Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 587269, 10 pages
http://dx.doi.org/10.1155/2013/587269

Research Article

The Discrete-Time Bulk-Service *Geo/Geo/1* Queue with Multiple Working Vacations

Jiang Cheng,[1,2] Yinghui Tang,[1] and Miaomiao Yu[1,3]

[1] *School of Mathematics & Software Science, Sichuan Normal University, Chengdu, Sichuan 610066, China*
[2] *College of Computer Science and Technology, Southwest University for Nationalities, Chengdu, Sichuan 610041, China*
[3] *School of Science, Sichuan University of Science and Engineering, Zigong, Sichuan 643000, China*

Correspondence should be addressed to Jiang Cheng; jiangcheng_uestc@163.com and Yinghui Tang; tangyh@uestc.edu.cn

Received 3 July 2012; Accepted 14 December 2012

Academic Editor: Alvaro Valencia

This paper deals with a discrete-time bulk-service $Geo/Geo/1$ queueing system with infinite buffer space and multiple working vacations. Considering an early arrival system, as soon as the server empties the system in a regular busy period, he leaves the system and takes a working vacation for a random duration at time n. The service times both in a working vacation and in a busy period and the vacation times are assumed to be geometrically distributed. By using embedded Markov chain approach and difference operator method, queue length of the whole system at random slots and the waiting time for an arriving customer are obtained. The queue length distributions of the outside observer's observation epoch are investigated. Numerical experiment is performed to validate the analytical results.

1. Introduction

Recently there has been a rapid increase in the literature on discrete-time queueing system with working vacations. These queueing models have been studied extensively and applied to computer networks, communication systems, and manufacturing systems. In the classical queueing system with server vacations, the server stops working during vacation periods. Suppose, however, that a system can be staffed with a substitute server during the times the main server is taking vacations. The service rate of the substitute server is different from (and probably lower than) that of the main server. This is the notion of working vacations recently introduced by Servi and Finn [1]. They studied an $M/M/1$ queue with multiple working vacations ($M/M/1/WV$). Their work is motivated by the analysis of a reconfigurable wavelength-division multiplexing (WDM) optical access network. In 2006, Wu and Takagi [2] generalized Servi and Finn's $M/M/1/WV$ queue to an $M/G/1/WV$ queue. Baba [3] extended Wu and Takagi's work to a renewal input $GI/M/1$ queue with working vacations and derived the steady-state system length distributions at an arrival and arbitrary epochs. The $Geo/Geo/1$ queue with single and multiple working vacations have been discussed in Li and Tian [4] and Tian et al. [5], respectively. Chae et al. [6] studied the $GI/M/1$ queue and $GI/Geo/1$ queue with single working vacation (SWV). The discrete-time infinite buffer $GI/Geo/1$ queue with multiple working vacations and vacation interruption has been studied in Li et al. [7, 8]. The discrete-time finite buffer $GI/Geo/1$ queue with multiple working vacations has been discussed by Goswami and Mund [9]. All the above studies on discrete-time single server queues have been carried out under the assumption that a server serves singly at a time. However, there are many instances where the servers are carried out in batches to enhance the performance of the system. Over the last several years the discrete-time single server queues in batch service without vacations have been studied in Gupta and Goswami [10], Chaudhry and Chang [11], Alfa and He [12], and Yi et al. [13]. Lately, This type of queueing systems raise interest once more by many scholars such as Banerjee et al. [14, 15], Claeys et al. [16, 17].

The continuous-time infinite buffer single server batch service queue with multiple vacations has been analyzed by Choi and Han [18], Chang and Takine [19]. The $M/G/1$ queue with bulk service and single vacation has been investigated by Sikdar and Gupta [20].

