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Research Article

A Two-Parametric Class of Merit Functions for the Second-Order Cone Complementarity Problem

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We propose a two-parametric class of merit functions for the second-order cone complementarity problem (SOCCP) based on the one-parametric class of complementarity functions. By the new class of merit functions, the SOCCP can be reformulated as an unconstrained minimization problem. The new class of merit functions is shown to possess some favorable properties. In particular, it provides a global error bound if F and G have the joint uniform Cartesian P-property. And it has bounded level sets under a weaker condition than the most available conditions. Some preliminary numerical results for solving the SOCCPs show the effectiveness of the merit function method via the new class of merit functions.

1. Introduction

We consider the following second-order cone complementarity problem (SOCCP) of finding $(x, y, \zeta) \in R^n \times R^n \times R^n$ such that

$$\begin{gathered} \langle x, y\rangle = 0, \quad x \in K, \ y \in K, \\ x = F(\zeta), \qquad y = G(\zeta), \end{gathered} \tag{1}$$

where $\langle\cdot,\cdot\rangle$ is the Euclidean inner product and $F : R^n \to R^n$ and $G : R^n \to R^n$ are continuously differentiable mappings. Here $K \subset R^n$ is the Cartesian product of second-order cones (SOC); that is, $K = K^{n_1} \times K^{n_2} \times \cdots \times K^{n_m}$ with $m, n_1, \ldots, n_m \geq 1$, $n = n_1 + n_2 + \cdots + n_m$, and the n_i-dimensional SOC defined by

$$K^{n_i} := \left\{x_i = (x_{i1}; x_{i2}) \in R \times R^{n_i-1} : x_{i1} - \|x_{i2}\| \geq 0\right\}, \tag{2}$$

with $\|\cdot\|$ denoting the Euclidean norm.

Recently great attention has been paid to the SOCCP, since it has a variety of engineering and management applications, such as filter design, antenna array weight design, truss design, and grasping force optimization in robotics [1, 2]. Furthermore, the SOCCP contains a wide class of problems, such as nonlinear complementarity problems (NCP) and second-order cone programming (SOCP) [3, 4]. For example, the SOCCP with $n_1 = n_2 = \cdots = n_m = 1$ and $G(\zeta) = \zeta$ for any $\zeta \in R^n$ is the NCP, and the KKT conditions for the SOCP reduce to the SOCCP.

There have been various methods for solving SOCCPs [5], such as interior point methods [6–8], (noninterior continuation) smoothing Newton methods [4, 9–12], and smoothing-regularization methods [13]. Recently, there is an alternative approach [14, 15] based on reformulating the SOCCP as an unconstrained smooth minimization problem. In that approach, it aims to find a smooth function $\psi : R^n \times R^n \to R_+$ such that

$$\psi(x, y) = 0 \Longleftrightarrow \langle x, y\rangle = 0, \quad x \in K, \ y \in K. \tag{3}$$

Such a ψ is called a merit function for the SOCCP. Thus the SOCCP is equivalent to the following unconstrained smooth (global) minimization problem:

$$\min_{\zeta \in R^n} \psi(F(\zeta), G(\zeta)). \tag{4}$$

A popular choice of ψ is the Fischer-Burmeister (FB) merit function

$$\psi_{\mathrm{FB}}(x,y) := \frac{1}{2}\left\|\phi_{\mathrm{FB}}(x,y)\right\|^2, \quad (5)$$

where $\phi_{\mathrm{FB}} : R^n \times R^n \to R^n$ is the vector-valued FB function defined by

$$\phi_{\mathrm{FB}}(x,y) := \sqrt{x^2+y^2} - x - y, \quad (6)$$

with $x^2 = x \circ x$ denoting the Jordan product between x and itself and $\sqrt{x}$ being a vector such that $(\sqrt{x})^2 = x$. The function ψ_{FB} is shown to be a merit function for the SOCCP in [14].

In this paper, we consider the two-parametric class of merit functions defined by

$$\psi_{\tau_1,\tau_2}(x,y) := \tau_1\psi_0(x,y) + \psi_{\tau_2}(x,y), \quad (7)$$

where $\psi_0 : R^n \times R^n \to R_+$ and $\psi_\tau : R^n \times R^n \to R_+$ are given, respectively, by

$$\psi_0(x,y) := \frac{1}{2}\left\|(x\circ y)_+\right\|^2, \quad (8)$$

$$\psi_\tau(x,y) := \frac{1}{2}\left\|\phi_\tau(x,y)_+\right\|^2, \quad (9)$$

with $(\cdot)_+$ denoting the metric projection on the second-order cone K, $\tau_1 > 0$ and $\tau_2 \in (0,4)$. Here $\phi_\tau : R^n \to R^n$ is the one-parametric class of SOC complementarity functions [16] defined by

$$\phi_\tau(x,y) := \sqrt{x^2+y^2+(\tau-2)(x\circ y)} - x - y, \quad (10)$$

where $\tau \in (0,4)$ is an arbitrary but fixed parameter. When $\tau = 2$, ϕ_τ reduces to the vector-valued FB function given by (6), and as $\tau \to 0$, it becomes a multiple of the vector-valued residual function

$$\phi_{\mathrm{NR}}(x,y) := x - (x-y)_+. \quad (11)$$

Thus, the one-parametric class of vector-valued functions (10) covers two popular second-order cone complementarity functions. Hence the two-parametric class of merit functions defined as (7)–(10) includes a broad class of merit functions.

We will show that the SOCCP can be reformulated as the following unconstrained smooth (global) minimization problem:

$$\min_{\zeta\in R^n} f(\zeta) := \psi_{\tau_1,\tau_2}(F(\zeta), G(\zeta)). \quad (12)$$

If $\tau_1 = 1$ and $\tau_2 = 2$, the function f in (12) induced by the new class of merit functions ψ_{τ_1,τ_2} reduces to [17]

$$\widehat{f_{\mathrm{LT}}}(\zeta) := \psi_0(F(\zeta), G(\zeta)) + \frac{1}{2}\left\|\phi_{\mathrm{FB}}(F(\zeta), G(\zeta))_+\right\|^2, \quad (13)$$

with ψ_0 given as (8). It has been shown that $\widehat{f_{\mathrm{LT}}}$ provides a global error bound if F and G are jointly strongly monotone, and it has bounded level sets if F and G are jointly monotone and a strictly feasible solution exists [17]. In contrast, the merit function ψ_{FB} lacks these properties.

Motivated by these works, we aim to study the two-parametric class of merit functions for the SOCCP defined as (7)–(10) and its favorable properties in this paper. We also prove that the class of merit functions provides a global error bound if F and G have the joint uniform Cartesian P-property, which will play an important role in analyzing the convergence rate of some iterative methods for solving the SOCCP. And it has bounded level sets under a rather weak condition, which ensures that the sequence generated by a descent method has at least one accumulation point.

