltonian systems when a more general form of twist condition between the origin and infinity holds for the Hamiltonian function. More precisely consider the system

$$\begin{aligned} \dot{x} &= JH'(t, x), \\ Px(0) = 0 &= Px(1), \end{aligned} \quad (1)$$

where $H \in C(\mathbf{R} \times \mathbf{R}^{2n}, \mathbf{R})$, and $H' \in C(\mathbf{R} \times \mathbf{R}^{2n}, \mathbf{R}^{2n})$ and H' satisfies the following:

$$(H_1) \quad H'(t, x) = B_0(t, x)x + o(|x|), \text{ as } |x| \rightarrow 0 \text{ uniformly in } t \in [0, 1],$$

$$(H_2) \quad H'(t, x) = B_\infty(t, x)x + o(|x|), \text{ as } |x| \rightarrow \infty \text{ uniformly in } t \in [0, 1],$$

for $B_0(t, x), B_\infty \in L^\infty((0, 1) \times \mathbf{R}^{2n}; \text{GLs}(\mathbf{R}^{2n}))$. Here and below, we use H' to denote the first derivative of H with respect to $x \in \mathbf{R}^{2n}$.

A quantitative way to measure the twisting is given by the Maslov-type index. As in [1, 2], an index for the second-order and first-order linear Hamiltonian systems was defined and developed in [3] for the study of linear operator equation with infinite Morse index. In [4, 5], by Conley and Zehnder and

Long, an index theory for symplectic path was defined. We refer to two excellent books [6, 7] for systematical treatment.

In [2], Dong discussed the classification of the linear Hamiltonian systems with Bolza boundary conditions as follows:

$$\begin{aligned} \dot{x} &= JB(t)x, \\ Px(0) = 0 &= Px(1), \end{aligned} \quad (2)$$

where $B \in L^\infty((0, 1); \text{GLs}(\mathbf{R}^{2n}))$ and $Px = (x_1, \dots, x_n) \in \mathbf{R}^n$, for any $x = (x_1, \dots, x_{2n})$. That is, for any $B \in L^\infty((0, 1); \text{GLs}(\mathbf{R}^{2n}))$, he associated it with a pair of numbers $(i(B), \nu(B)) \in \mathbf{Z} \times \{0, 1, \dots, n\}$. This pair of integers is called index and nullity of B , respectively. And he defined the nullity $\nu(B)$ as the dimension of the solution space of (1). Let $E[\alpha]$ be the integer $a \in \mathbf{Z}$ such that $a < \alpha \leq a + 1$. So that $E[a] = a - 1$ for all integer a defined as in [6]. In order to process the definition of $i(B)$, he defined $i(\lambda I_{2n}) = nE[\lambda/\pi]$. In particular, when $\lambda \in (k\pi, (k+1)\pi)$, $i(\lambda I_{2n}) = nk$. We will introduce the definitions and properties of $i(B)$ in detail in Section 2. After the discussion of the index theory, we will prove our main result in Section 3.

Throughout this paper, $\|x\|_C$ denotes the usual norm in $C[0, 1]$. For any $A_1, A_2 \in \text{GLs}(\mathbf{R}^{2n})$, we write $A_1 \leq A_2$ if $A_2 - A_1$ is positively semidefinite, and we write $A_1 < A_2$ if $A_2 - A_1$ is positive definite. For any $A_1, A_2 \in L^\infty((0, 1); \text{GLs}(\mathbf{R}^{2n}))$, we write $A_1 \leq A_2$ if $A_1(t) \leq A_2(t)$ for a.e. $t \in (0, 1)$, and we write

$A_1 < A_2$ if $A_1 \leq A_2$ and $A_1(t) < A_2(t)$ on a subset of $(0, 1)$ with a nonzero measure.

Remark 1. Let $C_1, C_2 \in (k\pi, (k+1)\pi)$, $C_3, C_4 \in ((k+1)\pi, (k+2)\pi)$ with $C_1 < C_2, C_3 < C_4$. Assume that $A_i = C_i I_{2n}, i = 1, 2$, and $B_i = C_i I_{2n}, i = 3, 4$. Then, we have $i(A_1) = i(A_2) = nk$, $i(B_1) = i(B_2) = n(k+1)$, and $\nu(A_i) = \nu(B_i) = 0$, for $i = 1, 2$; (1) has at least n pairs of solutions.