As to the research to queueing systems with batch service and working vacations (or vacation), Yu et al. [21] considered a finite capacity and bulk-arrival and bulk-service continuous-time queueing system with multiple working vacations and partial batch rejection. Vijaya Laxmi and Yesuf [22] studied a renewal input infinite buffer batch service queue with single exponential working vacation and accessibility to batches. Goswami and Mund [23] investigated a discrete-time batch service renewal input queue with multiple working vacations. Note that although all the above studies on discrete-time bulk-service queues have been carried out, but, from their models, we can only obtain the average length of the waiting for service; the average length of the whole system cannot be obtained at prearrival, arbitrary and outside observer's observation epochs. In fact, the number of customers being served in batches is a random variable; it is difficult to compute the average length of the whole system.

This paper focuses on a discrete-time batch-service infinite buffer *Geo*/*Geo*/1 queueing system with multiple working vacations in which arrivals occur according to a geometrical input. By using embedded Markov chain approach, we obtained the operation rules of the one-step transition probability matrix, the average length at random slots, and the average waiting time for an arriving customer. This model has potential applications in computer networks where jobs are processed in batches.

The rest of the paper is arranged as follows. In the next section, the model of the considered queueing system is described. In Section 3, the stationary distribution of queue length at arbitrary slot epochs is discussed. In Section 4, we study the waiting time distribution. In Section 5, we discuss the outside observer's observation epoch probabilities distribution. In Section 6, some numerical results and the sensitivity analysis of this system are given. Section 7 concludes this paper.

2. System Description

We consider a discrete-time bulk-service infinite buffer space queueing system with server multiple working vacations according to the rule of an early arrival system. Assume that the time axis is slotted into intervals of equal length with the length of a slot being unity, and it is marked as $0, 1, 2, \ldots, n, \ldots$, a potential arrival occurs in the interval (n, n^+) and potential batch departures occur in (n^-, n). In the meantime, the interarrival times T of customers are independent and geometrically distributed with probability mass function (p.m.f.) $P\{T = k\} = p\overline{p}^{k-1},\ k \geq 1,\ \overline{p} = 1 - p$.

The customers are served in batches of variable capacity, the maximum service capacity for the server being a ($a \geq 1$). Service times S_b during normal busy period and service times S_v during a working vacation are assumed independent and geometrically distributed with p.m.f. $P\{S_b = k\} = \mu_b\overline{\mu}_b^{k-1},\ k \geq 1,\ \overline{\mu}_b = 1 - \mu_b$ and p.m.f. $P\{S_v = k\} = \mu_v\overline{\mu}_v^{k-1},\ k \geq 1,\ \overline{\mu}_v = 1 - \mu_v$, respectively. A new arriving customer cannot go into the queue being served immediately in spite of the working vacation period and the normal busy period. The server leaves the system and takes a working vacation at epoch n as soon as system becomes empty in regular period. The working vacation time follows a geometric distribution with parameter θ ($0 < \theta < 1$) and its p.m.f. is $P\{V = k\} = \theta\overline{\theta}^{k-1},\ k \geq 1,\ \overline{\theta} = 1-\theta$. If there are some customers being served after the server finishes a working vacation, the service interrupted at the end of a vacation is lost and it is restarted with service rate μ_b at the beginning of the following service period, which means that the regular busy period starts. The various time epochs at which events occur are depicted in Figure 1.

We assume that interarrival times, service times, and working vacation times are mutually independent.

In addition, the service discipline is first in first out (FIFO). Let Q_n be the number of customers in the system at time n, and

$$J_n = \begin{cases} 0, & \text{the system is in a working vacation} \\ & \text{period at time } n, \\ 1, & \text{the system is in a regular busy} \\ & \text{period at time } n. \end{cases} \tag{1}$$

Then $\{Q_n, J_n\}$ is a Markov chain with the state space

$$\Omega = \{(0,0)\} \bigcup \{(k, j) : k \geq 1, j = 0, 1\}, \tag{2}$$

where state $(k, 1)$, $k \geq 1$ indicates that the system is in a regular busy period; state $(k, 0)$, $k \geq 0$ indicates that the system is in a working vacation period and there k customers in the system.