The organization of this paper is as follows. In Section 2, we review some preliminaries including the Euclidean Jordan algebra associated with SOC and some results about the one-parametric class of SOC complementarity functions. In Section 3, based on the one-parametric class of SOC complementarity functions, we propose a two-parametric class of merit functions for the second-order cone complementarity problem (SOCCP), which is shown to possess some favorable properties. In Section 4, we show that the class of merit functions provides a global error bound if F and G have the joint uniform Cartesian P-property, and it has bounded level sets under a rather weak condition. Some preliminary numerical results are reported in Section 5. And we close this paper with some conclusions in Section 6.

In what follows, we denote the nonnegative orthant of R by R_+. We use the symbol $\|\cdot\|$ to denote the Euclidean norm defined by $\|x\| := \sqrt{x^Tx}$ for a vector x or the corresponding induced matrix norm. For simplicity, we often use $x = (x_1; x_2)$ for the column vector $x = (x_1, x_2^T)^T$. For the SOC K^n, $\mathrm{int}\,K^n$, and $\mathrm{bd}\,K^n$ mean the topological interior and the boundary of K^n, respectively.

2. Preliminaries

In this section, we recall some preliminaries, which include Euclidean Jordan algebra [3, 18] associated with the SOC K and some results used in the subsequent analysis.

Without loss of generality, we may assume that $m = 1$ and $K = K^n$ in Sections 2 and 3.

First, we recall the Euclidean Jordan algebra associated with the SOC and some useful definitions. The Euclidean Jordan algebra for the SOC K^n is the algebra defined by

$$x \circ y = \left(x^Ty; x_1y_2 + y_1x_2\right), \quad \forall x, y \in R^n, \quad (14)$$

with $e = (1, 0, \ldots, 0) \in R^n$ being its unit element. Given an element $x = (x_1; x_2) \in R \times R^{n-1}$, we define

$$L(x) = \begin{pmatrix} x_1 & x_2^T \\ x_2 & x_1 I \end{pmatrix}, \quad (15)$$

where I represents the $(n-1)\times(n-1)$ identity matrix. It is easy to verify that $x \circ y = L(x)y$ for any $y \in R^n$. Moreover, $L(x)$ is symmetric positive definite (and hence invertible) if and only if $x \in \mathrm{int}\,K^n$.

Now we give the spectral factorization of vectors in R^n associated with the SOC K^n. Let $x = (x_1; x_2) \in R \times R^{n-1}$. Then x can be decomposed as

$$x = \lambda_1 u^{(1)} + \lambda_2 u^{(2)}, \tag{16}$$

where λ_1, λ_2 and $u^{(1)}$, $u^{(2)}$ are the spectral values and the associated spectral vectors of x given by

$$\lambda_i = x_1 + (-1)^i \|x_2\|,$$
$$u^{(i)} = \begin{cases} \frac{1}{2}\left(1; (-1)^i \frac{x_2}{\|x_2\|}\right), & \text{if } x_2 \neq 0, \\ \frac{1}{2}\left(1; (-1)^i \omega\right), & \text{otherwise}, \end{cases} \tag{17}$$

for $i = 1, 2$, with any $\omega \in R^{n-1}$ such that $\|\omega\| = 1$. It is obvious that $\|u^{(1)}\| = \|u^{(2)}\| = 1/\sqrt{2}$. By the spectral factorization, a scalar function can be extended to a function for the SOC. For any $x \in R^n$, we define

$$x^2 = \lambda_1^2 u^{(1)} + \lambda_2^2 u^{(2)}. \tag{18}$$

Since both eigenvalues of any $x \in K^n$ are nonnegative, we define

$$\sqrt{x} = \sqrt{\lambda_1} u^{(1)} + \sqrt{\lambda_2} u^{(2)}. \tag{19}$$

Lemma 1 (see [14]). *Let C be any closed convex cone in R^n. For each $x \in R^n$, let x_C^+ and x_C^- denote the nearest point (in the Euclidean norm) projection of x onto C and $-C^*$, respectively. The following results hold.*

(i) *For any $x \in R^n$, we have $x = x_C^+ + x_C^-$ and $\|x\|^2 = \|x_C^+\|^2 + \|x_C^-\|^2$.*

(ii) *For any $x \in R^n$ and $y \in C$, we have $\langle x, y\rangle \leq \langle x_C^+, y\rangle$.*

Lemma 2 (see [19]). *Let $x = (x_1; x_2) \in R \times R^{n-1}$ and $y = (y_1; y_2) \in R \times R^{n-1}$. Then we have*

$$\langle x, y\rangle \leq \sqrt{2}\,\|(x \circ y)_+\|. \tag{20}$$

The following results, describing the special properties of the function ϕ_τ given as (10), will play an important role in the subsequent analysis.

Lemma 3 (see [16]). *For any $x = (x_1; x_2)$, $y = (y_1; y_2) \in R \times R^{n-1}$, if $z = (z_1; z_2) = x^2 + y^2 + (\tau - 2)(x \circ y) \notin \operatorname{int} K^n$, then*

$$\begin{aligned} x_1^2 &= \|x_2\|^2, \qquad y_1^2 = \|y_2\|^2, \\ x_1 y_1 &= x_2^T y_2, \qquad x_1 y_2 = y_1 x_2, \\ x_1^2 + y_1^2 &+ (\tau - 2) x_1 y_1 \\ &= \|x_1 x_2 + y_1 y_2 + (\tau - 2) x_1 y_2\| \\ &= \|x_2\|^2 + \|y_2\|^2 + (\tau - 2) x_2^T y_2. \end{aligned} \tag{21}$$

If, in addition, $(x, y) \neq (0, 0)$, then $z_2 \neq 0$, and furthermore,

$$\begin{aligned} x_2^T \frac{z_2}{\|z_2\|} &= x_1, \qquad x_1 \frac{z_2}{\|z_2\|} = x_2, \\ y_2^T \frac{z_2}{\|z_2\|} &= y_1, \qquad y_1 \frac{z_2}{\|z_2\|} = y_2. \end{aligned} \tag{22}$$

Lemma 4 (see [20]). *For any $x, y \in R^n$, let ϕ_τ be defined as in (10). Then,*

$$2\|\phi_\tau(x, y)\|^2 \geq 2\|\phi_\tau(x, y)_+\|^2 \geq \frac{4 - \tau}{2}\left[\|(-x)_+\|^2 + \|(-y)_+\|^2\right]. \tag{23}$$

3. A Two-Parametric Class of Merit Functions

In this section, we study the two-parametric class of merit functions ψ_{τ_1,τ_2} given by (7)–(10). As we will see, ψ_{τ_1,τ_2} has some favorable properties. The most important property is that the SOCCP can be reformulated as the global minimization of the function $f(\zeta)$ given as (12). Moreover, the function f provides a global error bound and bounded level sets under weak conditions, which will be shown in the next section.