We make use of the critical point theory [8, 9] to prove Theorem 8. The novelty of our result is that it suffices to assume that $H(t, x) + (1/2)\mu|x|^2$ is convex. The twisting between origin and infinity is reflected in (H_1^*) and (H_2^*) . Thus, our results complement with Theorem 3.8 in [2] and Theorem 1.1 in [10]. For other results, we refer to [11–15].

This paper is organized as follows. In Section 2, we introduce some preliminaries including index theory, and establish the μ -index theory which is needed in the proofs. In Section 3, we present the proofs of the results.

2. Index Theory for Linear Hamiltonian Systems Satisfying Bolza Boundary Value Conditions

First, we recall some definitions and propositions in [2]. We consider the following system:

$$\dot{x} = JB(t)x, \quad (3)$$

$$Px(0) = 0 = Px(1). \quad (4)$$

Let $H = \{x : [0, 1] \rightarrow \mathbf{R}^{2n} | x \text{ is continuous on } [0, 1], \text{ satisfies (4), and } \dot{x} \in L^2\}$ with the norm $\|x\|_H = (\int_0^1 (|\dot{x}'(t)|^2 + |x(t)|^2) dt)^{1/2}$, $(\Lambda_1 x)(t) = Jx'(t)$, $(\bar{B}x)(t) = B(t)x(t)$, where $L^2 := L^2((0, 1); \mathbf{R}^{2n})$, and let $|x| = (\sum_{i=1}^{2n} |x_i|)^{1/2}$ for any $x = (x_1, \dots, x_{2n}) \in \mathbf{R}^{2n}$. Then, Λ_1 and \bar{B} are self-adjoint operators, and \bar{B} is bounded.

Definition 2 (see [2, Definitions 2.1, 2.3, 2.4, and 2.7]). For any $B_1, B_2, B \in L^\infty((0, 1), \text{GLs}(\mathbf{R}^{2n}))$, one defines that

$$(1) \nu(B) = \dim \ker(\Lambda_1 + \bar{B}),$$

$$(2) I(B_1, B_2) = \sum_{\lambda \in [0, 1]} \nu((1-\lambda)B_1 + \lambda B_2) \text{ for } B_1 < B_2,$$

$$(3) i(B) = i(\lambda I_{2n}) - I(B, \lambda I_{2n}) \text{ for } \lambda \in \mathbf{R} \text{ satisfies } B < \lambda I_{2n}.$$

Using spectral theory, a Morse-type index $i_\mu(B)$ was established in [2]. More precisely, for any $B \in L^\infty((0, 1), \text{GLs}(\mathbf{R}^{2n}))$, let $\mu \in \mathbf{R} \setminus \pi\mathbf{Z}$ with $B + \mu I_{2n} \geq I_{2n}$. Then, $\nu(-\mu I_{2n}) = 0$, $\Lambda x := J\dot{x}(t) - \mu x(t)$ is invertible, and the

inverse $\Lambda^{-1} : L^2 \rightarrow L^2$ is self-adjoint and compact. He put a quadratic form:

$$q_{\mu, B}(u, u) = \frac{1}{2} \int_0^1 (\Lambda^{-1} u(t), u(t)) + (C_\mu(t) u(t), u(t)) dt \quad \forall u \in L^2, \quad (5)$$

$$(C_\mu(t) u(t), u(t)) := \int_0^1 (C_\mu(t) u(t), u(t)), \quad (6)$$

where $C_\mu(t) := (\mu I_{2n} + B(t))^{-1}$. Then, $(\bar{C}_\mu(t) u(t), u(t))$ defines a Hilbert space structure on L^2 . $C_\mu^{-1} \Lambda^{-1}$ is a self-adjoint and compact operator under this interior product. By the spectral theory, there is a basis $e_j, j \in \mathbf{N}$ of L^2 , and a sequence $\lambda_j \rightarrow 0$ in \mathbf{R} such that