Using the lexicographical sequence for the states, the one-step transition block matrix can be written as

$$\mathbf{P} = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \vdots \\ a \\ a+1 \\ \vdots \end{array} \left[\begin{array}{cccccccccc} \mathbf{A_{00}} & \mathbf{B} & & & & & & & & \\ \mathbf{C} & \mathbf{D} & \mathbf{E} & & & & & & & \\ \mathbf{C} & \mathbf{F} & \mathbf{G} & \mathbf{E} & & & & & & \\ \mathbf{C} & \mathbf{F} & \mathbf{0} & \mathbf{G} & \mathbf{E} & & & & & \\ \mathbf{C} & \mathbf{F} & \mathbf{0} & \mathbf{0} & \mathbf{G} & \mathbf{E} & & & & \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & & & \\ \mathbf{C} & \mathbf{F} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{G} & \mathbf{E} & & \\ \mathbf{0} & \mathbf{H} & \mathbf{F} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{G} & \mathbf{E} & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{array}\right], \tag{3}$$

where the first column in front of the matrix denotes the number of the customers for the first column block matrix in matrix $\mathbf{P}$, and $\mathbf{A_{00}}$ is transition probability from $(0,0)$ to $(0,0)$; $\mathbf{B}$ is transition probability matrix from $(0,0)$ to $\{(1,0),(1,1)\}$; $\mathbf{C}$ is transition probability matrix from $\{(i,0),(i,1)\}$ $(1 \leq i \leq a)$ to $(0,0)$; $\mathbf{D}$ is transition probability matrix from $\{(1,0),(1,1)\}$ to $\{(1,0),(1,1)\}$; $\mathbf{E}$ is transition probability matrix from $\{(i,0),(i,1)\}$ $(i \geq 1)$ to $\{(i+1,0),(i+1,1)\}$ $(i \geq 1)$; $\mathbf{F}$ is transition probability matrix from $\{(i,0),(i,1)\}$ $(2 \leq i \leq a)$ to $\{(1,0),(1,1)\}$ and from $\{(i,0),(i,1)\}$ $(i \geq a+1)$ to $\{(i-a+1,0),(i-a+1,1)\}$ $(i \geq a+1)$; $\mathbf{G}$ is transition probability

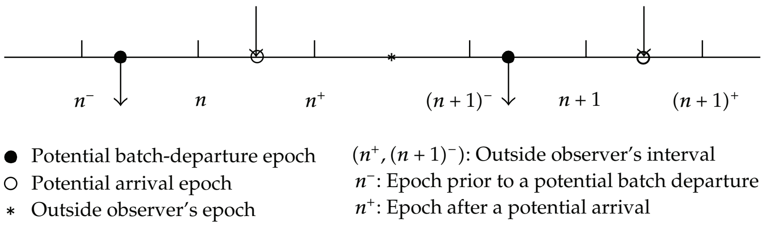

FIGURE 1: Various time epochs in EAS.

matrix from $\{(i,0),(i,1)\}$ $(i \geq 2)$ to $\{(i,0),(i,1)\}$ $(i \geq 2)$; **H** is transition probability matrix from $\{(i,0),(i,1)\}$ $(i \geq a+1)$ to $\{(i-a,0),(i-a,1)\}$ $(i \geq a+1)$.

We have

$$\mathbf{A_{00}} = \overline{p} + p\overline{\theta}\mu_v, \qquad \mathbf{B} = \begin{bmatrix} p\overline{\theta}\overline{\mu}_v & p\theta \end{bmatrix},$$
$$\mathbf{C} = \begin{bmatrix} \overline{p}\overline{\theta}\mu_v \\ \overline{p}\mu_b \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} \overline{p}\overline{\theta}\overline{\mu}_v + p\overline{\theta}\mu_v & \overline{p}\theta \\ 0 & \overline{p}\,\overline{\mu}_b + p\mu_b \end{bmatrix},$$
$$\mathbf{E} = \begin{bmatrix} p\overline{\theta}\overline{\mu}_v & p\theta \\ 0 & p\overline{\mu}_b \end{bmatrix}, \qquad \mathbf{F} = \begin{bmatrix} p\overline{\theta}\mu_v & 0 \\ 0 & p\mu_b \end{bmatrix}, \tag{4}$$
$$\mathbf{G} = \begin{bmatrix} \overline{p}\overline{\theta}\overline{\mu}_v & \overline{p}\theta \\ 0 & \overline{p}\,\overline{\mu}_b \end{bmatrix}, \qquad \mathbf{H} = \begin{bmatrix} \overline{p}\overline{\theta}\mu_v & 0 \\ 0 & \overline{p}\mu_b \end{bmatrix}.$$