Proposition 5. *Let ψ_τ be given by (9). Then,*

$$\begin{aligned} &\psi_\tau(x, y) = 0, \\ &\langle x, y\rangle \leq 0 \Longleftrightarrow \langle x, y\rangle = 0, \quad x \in K^n, \ y \in K^n. \end{aligned} \tag{24}$$

Proof. Suppose $x \in K^n$, $y \in K^n$, and $\langle x, y\rangle = 0$. Thus by Proposition 3.1 [16], we have $\phi_\tau(x, y) = 0$ and therefore $\psi_\tau(x, y) = (1/2)\|\phi_\tau(x, y)_+\|^2 = 0$, $\langle x, y\rangle \leq 0$. Conversely, we assume $\psi_\tau(x, y) = 0$ and $\langle x, y\rangle \leq 0$. Then $\phi_\tau(x, y)_+ = 0$ implies $\phi_\tau := \phi_\tau(x, y) \in -K^n$. From (10), we obtain

$$x + y = \sqrt{x^2 + y^2 + (\tau - 2)(x \circ y)} - \phi_\tau. \tag{25}$$

Squaring both sides yields

$$\begin{aligned} &(4 - \tau)(x \circ y) \\ &\quad = -2\left(\sqrt{x^2 + y^2 + (\tau - 2)(x \circ y)} \circ \phi_\tau\right) + (\phi_\tau)^2. \end{aligned} \tag{26}$$

Taking the trace of both sides and using the fact $\operatorname{tr}(x \circ y) = 2\langle x, y\rangle$, we have

$$\begin{aligned} &2(4 - \tau)\langle x, y\rangle \\ &\quad = -4\left\langle \sqrt{x^2 + y^2 + (\tau - 2)(x \circ y)}, \phi_\tau \right\rangle + 2\|\phi_\tau\|^2. \end{aligned} \tag{27}$$

Since $\sqrt{x^2 + y^2 + (\tau - 2)(x \circ y)} \in K^n$ and $\phi_\tau \in -K^n$, we obtain

$$-4\left\langle \sqrt{x^2 + y^2 + (\tau - 2)(x \circ y)}, \phi_\tau \right\rangle \geq 0, \tag{28}$$

and thus the right hand side of (27) is nonnegative. Then by the assumption $\langle x, y\rangle \leq 0$, we have $\langle x, y\rangle = 0$. This together with (27) implies $\phi_\tau(x, y) = 0$. Therefore, it follows from Proposition 3.1 [16] that $x \in K^n$ and $y \in K^n$. □

Proposition 6. *The function ψ_τ given by (9) is differentiable at any $(x,y)\in R^n\times R^n$. Moreover, $\nabla_x\psi_\tau(0,0)=\nabla_y\psi_\tau(0,0)=0$; if $x^2+y^2+(\tau-2)(x\circ y)\in \operatorname{int}K^n$, then*

$$\begin{aligned}\nabla_x\psi_\tau(x,y)&=\left(L_{x+((\tau-2)/2)y}L_z^{-1}-I\right)\phi_\tau(x,y)_+,\\ \nabla_y\psi_\tau(x,y)&=\left(L_{y+((\tau-2)/2)x}L_z^{-1}-I\right)\phi_\tau(x,y)_+;\end{aligned}\quad(29)$$

if $(x,y)\neq(0,0)$ and $x^2+y^2+(\tau-2)(x\circ y)\notin \operatorname{int}K^n$, then $x_1^2+y_1^2+(\tau-2)x_1y_1\neq 0$, and

$$\begin{aligned}\nabla_x\psi_\tau(x,y)&=\left[\frac{x_1+((\tau-2)/2)y_1}{\sqrt{x_1^2+y_1^2+(\tau-2)x_1y_1}}-1\right]\phi_\tau(x,y)_+,\\ \nabla_y\psi_\tau(x,y)&=\left[\frac{y_1+((\tau-2)/2)x_1}{\sqrt{x_1^2+y_1^2+(\tau-2)x_1y_1}}-1\right]\phi_\tau(x,y)_+.\end{aligned}\quad(30)$$

Proof

Case 1. If $(x,y)=(0,0)$, then for any $h=(h_1;h_2)$, $k=(k_1;k_2)\in R\times R^{n-1}$, let $\mu_1\le\mu_2$ be the spectral values of $h^2+k^2+(\tau-2)(h\circ k)$ and let $v^{(1)},v^{(2)}$ be the corresponding spectral vectors. Then,

$$\begin{aligned}&\left\|\sqrt{h^2+k^2+(\tau-2)(h\circ k)}-h-k\right\|\\ &\quad=\left\|\sqrt{\mu_1}v^{(1)}+\sqrt{\mu_2}v^{(2)}-h-k\right\|\\ &\quad\le\sqrt{\mu_1}\left\|v^{(1)}\right\|+\sqrt{\mu_2}\left\|v^{(2)}\right\|+\|h\|+\|k\|\\ &\quad\le\sqrt{2\mu_2}+\|h\|+\|k\|.\end{aligned}\quad(31)$$

It follows from the definition of spectral value that

$$\begin{aligned}\mu_2&=\|h\|^2+\|k\|^2+(\tau-2)h^Tk\\ &\quad+\left\|2(h_1h_2+k_1k_2)+(\tau-2)(h_1k_2+k_1h_2)\right\|\\ &\le 2\|h\|^2+2\|k\|^2+3|\tau-2|\,\|h\|\,\|k\|\\ &\le 5\left(\|h\|^2+\|k\|^2\right).\end{aligned}\quad(32)$$

Combining (31) and (32) together with Lemma 4 yields that

$$\begin{aligned}&2\left[\psi_\tau(h,k)-\psi_\tau(0,0)\right]\\ &\quad=\left\|\phi_\tau(h,k)_+\right\|^2\\ &\quad\le\left\|\phi_\tau(h,k)\right\|^2\\ &\quad=\left\|\sqrt{h^2+k^2+(\tau-2)(h\circ k)}-h-k\right\|^2\\ &\quad\le\left[\sqrt{10\left(\|h\|^2+\|k\|^2\right)}+\|h\|+\|k\|\right]^2\\ &\quad=O\left(\|h\|^2+\|k\|^2\right).\end{aligned}\quad(33)$$

This shows that $\psi_\tau(x,y)$ is differentiable at $(0,0)$ with $\nabla_x\psi_\tau(0,0)=\nabla_y\psi_\tau(0,0)=0$.

Case 2. If $x^2+y^2+(\tau-2)(x\circ y)\in\operatorname{int}K^n$, let $g:R^n\to R_+$ be defined by $g(z)=(1/2)\|z_+\|^2$ for any $z\in R^n$. By the proof of Proposition 3.2 [17], $g(z)$ is continuously differentiable and $\nabla g(z)=z_+$. Since $\psi_\tau(x,y)=g(\phi_\tau(x,y))$ and $\phi_\tau(x,y)$ is differentiable at any (x,y) satisfying $x^2+y^2+(\tau-2)(x\circ y)\in\operatorname{int}K^n$ by Proposition 3.2 [16], ψ_τ is differentiable in this case and

$$\begin{aligned}\nabla_x\psi_\tau(x,y)&=\nabla_x\phi_\tau(x,y)\nabla g(\phi_\tau(x,y))\\ &=\left(L_{x+((\tau-2)/2)y}L_z^{-1}-I\right)\phi_\tau(x,y)_+,\\ \nabla_y\psi_\tau(x,y)&=\nabla_y\phi_\tau(x,y)\nabla g(\phi_\tau(x,y))\\ &=\left(L_{y+((\tau-2)/2)x}L_z^{-1}-I\right)\phi_\tau(x,y)_+.\end{aligned}\quad(34)$$