$$(C_\mu(t) e_i(t), e_j(t)) = \delta_{ij}, \quad (7)$$

$$(\Lambda^{-1} e_j, u) = (C_\mu \lambda_j e_j, u) \quad \forall u \in L^2.$$

For any $u \in L^2$ as $u = \sum_{j=1}^\infty \xi_j e_j$, (5) can be rewritten as follows:

$$q_{\mu, B}(u, u) = \frac{1}{2} \int_0^1 (\Lambda^{-1} u, u) + (C_\mu(t) u, u) dt = \frac{1}{2} \sum_{j=1}^\infty (1 + \lambda_j) \xi_j^2. \quad (8)$$

Define that

$$E_\mu^-(B) := \left\{ \sum_{j=1}^\infty \xi_j e_j \mid \xi_j = 0 \text{ if } 1 + \lambda_j \geq 0 \right\},$$

$$E_\mu^0(B) := \left\{ \sum_{j=1}^\infty \xi_j e_j \mid \xi_j = 0 \text{ if } 1 + \lambda_j \neq 0 \right\}, \quad (9)$$

$$E_\mu^+(B) := \left\{ \sum_{j=1}^\infty \xi_j e_j \mid \xi_j = 0 \text{ if } 1 + \lambda_j \leq 0 \right\}.$$

$E_\mu^-(B), E_\mu^0(B)$, and $E_\mu^+(B)$ are $q_{\mu, B}$ -orthogonal, and $E_\mu^-(B) \oplus E_\mu^0(B) \oplus E_\mu^+(B) = L^2$. Since $\lambda_j \rightarrow 0$ as $j \rightarrow \infty$, $E_\mu^-(B)$ and $E_\mu^0(B)$ are two finite-dimensional subspaces.

Definition 3 (see [2, Definition 2.9]). For any $B \in L^\infty((0, 1), \text{GLs}(\mathbf{R}^{2n}))$, $\mu \in \mathbf{R}$ with $\mu I_{2n} + B \geq I_{2n}$, one defines that

$$\nu_\mu(B) := \dim E_\mu^0(B), \quad i_\mu(B) := \dim E_\mu^-(B). \quad (10)$$

One calls $\nu_\mu(B)$ and $i_\mu(B)$ μ -nullity and μ -index of B , respectively.

Proposition 4 (see [2, Propositions 2.10, 2.13]). *The index, relative Morse index, and μ -index have the following properties.*

- (1) For any $B_1, B_2, B \in L^\infty((0, 1), GLs(\mathbf{R}^{2n}))$, with $B_1 < B_2$, one has

$$\begin{aligned} \nu_\mu(B) &= \nu(B), & I(B_1, B_2) &= i_\mu(B_2) - i_\mu(B_1), \\ i_\mu(B_1) - i_\mu(B_2) &= i(B_1) - i(B_2). \end{aligned} \tag{11}$$

- (2) There exists $\epsilon_0 > 0$, such that, for any $\epsilon \in (0, \epsilon_0]$, one gets

$$\begin{aligned} \nu(B + \epsilon I_{2n}) &= 0 = \nu(B - \epsilon I_{2n}), \\ i(B - \epsilon I_{2n}) &= i(B), & i(B + \epsilon I_{2n}) &= i(B) + \nu(B). \end{aligned} \tag{12}$$

In particular, if $\nu(B) = 0$, one obtains $i(B + \epsilon I_{2n}) = i(B)$ for $\epsilon \in (0, \epsilon_0]$.

- (3) $i_\mu(B) - i(B)$ is a constant for B satisfying $B + \mu I_{2n} \geq I_{2n}$, that is, $i_\mu(B) - i(B) = i_\mu(B_1) - i(B_1)$ for any other $B_1 \in L^\infty((0, 1), GLs(\mathbf{R}^{2n}))$ with $B_1 + \mu I_{2n} \geq I_{2n}$. For any $\mu > 1$, one has

$$\begin{aligned} i_\mu(0) &= nE \left[\frac{\mu}{\pi} \right], \\ i_\mu(B) &= nE \left[\frac{\mu}{\pi} \right] + n + i(B), \end{aligned} \tag{13}$$

for any $B \in L^\infty((0, 1), GLs(\mathbf{R}^{2n}))$ with $B + \mu I_{2n} \geq I_{2n}$.