3. The Stationary Queue Size Distributions at Random Slots

Assume that (Q,J) is the stationary limit of $\{Q_n, J_n\}$, and its distribution is denoted as $\pi_{k,j} = \lim_{n\to\infty} P\{Q_n = k, J_n = j\} = P\{Q = k, J = j\}$, $(k,j) \in \Omega$, we have the following theorem.

Theorem 1. *If* $\rho = p/a\mu_b < 1$, $\rho_0 = p/a\mu_v < 1$, $\pi_{k,j}$ $(k \geq 1, j = 0,1)$ *are given by*

$$\pi_{k,0} = \frac{p\overline{\theta}\overline{\mu}_v(1-\xi)\xi^{k-1}}{\omega}\pi_{0,0},$$
$$\pi_{k,1} = c_0' r^k + \frac{p\theta\overline{\theta}\overline{\mu}_v(p+\overline{p}\xi)(1-\xi)\xi^{k-1}\pi_{0,0}}{\omega\varpi}, \quad (k \geq 1), \tag{5}$$

where

$$\omega = 1 - \overline{\theta}\left[1 - \overline{p}\mu_v\right] - \overline{p}\overline{\theta}\mu_v\xi^a,$$
$$\varpi = \left[(1-\overline{p}\,\overline{\mu}_b)\xi - \left(p\overline{\mu}_b + p\mu_b\xi^a + \overline{p}\mu_b\xi^{a+1}\right)\right],$$
$$c_0' = \Big(p\theta\Big\{\omega\varpi + \overline{\theta}\overline{\mu}_v \times \Big\{p\varpi(1-\xi) + (p+\overline{p}\xi)\left[\varpi - \overline{p}\mu_b(1-\xi^a)\right]\Big\}\Big\} \times (1-r)\Big) \times \left(r\omega\varpi\overline{p}\mu_b(1-r^a)\right)^{-1}\pi_{0,0},$$

$$\pi_{0,0} = \Big\{1 + \left(p\overline{\theta}\overline{\mu}_v\varpi + p\theta\overline{\theta}\overline{\mu}_v(p+\overline{p}\xi)\right)(\omega\varpi)^{-1} + \Big(p\theta\Big\{\omega\varpi + \overline{\theta}\overline{\mu}_v \times \{p\varpi(1-\xi) + (p+\overline{p}\xi)\left[\varpi - \overline{p}\mu_b(1-\xi^a)\right]\}\Big\}\Big) \times \left(\omega\varpi\overline{p}\mu_b(1-r^a)\right)^{-1}\Big\}^{-1}, \tag{6}$$

$0 < \xi < 1$ *and* ξ *is the root of the equation* $\overline{p}\mu_v\overline{\theta}z^{a+1} + p\mu_v\overline{\theta}z^a - (1-\overline{p}\,\overline{\mu}_v\overline{\theta})z + p\overline{\mu}_v\overline{\theta} = 0$, $0 < r < 1$ *and* r *is the root of the equation* $\overline{p}\mu_b z^{a+1} + p\mu_b z^a + p\overline{\mu}_b - (1-\overline{p}\,\overline{\mu}_b)z = 0$.