Case 3. If $(x,y)\neq(0,0)$ and $x^2+y^2+(\tau-2)(x\circ y)\notin\operatorname{int}K^n$, it follows from Lemma 3 that

$$x_1^2+y_1^2+(\tau-2)x_1y_1=\left\|x_1x_2+y_1y_2+(\tau-2)x_1y_2\right\|\neq 0.\quad(35)$$

In this case, direct calculations together with Lemma 3 yield

$$\begin{aligned}&\phi_\tau(x,y)\\ &=\left(\begin{array}{c}\sqrt{x_1^2+y_1^2+(\tau-2)x_1y_1}-x_1-y_1\\ \dfrac{x_1x_2+y_1y_2+((\tau-2)/2)x_1y_2+((\tau-2)/2)y_1x_2}{\sqrt{x_1^2+y_1^2+(\tau-2)x_1y_1}}-x_2-y_2\end{array}\right).\end{aligned}\quad(36)$$

Thus, the bigger spectral value λ_2 of $\phi_\tau(x,y)$ and its corresponding spectral vector $u^{(2)}$ are given as

$$\begin{aligned}\lambda_2&=\sqrt{x_1^2+y_1^2+(\tau-2)x_1y_1}-x_1-y_1+\|w_2\|,\\ u^{(2)}&=\frac{1}{2}\left(1;\frac{w_2}{\|w_2\|}\right),\end{aligned}\quad(37)$$

where

$$\begin{aligned}w_2=&\frac{x_1x_2+y_1y_2+((\tau-2)/2)x_1y_2+((\tau-2)/2)y_1x_2}{\sqrt{x_1^2+y_1^2+(\tau-2)x_1y_1}}\\ &-x_2-y_2.\end{aligned}\quad(38)$$

By the spectral factorization, we have

$$\begin{aligned}\phi_\tau(x,y)_+=\phi_\tau(x,y)&\Longleftrightarrow\phi_\tau(x,y)\in K^n,\\ \phi_\tau(x,y)_+=0&\Longleftrightarrow\phi_\tau(x,y)\in -K^n,\\ \phi_\tau(x,y)_+=\lambda_2u^{(2)}&\Longleftrightarrow\phi_\tau(x,y)\notin K^n\cup -K^n.\end{aligned}\quad(39)$$

Therefore, we prove the differentiability of ψ_τ in this case by considering the following three subcases.

(i) If $\phi_\tau(x,y) \notin K^n \cup -K^n$, then $\phi_\tau(x,y)_+ = \lambda_2 u^{(2)}$, where $\lambda_2, u^{(2)}$ are given by (37). Then we have

$$\begin{aligned}
&\psi_\tau(x,y) \\
&= \frac{1}{2}\|\phi_\tau(x,y)_+\|^2 = \frac{1}{4}\lambda_2^2 \\
&= \frac{1}{4}\Big[\Big(\sqrt{x_1^2+y_1^2+(\tau-2)x_1y_1} - x_1 - y_1\Big)^2 \\
&\quad + 2\Big(\sqrt{x_1^2+y_1^2+(\tau-2)x_1y_1} - x_1 - y_1\Big)\|w_2\| + \|w_2\|^2\Big].
\end{aligned} \tag{40}$$

It is obvious that ψ_τ is differentiable in this case. Moreover, by Lemma 3, we have

$$\begin{aligned}
&\nabla_{x_1} w_2 \\
&= \Bigg[\Big(x_2 + \frac{\tau-2}{2}y_2\Big)\sqrt{x_1^2+y_1^2+(\tau-2)x_1y_1} \\
&\quad - \Big(x_1x_2 + y_1y_2 + \frac{\tau-2}{2}x_1y_2 + \frac{\tau-2}{2}y_1x_2\Big) \\
&\quad \times \frac{x_1 + ((\tau-2)/2)y_1}{\sqrt{x_1^2+y_1^2+(\tau-2)x_1y_1}}\Bigg] \\
&\quad \times \big(x_1^2+y_1^2+(\tau-2)x_1y_1\big)^{-1} \\
&= \Big[\Big(x_2 + \frac{\tau-2}{2}y_2\Big)\big(x_1^2+y_1^2+(\tau-2)x_1y_1\big) \\
&\quad - \Big(x_1x_2 + y_1y_2 + \frac{\tau-2}{2}x_1y_2 + \frac{\tau-2}{2}y_1x_2\Big) \\
&\quad \times \Big(x_1 + \frac{\tau-2}{2}y_1\Big)\Big] \times \Big(\sqrt{x_1^2+y_1^2+(\tau-2)x_1y_1}\Big)^{-3} \\
&= 0, \\
&\qquad \nabla_{x_2} w_2 = \frac{x_1 + ((\tau-2)/2)y_1}{\sqrt{x_1^2+y_1^2+(\tau-2)x_1y_1}} - 1.
\end{aligned} \tag{41}$$

Therefore, the derivative of ψ_τ with respect to x_1 is

$$\begin{aligned}
&\frac{\partial}{\partial x_1}\psi_\tau(x,y) \\
&= \frac{1}{4}\Bigg[2\Big(\sqrt{x_1^2+y_1^2+(\tau-2)x_1y_1} - x_1 - y_1\Big) \\
&\qquad \times \Bigg(\frac{x_1 + ((\tau-2)/2)y_1}{\sqrt{x_1^2+y_1^2+(\tau-2)x_1y_1}} - 1\Bigg) \\
&\qquad + 2\Bigg(\frac{x_1 + ((\tau-2)/2)y_1}{\sqrt{x_1^2+y_1^2+(\tau-2)x_1y_1}} - 1\Bigg)\|w_2\| \\
&\qquad + 2\Big(\sqrt{x_1^2+y_1^2+(\tau-2)x_1y_1} - x_1 - y_1\Big) \\
&\qquad \times \frac{w_2^T \nabla_{x_1} w_2}{\|w_2\|} + 2w_2^T \nabla_{x_1} w_2\Bigg] \\
&= \frac{1}{2}\Bigg[\Bigg(\frac{x_1 + ((\tau-2)/2)y_1}{\sqrt{x_1^2+y_1^2+(\tau-2)x_1y_1}} - 1\Bigg) \\
&\qquad \times \Big(\sqrt{x_1^2+y_1^2+(\tau-2)x_1y_1} - x_1 - y_1 + \|w_2\|\Big)\Bigg],
\end{aligned} \tag{42}$$