Remark 5. From this proposition, there is $\epsilon_0 > 0$, $\epsilon \in (0, \epsilon_0]$ such that $\nu_\mu(B + \epsilon I_{2n}) = 0 = \nu_\mu(B - \epsilon I_{2n})$ and $i_\mu(B - \epsilon I_{2n}) = i_\mu(B)$, $i_\mu(B + \epsilon I_{2n}) = i_\mu(B) + \nu_\mu(B)$.

In order to prove Theorem 8, we make use of minimax arguments for the multiplicity of critical points in the presence of symmetry. We state two results of this type from [8, 9].

Lemma 6 (cf. Chang [9, Theorem 4.3.4]). *Assume that $f \in C^1(X, \mathbf{R}^1)$ satisfies the (PS) condition, $f(\theta) = 0$, $f(-x) = f(x)$, and*

- (1) *there are an m -dimensional subspace X_1 and a constant $\rho > 0$ such that*

$$\sup_{x \in X_1 \cap S_\rho} f(x) < 0, \tag{14}$$

- (2) *there is a j -dimensional subspace X_2 such that*

$$\inf_{x \in X_2^\perp} f(x) > -\infty, \tag{15}$$

where X_2^\perp is a subset of X such that $X_2^\perp \oplus X_2 = X$. Then, f has at least $m - j$ pairs of critical points if $m - j > 0$.

Lemma 7 (see [8, Theorem 2.2.8]). *Suppose that $f \in C^1(H, \mathbf{R})$ satisfies (C) condition, $f(\theta) \geq 0$, even, and there exist*

two closed subspaces H^+H^- of H , with $\text{codim } H^- < +\infty$, and two constants $c_\infty > c_0 \geq f(0)$ such that

- (a) $f(u) \geq c_0$ for all $u \in S_\rho \cap H^+$,
 (b) $f(u) \leq c_\infty$ for all $u \in H^-$.

Then, if $\dim H^- \geq \text{codim } H^+$, f possesses at least $m = \dim H^- - \text{codim } H^+$ distinct pairs of critical points whose corresponding critical values belong to $[c_0, c_\infty]$.

3. Proof of Main Results

We state the main result in this paper. We further make the following assumptions.

- (H_1^*) There exists $A_1, A_2 \in L^\infty((0, 1); GLs(\mathbf{R}^{2n}))$ such that $A_1(t) \leq B_0(t, x) \leq A_2(t)$ with $i(A_1) = i(A_2)$, $\nu(A_2) = 0$.
 (H_2^*) There exists $B_1, B_2 \in L^\infty((0, 1); GLs(\mathbf{R}^{2n}))$ such that $B_1(t) \leq B_\infty(t, x) \leq B_2(t)$ with $i(B_1) = i(B_2)$, $\nu(B_2) = 0$.
 (H_3) $H(t, \theta) = 0$, $H(t, x) = H(t, -x)$.
 (H_4) $H(t, x) + (1/2)\mu|x|^2$ is convex.

Our main result reads as follows.

Theorem 8. *Assume that (H_1), (H_1^*), (H_2), (H_2^*), (H_3), (H_4) are satisfied, then (1) has at least $|i(A_1) - i(B_1)|$ pairs of solutions.*

Theorem 9. *If $i(A_1) > i(B_1)$, then (1) has at least $i(A_1) - i(B_1)$ pairs of solutions.*

If $i(A_1) > i(B_1)$, let $\mu \in \mathbf{R} \setminus \pi\mathbf{Z}$ with $A_1 + \mu I_{2n} \geq I_{2n}$, $B_1 + \mu I_{2n} \geq I_{2n}$. Recalling that $\Lambda x = J\dot{x} - \mu I_{2n}$, one denotes $H_\mu(t, x) = H(t, x) + (1/2)\mu|x|^2$. Then, one obtains

$$H_\mu^*(t, x) = \sup_{y \in \mathbf{R}^{2n}} \{(x, y) - H_\mu(t, y)\}. \tag{16}$$