Proof. According to the one-step transition probability matrix, we can see which is not **QBD** process, the method of matrix-geometric solution is invalid. Based on the stationary equations obtained directly by stochastic balance, we have

$$\pi_{0,0} = \pi_{0,0}\mathbf{A}_{00} + \sum_{i=1}^{a}(\pi_{i,0},\pi_{i,1})\mathbf{C},$$
$$(\pi_{1,0},\pi_{1,1}) = \pi_{0,0}\mathbf{B} + (\pi_{1,0},\pi_{1,1})\mathbf{D} + \sum_{i=2}^{a}(\pi_{i,0},\pi_{i,1})\mathbf{F} + (\pi_{a+1,0},\pi_{a+1,1})\mathbf{H},$$
$$(\pi_{k,0},\pi_{k,1}) = (\pi_{k-1,0},\pi_{k-1,1})\mathbf{E} + (\pi_{k,0},\pi_{k,1})\mathbf{G} + (\pi_{a+k-1,0},\pi_{a+k-1,1})\mathbf{F} + (\pi_{a+k,0},\pi_{a+k,1})\mathbf{H}, \quad k \geq 2. \tag{7}$$

Then, we obtain the following equations:

$$\pi_{0,0} = \pi_{0,0}\left(\overline{p} + p\overline{\theta}\mu_v\right) + \overline{p}\overline{\theta}\mu_v\sum_{i=1}^{a}\pi_{i,0} + \overline{p}\mu_b\sum_{i=1}^{a}\pi_{i,1}, \tag{8}$$

$$\pi_{1,0} = p\overline{\theta}\overline{\mu}_v\pi_{0,0} + \overline{p}\overline{\theta}\overline{\mu}_v\pi_{1,0} + p\overline{\theta}\mu_v\sum_{i=1}^{a}\pi_{i,0} + \overline{p}\overline{\theta}\mu_v\pi_{a+1,0}, \tag{9}$$

$$\pi_{k,0} = p\overline{\theta}\overline{\mu}_v\pi_{k-1,0} + \overline{p}\overline{\theta}\overline{\mu}_v\pi_{k,0} + p\overline{\theta}\mu_v\pi_{a+k-1,0} + \overline{p}\overline{\theta}\mu_v\pi_{a+k,0}, \quad k \geq 2, \tag{10}$$

$$\pi_{1,1} = p\theta\pi_{0,0} + \overline{p}\theta\pi_{1,0} + \overline{p}\,\overline{\mu}_b\pi_{1,1} + p\mu_b\sum_{i=1}^{a}\pi_{i,1} + \overline{p}\mu_b\pi_{a+1,1}, \tag{11}$$

$$\begin{aligned}\pi_{k,1} ={}& p\theta\pi_{k-1,0} + p\overline{\mu}_b\pi_{k-1,1} + \overline{p}\theta\pi_{k,0} + \overline{p}\,\overline{\mu}_b\pi_{k,1} \\ &+ p\mu_b\pi_{a+k-1,1} + \overline{p}\mu_b\pi_{a+k,1}, \quad k \geq 2.\end{aligned} \tag{12}$$

According to the characteristic of difference equations, let $\pi_{k+j,0} = E^{j}\pi_{k,0}$ [24], $j \in Z$; $k = 0, 1, 2, \ldots$, where E denotes the difference operator for the difference equation, substituting it into (10), (10) can be written as

$$\left(\overline{p}\overline{\theta}\mu_v E^{a+1} + p\overline{\theta}\mu_v E^{a} - \left(1 - \overline{p}\overline{\theta}\overline{\mu}_v\right)E + p\overline{\theta}\overline{\mu}_v\right)\pi_{k-1,0} = 0. \tag{13}$$

The auxiliary equation is given by

$$\overline{p}\overline{\theta}\mu_v z^{a+1} + p\overline{\theta}\mu_v z^{a} - \left(1 - \overline{p}\overline{\theta}\overline{\mu}_v\right)z + p\overline{\theta}\overline{\mu}_v = 0. \tag{14}$$