and the gradient of ψ_τ with respect to x_2 is

$$\begin{aligned}
&\nabla_{x_2}\psi_\tau(x,y) \\
&= \frac{1}{4}\Bigg[2\Big(\sqrt{x_1^2+y_1^2+(\tau-2)x_1y_1} - x_1 - y_1\Big) \\
&\qquad \times \frac{\nabla_{x_2} w_2 \cdot w_2}{\|w_2\|} + 2\nabla_{x_2} w_2 \cdot w_2\Bigg] \\
&= \frac{1}{2}\Bigg[\Big(\sqrt{x_1^2+y_1^2+(\tau-2)x_1y_1} - x_1 - y_1\Big) \\
&\qquad \times \Bigg(\frac{x_1 + ((\tau-2)/2)y_1}{\sqrt{x_1^2+y_1^2+(\tau-2)x_1y_1}} - 1\Bigg)\frac{w_2}{\|w_2\|} \\
&\qquad + \Bigg(\frac{x_1 + ((\tau-2)/2)y_1}{\sqrt{x_1^2+y_1^2+(\tau-2)x_1y_1}} - 1\Bigg)w_2\Bigg] \\
&= \frac{1}{2}\Bigg[\Bigg(\frac{x_1 + ((\tau-2)/2)y_1}{\sqrt{x_1^2+y_1^2+(\tau-2)x_1y_1}} - 1\Bigg) \\
&\qquad \times \Big(\sqrt{x_1^2+y_1^2+(\tau-2)x_1y_1} - x_1 - y_1 + \|w_2\|\Big)\frac{w_2}{\|w_2\|}\Bigg].
\end{aligned} \tag{43}$$

Then it follows from (37), (42), and (43) that $\nabla_x \psi_\tau$ can be rewritten as

$$\nabla_x \psi_\tau(x,y) = \begin{pmatrix} \frac{\partial}{\partial x_1}\psi_\tau(x,y) \\ \nabla_{x_2}\psi_\tau(x,y) \end{pmatrix}$$

$$= \left(\frac{x_1 + ((\tau-2)/2)\, y_1}{\sqrt{x_1^2 + y_1^2 + (\tau-2)\, x_1 y_1}} - 1 \right) \lambda_2 u^{(2)}$$

$$= \left(\frac{x_1 + ((\tau-2)/2)\, y_1}{\sqrt{x_1^2 + y_1^2 + (\tau-2)\, x_1 y_1}} - 1 \right) \phi_\tau(x,y)_+. \tag{44}$$

Similarly, we can show that

$$\nabla_y \psi_\tau(x,y) = \left(\frac{y_1 + ((\tau-2)/2)\, x_1}{\sqrt{x_1^2 + y_1^2 + (\tau-2)\, x_1 y_1}} - 1 \right) \phi_\tau(x,y)_+. \tag{45}$$

(ii) If $\phi_\tau(x,y) \in K^n$, we have $\phi_\tau(x,y)_+ = \phi_\tau(x,y)$ and thus $\psi_\tau(x,y) = (1/2)\|\phi_\tau(x,y)\|^2$. Then by Proposition 3.2 [16], the gradient of ψ_τ is

$$\nabla_x \psi_\tau(x,y) = \left[\frac{x_1 + ((\tau-2)/2)\, y_1}{\sqrt{x_1^2 + y_1^2 + (\tau-2)\, x_1 y_1}} - 1 \right] \phi_\tau(x,y)$$

$$= \left[\frac{x_1 + ((\tau-2)/2)\, y_1}{\sqrt{x_1^2 + y_1^2 + (\tau-2)\, x_1 y_1}} - 1 \right] \phi_\tau(x,y)_+,$$

$$\nabla_y \psi_\tau(x,y) = \left[\frac{y_1 + ((\tau-2)/2)\, x_1}{\sqrt{x_1^2 + y_1^2 + (\tau-2)\, x_1 y_1}} - 1 \right] \phi_\tau(x,y)$$

$$= \left[\frac{y_1 + ((\tau-2)/2)\, x_1}{\sqrt{x_1^2 + y_1^2 + (\tau-2)\, x_1 y_1}} - 1 \right] \phi_\tau(x,y)_+. \tag{46}$$

If there exists (x', y') such that $\phi_\tau(x',y') \notin K^n \cup -K^n$ and $\phi_\tau(x',y') \to \phi_\tau(x,y)$, it follows from (44)–(46) that

$$\begin{aligned} \nabla_x \psi_\tau(x',y') &\longrightarrow \nabla_x \psi_\tau(x,y), \\ \nabla_y \psi_\tau(x',y') &\longrightarrow \nabla_y \psi_\tau(x,y). \end{aligned} \tag{47}$$

Therefore, ψ_τ is differentiable in this subcase.

(iii) If $\phi_\tau(x,y) \in -K^n$, we have $\phi_\tau(x,y)_+ = 0$ and thus $\psi_\tau(x,y) = (1/2)\|\phi_\tau(x,y)_+\|^2 = 0$. Then it is obvious that the gradient of ψ_τ is

$$\nabla_x \psi_\tau(x,y) = 0 = \left[\frac{x_1 + ((\tau-2)/2)\, y_1}{\sqrt{x_1^2 + y_1^2 + (\tau-2)\, x_1 y_1}} - 1 \right] \phi_\tau(x,y)_+,$$

$$\nabla_y \psi_\tau(x,y) = 0 = \left[\frac{y_1 + ((\tau-2)/2)\, x_1}{\sqrt{x_1^2 + y_1^2 + (\tau-2)\, x_1 y_1}} - 1 \right] \phi_\tau(x,y)_+. \tag{48}$$

If there exists (x', y') such that $\phi_\tau(x',y') \notin K^n \cup -K^n$ and $\phi_\tau(x',y') \to \phi_\tau(x,y)$, it follows from (44), (45), and (48) that

$$\begin{aligned} \nabla_x \psi_\tau(x',y') &\longrightarrow 0 = \nabla_x \psi_\tau(x,y), \\ \nabla_y \psi_\tau(x',y') &\longrightarrow 0 = \nabla_y \psi_\tau(x,y). \end{aligned} \tag{49}$$

Therefore, ψ_τ is differentiable in this subcase. □

Proposition 7. *Let ψ_τ be given by* (9)*. For any $x = (x_1, x_2)$, $y = (y_1, y_2) \in R \times R^{n-1}$, we have*

$$\langle x, \nabla_x \psi_\tau(x,y) \rangle + \langle y, \nabla_y \psi_\tau(x,y) \rangle = \|\phi_\tau(x,y)_+\|^2, \tag{50}$$

$$\langle \nabla_x \psi_\tau(x,y), \nabla_y \psi_\tau(x,y) \rangle \geq 0, \tag{51}$$

and the equality in (51) *holds whenever $\psi_\tau(x,y) = 0$.*

Proof. By following the proof of Lemma 4.1 [16] and using Proposition 6, we can show that the desired results hold. □

Proposition 8. *Let $f : R^n \to R_+$ be given by* (7)–(10) *and* (12)*. Then, the following results hold.*

(i) *For all $\zeta \in R^n$, we have $f(\zeta) \geq 0$ and $f(\zeta) = 0$ if and only if ζ solves the SOCCP.*

(ii) *If F and G are differentiable, the function f is differentiable with*

$$\begin{aligned} &\nabla f(\zeta) \\ &= \tau_1 \left[\nabla F(\zeta) L_{G(\zeta)} + \nabla G(\zeta) L_{F(\zeta)} \right] (F(\zeta) \circ G(\zeta))_+ \\ &\quad + \nabla F(\zeta) \nabla_x \psi_{\tau_2}(F(\zeta), G(\zeta)) + \nabla G(\zeta) \nabla_y \psi_{\tau_2}(F(\zeta), G(\zeta)). \end{aligned} \tag{52}$$