Thus, $H_\mu^*(t, -x) = H^*(t, x)$ and

$$x = H_\mu^{*'}(t, x^*) \quad \text{iff } x^* = H_\mu'(t, x) \tag{17}$$

(by the Fenchel conjugate formula; see [6, Proposition II]). Hence, $|x| \rightarrow \infty$ if and only if $|x^*| \rightarrow \infty$. Consider the functional defined by

$$\begin{aligned} \psi(u) &= \int_0^1 \left(\frac{1}{2} \Lambda^{-1} u(t), u(t) \right) + H_\mu^*(t, u(t)) dt \\ &\quad \forall u \in L^2. \end{aligned} \tag{18}$$

In order to prove Theorem 9, we need the following lemmas.

Lemma 10. *ψ satisfies the (PS) condition.*

Proof. Assume that $\{u_j\}$ is a sequence in L^2 such that $\psi(u_j)$ is bounded and $\psi'(u_j) \rightarrow 0$ in L^2 . It suffices to prove that $\{u_j\}$ has convergent sequence. By (18), we have

$$(\psi'(u), v) = \int_0^1 (\Lambda^{-1}u(t), v(t)) + (H_\mu^{*'}(t, u(t)), v(t)) dt, \quad (19)$$

for all $v \in L^2$. Hence, we get

$$\psi'(u_j) = \Lambda^{-1}u_j + H_\mu^{*'}(t, u_j(t)) \rightarrow \theta, \quad \text{in } L^2. \quad (20)$$

Let $x_j = \Lambda^{-1}u_j$, $y_j = u_j/\|x_j\|_C$, and $z_j = \psi'(u_j) - \Lambda^{-1}(u_j)$. If $\|x_j\|_C \rightarrow \infty$, then $\|x_j\|_{L^2} \rightarrow \infty$. Joining (H_2) with (17) and (20), we have

$$\begin{aligned} u_j(t) &= H_\mu'(t, z_j(t)) \\ &= (B_\infty(t, z_j(t)) + \mu I_{2n}) z_j(t) \\ &\quad + o(|z_j(t)|) \\ &= -(B_\infty(t, z_j(t)) + \mu I_{2n}) \Lambda^{-1}u_j \\ &\quad + (B_\infty(t, z_j(t)) + \mu I_{2n}) \psi'(u_j) \\ &\quad + o(|z_j(t)|). \end{aligned} \quad (21)$$

Using the preceding notations, we have

$$\begin{aligned} y_j(t) &= -(B_\infty(t, z_j(t)) + \mu I_{2n}) \Lambda^{-1}y_j \\ &\quad + (B_\infty(t, z_j(t)) + \mu I_{2n}) \psi'(u_j) \|x_j\|_C^{-1} \\ &\quad + o(|z_j(t)|) \|x_j\|_C^{-1}. \end{aligned} \quad (22)$$

So $y_j(t)$ is bounded in L^2 . We assume that $y_j \rightarrow \bar{y}$ in L^2 , and hence $\Lambda^{-1}y_j \rightarrow \Lambda^{-1}\bar{y}$. From (22), there exists $\bar{B} \in L^\infty((0, 1); \text{GLs}(\mathbf{R}^{2n}))$, such that

$$B_\infty(t, z_j(t)) v(t) \rightarrow \bar{B}(t) v(t) \quad \text{for any } v \in L^2. \quad (23)$$

And we have $B_1(t) \leq \bar{B}(t) \leq B_2(t)$. Taking the limit as $j \rightarrow \infty$ in (22), we obtain $\bar{y}(t) = -(\bar{B}(t) + \mu I_{2n}) \Lambda^{-1}\bar{y}$. Let $\bar{x} = \Lambda^{-1}\bar{y}$. We get

$$J\bar{x} + \bar{B}(t)\bar{x} = 0, \quad P\bar{x}(0) = 0 = P\bar{x}(1). \quad (24)$$