Let $f(z) = \overline{p}\overline{\theta}\mu_v z^{a+1} + p\overline{\theta}\mu_v z^{a} + \overline{p}\overline{\theta}\overline{\mu}_v z + p\overline{\theta}\overline{\mu}_v$ and $g(z) = -z$. Using Rouché's theorem, it can be shown that there is only one zero real root falls in the unit circle (Note: the root must be real root; otherwise, there are two roots at least fall in the unit circle, because the imaginary roots of an equation appear in pairs.) We denote this root by ξ $(0 < \xi < 1)$ and the other a roots by ξ_i, $|\xi_i| \geq 1$ $(i = 1, 2, 3, \ldots, a)$. So ξ satisfies $f(\xi) + g(\xi) = 0$. Therefore, the solution of (10) can be written as

$$\pi_{k,0} = c_0\xi^{k} + \sum_{i=1}^{a}c_i\xi_i^{k}, \quad k \geq 1. \tag{15}$$

Since c_i $(i = 1, 2, 3, \ldots, a) = 0$ (otherwise, the probability $\pi_{k,0}$ tends to ∞ when k tends to ∞), we get $\pi_{k,0} = c_0\xi^{k}$ $(k \geq 1)$.

Let $k = 1$, we get $c_0 = \xi^{-1}\pi_{1,0}$, then

$$\pi_{k,0} = \pi_{1,0}\xi^{k-1}, \quad k \geq 1. \tag{16}$$

Substituting (16) into (9), we obtain

$$\pi_{1,0} = \frac{p\overline{\theta}\overline{\mu}_v(1-\xi)}{\omega}\pi_{0,0}, \tag{17}$$

where $\omega = 1 - \overline{\theta}[1 - \overline{p}\mu_v] - \overline{p}\overline{\theta}\mu_v\xi^{a}$.

Hence,

$$\pi_{k,0} = \frac{p\overline{\theta}\overline{\mu}_v(1-\xi)\xi^{k-1}}{\omega}\pi_{0,0}, \quad k \geq 1. \tag{18}$$

For $k \geq 2$, the difference equation (12) can be written as

$$\begin{aligned}&p\overline{\mu}_b\pi_{k-1,1} + \left(\overline{p}\,\overline{\mu}_b - 1\right)\pi_{k,1} + p\mu_b\pi_{a+k-1,1} + \overline{p}\mu_b\pi_{a+k,1} \\ &\quad + \frac{p\theta\overline{\theta}\overline{\mu}_v\left(p+\overline{p}\xi\right)(1-\xi)\xi^{k-2}}{\omega}\pi_{0,0} = 0.\end{aligned} \tag{19}$$

Using $\pi_{k+j,1} = E^{j}\pi_{k,1}$, $j \in Z$; $k = 1, 2, \ldots$, the auxiliary equation of (19) such that

$$\overline{p}\mu_b z^{a+1} + p\mu_b z^{a} - \left(1 - \overline{p}\,\overline{\mu}_b\right)z + p\overline{\mu}_b = 0. \tag{20}$$

Let $G(z) = \overline{p}\mu_b z^{a+1} + p\mu_b z^{a} + \overline{p}\,\overline{\mu}_b z + p\overline{\mu}_b$, obviously $G(1) = 1$, $G'(1) = (a+1)\overline{p}\mu_b + ap\mu_b + \overline{p}\,\overline{\mu}_b = a\mu_b + 1 - p$. Since $\rho = p/(a\mu_b) < 1$, that is, $p < a\mu_b$, we can see that $G'(1) > 1$. According to Hunter [25], the equation $z = G(z)$ has a unique real root in unit circle, which can be denoted by r; the other a roots can be denoted by r_i, $|r_i| \geq 1$ $(i = 1, 2, \ldots, a)$. As mentioned above, the general solution of (20) can be written as

$$z^{*} = c_0' r^{k}, \quad k \geq 1. \tag{21}$$

Using the method of nonhomogeneous difference operator, the special solution of (19) can be written as

$$\pi_{k,1}^{*} = \frac{p\theta\overline{\theta}\overline{\mu}_v\left(p+\overline{p}\xi\right)(1-\xi)\xi^{k-1}\pi_{0,0}}{\omega\varpi}, \quad k \geq 1, \tag{22}$$

where $\varpi = [(1 - \overline{p}\,\overline{\mu}_b)\xi - (p\overline{\mu}_b + p\mu_b\xi^{a} + \overline{p}\mu_b\xi^{a+1})]$.