Proof. (i) It is obvious that $f(\zeta) \geq 0$ for all $\zeta \in R^n$. Now we prove $f(\zeta) = 0$ if and only if ζ solves the SOCCP. Suppose $f(\zeta) = 0$. Then we have $\psi_{\tau_2}(x,y) = 0$, and $\psi_0(x,y) = 0$ which implies $x \circ y \in -K^n$ and therefore $\langle x, y \rangle \leq 0$. By Proposition 5, the last relation together with $\psi_{\tau_2}(x,y) = 0$ yields $\langle x, y \rangle = 0$, $x \in K^n$, $y \in K^n$. Therefore, ζ solves the SOCCP. On the other hand, suppose that ζ solves the SOCCP. Then $\langle x, y \rangle = 0$, $x \in K^n$, $y \in K^n$, which are equivalent to $x \circ y = 0$, $x \in K^n$, $y \in K^n$ [4]. By Proposition 5, (7), (8), and (12), we have $\psi_{\tau_2}(x,y) = 0$, $\psi_0(x,y) = 0$, and therefore $f(\zeta) = 0$.

(ii) From Lemma 3.1 [19], we have that the function ψ_0 is differentiable for all $(x,y) \in R^n \times R^n$ with $\nabla_x \psi_0(x,y) = L_y \cdot (x \circ y)_+$ and $\nabla_y \psi_0(x,y) = L_x \cdot (x \circ y)_+$. Then, by the chain rule and direct calculations, the result follows. □

4. Error Bound and Bounded Level Sets

By Proposition 8, we see that the SOCCP is equivalent to the global minimization of the function $f(\zeta)$. In this section, we show that the function f provides a global error bound for the solution of the SOCCP and has bounded level sets, under rather weak conditions.

In this section, we consider the general case that $K \subset R^n$ is the Cartesian product of SOCs; that is, $K = K^{n_1} \times K^{n_2} \times \cdots \times K^{n_m}$ with $m, n_1, \ldots, n_m \geq 1$, $n = n_1 + n_2 + \cdots + n_m$. Thus, we obtain

$$\psi_{\tau_1,\tau_2}(x,y) = \sum_{i=1}^{m} \psi_{\tau_1,\tau_2}(x_i, y_i),$$
$$f(\zeta) = \sum_{i=1}^{m} \psi_{\tau_1,\tau_2}(F_i(\zeta), G_i(\zeta)), \tag{53}$$

and therefore the results in Sections 2 and 3 can be easily extended to the general case.

First, we discuss under what condition the function f provides a global error bound for the solution of the SOCCP. To this end, we need the concepts of Cartesian P-properties introduced in [21] for a nonlinear transformation, which are natural extensions of the P-properties on Cartesian products in R^n established by Facchinei and Pang [22]. Recently, the Cartesian P-properties are extended to the context of general Euclidean Jordan algebra associated with symmetric cones [20].

Definition 9. The mappings $F = (F_1, \ldots, F_m)$ and $G = (G_1, \ldots, G_m)$ are said to have

(i) the joint uniform Cartesian P-property if there exists a constant $\rho > 0$ such that, for every $\zeta, \xi \in R^n$, there is an index $i \in \{1, 2, \ldots, m\}$ such that

$$\langle F_i(\zeta) - F_i(\xi), G_i(\zeta) - G_i(\xi)\rangle \geq \rho\|\zeta - \xi\|^2; \tag{54}$$

(ii) the joint Cartesian P-property if, for every $\zeta, \xi \in R^n$ with $\zeta \neq \xi$, there is an index $i \in \{1, 2, \ldots, m\}$ such that

$$\zeta_i \neq \xi_i, \quad \langle F_i(\zeta) - F_i(\xi), G_i(\zeta) - G_i(\xi)\rangle > 0. \tag{55}$$

Now we show that the function f provides a global error bound for the solution of the SOCCP if F and G have the joint uniform Cartesian P-property.

Proposition 10. *Let f be given by* (7)–(10) *and* (12). *Suppose that F and G have the joint uniform Cartesian P-property and the SOCCP has a solution ζ^*. Then there exists a constant $\kappa > 0$ such that, for any $\zeta \in R^n$,*

$$\kappa\|\zeta - \zeta^*\|^2 \leq \left(\sqrt{\frac{2}{\tau_1}} + \frac{4}{\sqrt{4-\tau_2}}\right) f(\zeta)^{1/2}. \tag{56}$$

Proof. Since F and G have the joint uniform Cartesian P-property, there exists a constant $\rho > 0$ such that, for any $\zeta \in R^n$, there is an index $i \in \{1, 2, \ldots, m\}$ such that

$$\begin{aligned}
&\rho\|\zeta - \zeta^*\|^2 \\
&\leq \langle F_i(\zeta) - F_i(\zeta^*), G_i(\zeta) - G_i(\zeta^*)\rangle \\
&= \langle F_i(\zeta), G_i(\zeta)\rangle + \langle F_i(\zeta^*), -G_i(\zeta)\rangle \\
&\quad + \langle -F_i(\zeta), G_i(\zeta^*)\rangle \\
&\leq \langle F_i(\zeta), G_i(\zeta)\rangle + \langle F_i(\zeta^*), (-G_i(\zeta))_+\rangle \\
&\quad + \langle (-F_i(\zeta))_+, G_i(\zeta^*)\rangle \\
&\leq \sqrt{2}\,\|(F_i(\zeta) \circ G_i(\zeta))_+\| + \|F_i(\zeta^*)\|\,\|(-G_i(\zeta))_+\| \\
&\quad + \|(-F_i(\zeta))_+\|\,\|G_i(\zeta^*)\| \\
&\leq \max\left\{\sqrt{2}, \|F_i(\zeta^*)\|, \|G_i(\zeta^*)\|\right\} \\
&\quad \times [\|(F_i(\zeta) \circ G_i(\zeta))_+\| + \|(-F_i(\zeta))_+\| + \|(-G_i(\zeta))_+\|] \\
&\leq \max\left\{\sqrt{2}, \|F(\zeta^*)\|, \|G(\zeta^*)\|\right\} \\
&\quad \times [\|(F_i(\zeta) \circ G_i(\zeta))_+\| + \|(-F_i(\zeta))_+\| + \|(-G_i(\zeta))_+\|] \\
&\leq \max\left\{\sqrt{2}, \|F(\zeta^*)\|, \|G(\zeta^*)\|\right\} \\
&\quad \times [\|(F(\zeta) \circ G(\zeta))_+\| + \|(-F(\zeta))_+\| + \|(-G(\zeta))_+\|],
\end{aligned} \tag{57}$$

where the second inequality is due to Lemma 1(ii) and the third inequality follows from Lemma 2. Setting $\kappa := \rho/\max\{\sqrt{2}, \|F(\zeta^*)\|, \|G(\zeta^*)\|\}$, we obtain

$$\kappa\|\zeta - \zeta^*\|^2 \leq \|(F(\zeta) \circ G(\zeta))_+\| + \|(-G(\zeta))_+\| + \|(-F(\zeta))_+\|. \tag{58}$$