By assumption (H_2^*) , we have $i(\bar{B}) = 0$ and $\bar{x} = 0$, which is impossible. Since $\|\bar{x}\|_C = 1$, $\|\Lambda^{-1}u_j\|_C$ is bounded. From (17) and (20), $\|u_j\|_{L^2}$ is bounded. Assume that $u_j \rightarrow u_0$ in L^2 ; then $\Lambda^{-1}u_j \rightarrow \Lambda^{-1}u_0$. Let $\xi_j := \Lambda^{-1}u_j + H_\mu^{*'}(t, u_j)$; then $H_\mu^{*'}(t, u_j) = \xi_j - \Lambda^{-1}u_j \rightarrow -\Lambda^{-1}u_0$. Fenchel conjugate formula gives $u_j = H_\mu^{*'}(t, \xi_j - \Lambda^{-1}u_j) \rightarrow H_\mu^{*'}(t, -\Lambda^{-1}u_0)$ in L^2 (by [6, Proposition II, Theorem 4]). \square

Lemma 11. *The assumption (1) of Lemma 6 is valid, where f is defined as in (18).*

Proof. From (H_1) and (H_1^*) , we have that, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$((A_1(t) - \epsilon)x, x) \leq (H'(t, x), x) \leq ((A_2(t) + \epsilon)x, x), \quad \text{for any } |x| \leq \delta. \quad (25)$$

Thus,

$$\begin{aligned} H_\mu(t, y) &= \left(\int_0^1 H_\mu'(t, \theta y) d\theta, y \right) \\ &\geq \frac{1}{2} (((A_1(t) - \epsilon I_{2n}) + \mu I_{2n}) y, y), \end{aligned} \quad (26)$$

for any $|y| \leq \delta$.

Let $g(y) = (x, y) - H_\mu(t, y)$. By (16) and (H_4) , $g(y)$ is strictly concave. Then, $|y| \rightarrow \infty$ as $g(y) \rightarrow -\infty$. So, g achieves its maximum at a unique point. Because of $g'(y) = x - H_\mu'(t, y)$, we get that

$$x = H_\mu'(t, y) \quad \text{has a unique solution.} \quad (27)$$

And we can easily get $0 = H_\mu'(t, 0)$. Hence, we have $|y| \rightarrow 0$ when $|x| \rightarrow 0$. Otherwise, there exists $\epsilon_0 > 0$, $x_n, y_n \in \mathbf{R}^{2n}$, $t_n \in [0, 1]$ s.t. $|x_n| \rightarrow 0$. But $|y_n| \geq \epsilon_0$, $x_n = H_\mu'(t_n, y_n)$. Hence, $|y_n|$ is bounded, and there exists y_0, t_0 such that $y_n \rightarrow y_0$, $t_n \rightarrow t_0$, $x_n \rightarrow 0$. Thus, $0 = H_\mu'(t_0, y_0)$. This contradicts the uniqueness, which yields there exists $\rho(\rho < \delta)$ such that, for any $|x| \leq \rho$, we have $|y| \leq \delta$. So, for any $|x| \leq \rho$, we have

$$\begin{aligned} H_\mu^*(t, x) &= \sup_{y \in \mathbf{R}^{2n}} \{(x, y) - H_\mu(t, y)\} \\ &= \sup_{|y| \leq \delta} \{(x, y) - H_\mu(t, y)\} \\ &\leq \sup_{|y| \leq \delta} \left\{ (x, y) - \frac{1}{2} (((A_1(t) - \epsilon I_{2n}) + \mu I_{2n}) y, y) \right\} \\ &= \frac{1}{2} (((A_1(t) - \epsilon I_{2n}) + \mu I_{2n})^{-1} x, x). \end{aligned} \quad (28)$$