By (7), (8), and (12), we have

$$\|(F(\zeta) \circ G(\zeta))_+\| = \sqrt{2}\psi_0(F(\zeta), G(\zeta))^{1/2} \leq \sqrt{\frac{2}{\tau_1}} f(\zeta)^{1/2}. \tag{59}$$

Moreover, we obtain from Lemma 4 that

$$\begin{aligned}
&\|(-F(\zeta))_+\| + \|(-G(\zeta))_+\| \\
&\quad \leq \sqrt{2}\left[\|(-F(\zeta))_+\|^2 + \|(-G(\zeta))_+\|^2\right]^{1/2} \\
&\quad \leq \frac{4}{\sqrt{4-\tau_2}}\psi_{\tau_2}(F(\zeta), G(\zeta))^{1/2} \\
&\quad \leq \frac{4}{\sqrt{4-\tau_2}} f(\zeta)^{1/2}.
\end{aligned} \tag{60}$$

Combining (58), (59), and (60) yields the desired result. □

To guarantee the boundedness of the level sets

$$L_f(\gamma) := \{\zeta \in R^n \mid f(\zeta) \leq \gamma\} \tag{61}$$

for any $\gamma \geq 0$, we give the following condition.

Condition 11. For any sequence $\{\zeta^k\} \subseteq R^n$ such that

$$\|\zeta^k\| \longrightarrow +\infty,$$
$$\left\|\left(-F\left(\zeta^k\right)\right)_+\right\| < +\infty, \qquad \left\|\left(-G\left(\zeta^k\right)\right)_+\right\| < +\infty, \tag{62}$$

there holds that

$$\max_{1 \leq i \leq m} \lambda_{\max}\left[\left(F_i\left(\zeta^k\right) \circ G_i\left(\zeta^k\right)\right)_+\right] \longrightarrow +\infty. \tag{63}$$

Proposition 12. *If the mappings F and G satisfy Condition 11, then the level sets $L_f(\gamma)$ of f for any $\gamma \geq 0$ are bounded.*

Proof. On the contrary, we assume that there exists an unbounded sequence $\{\zeta^k\} \subseteq L_f(\widehat{\gamma})$ for some $\widehat{\gamma} \geq 0$. Thus $\|\zeta^k\| \to +\infty$ and $\psi_{\tau_2}(F(\zeta^k), G(\zeta^k)) \leq f(\zeta^k) \leq \widehat{\gamma}$ for all k. Then by Lemma 4, we have for all k that

$$\begin{aligned}\left\|\left(-F\left(\zeta^k\right)\right)_+\right\|^2 + \left\|\left(-G\left(\zeta^k\right)\right)_+\right\|^2 &\leq \frac{4}{4-\tau_2}\left\|\phi_{\tau_2}\left(F\left(\zeta^k\right), G\left(\zeta^k\right)\right)\right\|^2 \\ &= \frac{8}{4-\tau_2}\psi_{\tau_2}\left(F\left(\zeta^k\right), G\left(\zeta^k\right)\right) \\ &\leq \frac{8}{4-\tau_2}\widehat{\gamma},\end{aligned} \tag{64}$$

which implies $\|(-F(\zeta^k))_+\| < +\infty$ and $\|(-G(\zeta^k))_+\| + \infty$. Therefore, from Condition 11, there is $j \in \{1, 2, \ldots, m\}$ such that $\lambda_{\max}[(F_j(\zeta^k) \circ G_j(\zeta^k))_+] \to +\infty$. It follows from (7), (8), and (12) that

$$\begin{aligned}\lambda_{\max}\left[\left(F_j\left(\zeta^k\right) \circ G_j\left(\zeta^k\right)\right)_+\right] &\leq \sqrt{2}\left\|\left(F_j\left(\zeta^k\right) \circ G_j\left(\zeta^k\right)\right)_+\right\| \\ &= 2\psi_0\left(F_j\left(\zeta^k\right), G_j\left(\zeta^k\right)\right)^{1/2} \\ &\leq \frac{2}{\sqrt{\tau_1}} f\left(\zeta^k\right)^{1/2},\end{aligned} \tag{65}$$

and hence $f(\zeta^k) \to +\infty$. This contradicts the fact that $\{\zeta^k\} \subseteq L_f(\widehat{\gamma})$. □

It should be noted that Condition 11 is rather weak to guarantee the boundedness of level sets of f. As far as we know, the weakest condition available to ensure the boundedness of level sets is the following condition given by [20].

Condition 13 (see [20]). For any sequence $\{\zeta^k\} \subseteq R^n$ such that

$$\begin{gathered}\left\|\zeta^k\right\| \to +\infty, \\ \left\|\left(-F\left(\zeta^k\right)\right)_+\right\| < +\infty, \qquad \left\|\left(-G\left(\zeta^k\right)\right)_+\right\| < +\infty,\end{gathered} \tag{66}$$

there holds that

$$\max_{1 \leq i \leq m} \lambda_{\max}\left[F_i\left(\zeta^k\right) \circ G_i\left(\zeta^k\right)\right] \longrightarrow +\infty. \tag{67}$$

It is obvious that $\lambda_{\max}[F_i(\zeta^k) \circ G_i(\zeta^k)] \leq \lambda_{\max}[(F_i(\zeta^k) \circ G_i(\zeta^k))_+]$ for any $i \in \{1, 2, \ldots, m\}$, and therefore Condition 13 implies Condition 11. It has been shown that the symmetric cone complementarity problem (SCCP) with the jointly monotone mappings and a strictly feasible point, or the SCCP with joint Cartesian R_{02}-property [23], all imply Condition 13 [20]. Hence they all implies Condition 11, since the SCCP includes the SOCCP. Therefore, Condition 11 is a weaker condition than the most available conditions to guarantee the boundedness of level sets.

5. Numerical Results

In this section, we employ the merit function method based on the unconstrained minimization reformulation (12) to solve the SOCCPs (1). All the experiments were performed on a desktop computer with Intel Pentium Dual T2390 CPU 1.86 GHz and 1.00 GB memory. The operating system was Windows XP and the implementations were done in MATLAB 7.0.1.