By (18),

$$\begin{aligned} \psi(u) &\leq \int_0^1 \left(\frac{1}{2} \Lambda^{-1}u(t), u(t) \right) \\ &\quad + \frac{1}{2} (((A_1(t) - \epsilon I_{2n}) + \mu I_{2n})^{-1} u(t), u(t)) dt \\ &\quad \text{for any } |u| \leq \rho, \end{aligned} \quad (29)$$

that is,

$$\psi(u) \leq q_{\mu, (A_1 - \epsilon)}(u, u) \quad \text{for any } |u| \leq \rho. \quad (30)$$

Let $X_1 = E_\mu^-(A_1 - \epsilon)$. Then, $\dim X_1 = i_\mu(A_1 - \epsilon)$. For any $u \in X_1 \cap S_\rho$, we have $\sup \psi(u) < 0$. By Proposition 4, for ϵ being small enough, $i_\mu(A_1 - \epsilon) = i_\mu(A_1)$. \square

Lemma 12. *The assumption (2) of Lemma 6 is also satisfied.*

Proof. From (H_2) , let $h_2(t, x) = H^1(t, x) - B_{\infty}(t, x)x$. Then, for any $\epsilon > 0$, there exists a constant M_1 such that

$$|h_2(t, x)| \leq \epsilon |x| + M_1, \quad \forall (t, x) \in [0, 1] \times \mathbf{R}^{2n}. \quad (31)$$

Combining (H_2) and (31), there exists a constant M_2 such that

$$H_{\mu}(t, x) \leq \frac{1}{2} \left((B_2(t) + 4\epsilon I_{2n} + \mu I_{2n}) x, x \right) + M_2, \quad (32)$$

for all $(t, x) \in [0, 1] \times \mathbf{R}^{2n}$. By the definition of $H_{\mu}^*(t, x)$, we have

$$\begin{aligned} H_{\mu}^*(t, x) &\geq \sup_{y \in \mathbf{R}^{2n}} \left\{ (x, y) - \frac{1}{2} \left((B_2(t) + 4\epsilon I_{2n} + \mu I_{2n}) y, y \right) \right\} - M_2 \\ &= \frac{1}{2} \left(\left((B_2(t) + 4\epsilon I_{2n}) + \mu I_{2n} \right)^{-1} x, x \right) - M_2. \end{aligned} \quad (33)$$

Thus,

$$\begin{aligned} \psi(u) &\geq \int_0^1 \left(\frac{1}{2} \Lambda^{-1} u(t), u(t) \right) \\ &\quad + \frac{1}{2} \left((B_2(t) + 4\epsilon I_{2n}) + \mu I_{2n} \right)^{-1} (u(t), u(t)) dt \\ &\quad - M_2. \end{aligned} \quad (34)$$

So, we infer

$$\psi(u) \geq q_{\mu, (B_2 + 4\epsilon)}(u, u) - M_2. \quad (35)$$

Let $X_2 = E_{\mu}^-(B_2 + 4\epsilon)$. We have $\dim X_2 = i_{\mu}(B_2 + 4\epsilon)$ and $X_2^{\perp} = E_{\mu}^0(B_2(t) + 4\epsilon) \oplus E_{\mu}^+(B_2(t) + 4\epsilon)$. So, we have

$$\inf_{X_2^{\perp}} \psi(u) \geq -M_2. \quad (36)$$

By Proposition 4, we have $i_{\mu}(A_1) - i_{\mu}(B_2) = i(A_1) - i(B_2)$, and we can also let ϵ be small enough such that $i_{\mu}(B_2 + 4\epsilon) = i_{\mu}(B_2)$. This completes the proof. \square

Theorem 13. *If $i(A_1) < i(B_1)$, then (2) has at least $i(B_1) - i(A_1)$ pairs of solutions.*

The argument in Theorem 8 can also be used here by using Lemma 7. More precisely, if $H^+ = E_{\mu}^+(A_2 + \epsilon)$, then $\text{codim } H^+ = i(A_1)$ and $\psi(u) > c_0 > 0$ on $H^+ \cap S_{\rho}$. If $H^- = E_{\mu}^-(B_1 - 4\epsilon)$, then $\dim H^- = i(B_1)$ and $\psi(u) \leq c_{\infty}$ on H^- .

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