We adopt the L-BFGS method [24], a limited-memory quasi-Newton method, with 5 limited-memory vector-updates to solve the unconstrained minimization reformulation (12) where the two-parametric class of merit functions ψ_{τ_1,τ_2} is given as (7)–(10). For the scaling matrix $H^0 = \gamma I$ in the L-BFGS, we adopt $\gamma = p^T q / q^T q$ as recommended by [25], where

$$p := \zeta - \zeta^{\text{old}}, \qquad q := \nabla\psi_{\tau_1,\tau_2}(\zeta) - \nabla\psi_{\tau_1,\tau_2}\left(\zeta^{\text{old}}\right). \tag{68}$$

In the L-BFGS, we revert to the steepest descent direction $-\nabla\psi_{\tau_1,\tau_2}(\zeta)$ whenever $p^T q \leq 10^{-5}\|p\|\|q\|$. In addition, we use the nonmonotone line search [26] to seek a suitable steplength. In detail, we compute the smallest nonnegative integer l_k such that

$$\psi_{\tau_1,\tau_2}\left(\zeta^k + \rho^{l_k} d^k\right) \leq W_k + \sigma\rho^{l_k}\nabla\psi_{\tau_1,\tau_2}\left(\zeta^k\right)^T d^k, \tag{69}$$

where d^k denotes the direction in the kth iteration generated by the L-BFGS, ρ and σ are parameters in (0,1), and W_k is given by

$$W_k = \max_{j=k-m_k,\ldots,k} \psi_{\tau_1,\tau_2}\left(\zeta^j\right), \tag{70}$$

where, for a given nonnegative integer $\widehat{m}$ and s, we set

$$m_k = \begin{cases} 0, & \text{if } k \leq s, \\ \min\left\{m_{k-1} + 1, \widehat{m}\right\} & \text{otherwise.}\end{cases} \tag{71}$$

Throughout the numerical experiments, we choose the following parameters:

$$\rho = 0.8, \qquad \sigma = 0.01, \qquad \widehat{m} = 5, \qquad s = 5. \tag{72}$$

The algorithm is stopped whenever the number of function evaluations for ψ_{τ_1,τ_2} is over 10000 or $\max\{\psi_{\tau_1,\tau_2}(\zeta), |\langle F(\zeta), G(\zeta)\rangle|\} \leq 10^{-6}$ as the stopping criterion.

The test problems are the randomly generated linear SOCCPs (1), where

$$F(\zeta) = \zeta, \qquad G(\zeta) = M\zeta + q, \tag{73}$$

with $M \in R^{n \times n}$ and $q \in R^n$. In detail, we generate a random matrix $N = \text{rand}(n, n)$ and a random vector $q = \text{rand}(n, 1)$, and then let $M := N^T N$. Since the matrix M is semidefinite positive, the generated problems (1) are the monotone linear SOCCPs. In the tables of test results, n denotes the size of problems; NF denotes the (average) number of iterations; CPU(s) denotes the (average) CPU time in seconds;

Table 1: Numerical results for SOCCPs with $(\tau_1, \tau_2) = (0.1, 0.1)$.

n	NF	CPU (s)	Gap
50	155.1	0.063	$1.44929e-6$
100	162.0	0.078	$8.38541e-6$
150	166.0	0.171	$6.41033e-5$
200	168.1	0.297	$5.46509e-6$
250	170.3	0.563	$5.80518e-6$
300	172.2	0.953	$4.95267e-6$
350	174.2	1.453	$4.18082e-6$
400	175.0	2.094	$3.67026e-6$
450	176.0	2.875	$3.32159e-4$
500	177.0	3.766	$3.48330e-7$
550	178.0	4.984	$3.25888e-6$
600	178.2	6.234	$3.57938e-5$
650	179.1	7.938	$3.16528e-5$
700	180.3	9.687	$2.82810e-6$
750	181.6	11.594	$2.54843e-6$
800	181.0	14.297	$3.17068e-6$
850	182.7	16.625	$2.18459e-6$
900	182.6	19.703	$2.59598e-5$
950	183.0	22.922	$2.35290e-7$
1000	183.2	26.438	$2.41060e-6$

Table 2: Numerical results for SOCCPs with $(\tau_1, \tau_2) = (1, 2.0)$.

n	NF	CPU (s)	Gap
50	156.0	0.062	$1.29445e-6$
100	162.1	0.078	$7.64185e-6$
150	166.1	0.172	$6.51732e-6$
200	169.2	0.313	$5.83238e-6$
250	171.6	0.547	$4.96726e-5$
300	172.3	0.953	$4.72040e-7$
350	174.5	1.469	$4.43086e-6$
400	175.6	2.094	$4.02729e-5$
450	176.8	2.859	$3.28247e-5$
500	177.0	3.781	$3.49543e-6$
550	178.0	4.984	$3.22618e-6$
600	178.2	6.250	$3.29624e-6$
650	179.0	7.891	$3.23443e-6$
700	180.5	9.765	$3.21537e-4$
750	181.2	11.687	$2.73740e-6$
800	181.0	14.266	$3.06024e-6$
850	182.1	16.609	$2.85270e-7$
900	182.0	19.563	$3.09099e-5$
950	183.0	23.000	$2.55965e-6$
1000	183.0	26.360	$2.86313e-6$

and Gap denotes the (average) value of $|\langle F(\zeta), G(\zeta)\rangle|$ when the algorithm terminates.

We solve the linear SOCCPs of different dimensions with size n from 50 to 1000 and $m = 1$. The random problems of each size are generated 10 times, and the test results with different parameters $\tau_1 > 0$ and $\tau_2 \in (0, 4)$ are listed in Tables 1, 2, and 3. From the results of these tables, we give several observations.

(i) All the random problems have been solved in very short CPU time.

(ii) The problem size slightly affects the number of iterations.

(iii) For the same dimension of linear SOCCPs, choices of different parameters $\tau_1 > 0$ and $\tau_2 \in (0, 4)$ generally do not affect the number of iterations and the CPU time.

6. Conclusions

In this paper, we have studied a two-parametric class of merit functions for the second-order cone complementarity problem based on the one-parametric class of complementarity functions. The new proposed class of merit functions includes a broad class of merit functions, since the one-parametric class of complementarity functions is closely related to the famous natural residual function and Fischer-Burmeister function. The new class of merit functions has been shown to possess some favorable properties. In particular, it provides a global error bound if F and G have the joint uniform Cartesian P-property. And it has bounded level sets under a weaker condition than the most available conditions [20, 23]. Some preliminary numerical results for solving the SOCCPs show the effectiveness of the merit function method via the new class of merit functions.

Table 3: Numerical results for SOCCPs with $(\tau_1, \tau_2) = (10, 3.5)$.

n	NF	CPU (s)	Gap
50	157.0	0.063	$1.27879e-5$
100	162.1	0.094	$1.12054e-6$
150	166.1	0.172	$8.69632e-4$
200	169.0	0.328	$6.57170e-6$
250	171.2	0.562	$4.73086e-6$
300	172.2	0.969	$6.76156e-6$
350	174.3	1.437	$3.97485e-6$
400	175.0	2.125	$4.16500e-7$
450	176.0	2.875	$5.02327e-6$
500	177.3	3.781	$3.87474e-6$
550	178.2	4.937	$4.14691e-5$
600	178.5	6.266	$5.16964e-5$
650	179.6	7.829	$4.40620e-7$
700	180.2	9.641	$3.52956e-5$
750	181.0	11.625	$3.10725e-5$
800	181.0	14.312	$3.48511e-6$
850	182.0	16.610	$3.21915e-6$
900	182.0	19.578	$3.39945e-4$
950	183.0	22.922	$2.72912e-6$
1000	183.1	26.390	$3.36630e-6$

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