From (21) and (22), we have

$$\pi_{k,1} = c_0' r^{k} + \frac{p\theta\overline{\theta}\overline{\mu}_v\left(p+\overline{p}\xi\right)(1-\xi)\xi^{k-1}\pi_{0,0}}{\omega\varpi}, \quad k \geq 1. \tag{23}$$

Substituting (17), (18), and (23) into (11), we obtain

$$\begin{aligned}c_0' = {}&\Bigl(p\theta\left\{\omega\varpi + \overline{\theta}\overline{\mu}_v\left\{p\varpi(1-\xi) + \left(p+\overline{p}\xi\right)\left[\varpi - \overline{p}\mu_b\left(1-\xi^{a}\right)\right]\right\}\right\} \\ &\quad\times (1-r)\Bigr)\left(r\omega\varpi\overline{p}\mu_b\left(1-r^{a}\right)\right)^{-1}\pi_{0,0}.\end{aligned} \tag{24}$$

Substituting (24) into (23), we can obtain $\pi_{k,1}$ $(k \geq 1)$.

Using $\sum_{i=0}^{\infty}\pi_{i,0} + \sum_{i=1}^{\infty}\pi_{i,1} = 1$, we can get

$$\begin{aligned}\pi_{0,0} = {}&\Bigl\{1 + \left(p\overline{\theta}\overline{\mu}_v\varpi + p\theta\overline{\theta}\overline{\mu}_v\left(p+\overline{p}\xi\right)\right)(\omega\varpi)^{-1} \\ &\quad + \Bigl(p\theta\Bigl\{\omega\varpi + \overline{\theta}\overline{\mu}_v \\ &\qquad\times\left\{p\varpi(1-\xi) + \left(p+\overline{p}\xi\right)\left[\varpi - \overline{p}\mu_b\left(1-\xi^{a}\right)\right]\right\}\Bigr\}\Bigr) \\ &\quad\times\left(\omega\varpi\overline{p}\mu_b\left(1-r^{a}\right)\right)^{-1}\Bigr\}^{-1}.\end{aligned} \tag{25}$$

$\square$

Remark 2. If $\theta \to 0$ and $a = 1$, this queueing system is equivalent to *Geom*/*Geom*/1 queueing system with a server serves customers singly. Because of $\rho_0 = p/a\mu_v < 1$ and $1 \leq a$, then $p/\mu_v < 1$. Hence, $0 < \xi = (p\overline{\mu}_v/\overline{p}\mu_v) < 1$. We have

$$\pi_{0,0} = 1 - \xi, \qquad \pi_{k,0} = (1-\xi)\xi^{k}, \quad k \geq 1, \tag{26}$$

which are matched with the results given by Tian and Ma [26].

Corollary 3. *Define J as the state of system at random slots, the steady-state probability of this system at random slots can be written as*

$$
\begin{aligned}
P\{J=0\} &= \left(1+\frac{p\bar{\theta}\bar{\mu}_v}{\omega}\right)\pi_{0,0},\\
P\{J=1\} &= \Big\{\Big(p\theta\Big\{\omega\varpi+\bar{\theta}\bar{\mu}_v\\
&\qquad \times\{p\varpi(1-\xi)+(p+\bar{p}\xi)[\varpi-\bar{p}\mu_b(1-\xi^a)]\}\Big\}\Big)\\
&\quad \times\left(\omega\varpi\bar{p}\mu_b(1-r^a)\right)^{-1}\\
&\quad +\left(p\theta\bar{\theta}\bar{\mu}_v(p+\bar{p}\xi)\right)(\omega\varpi)^{-1}\Big\}\pi_{0,0}.
\end{aligned}
\tag{27}
$$

Theorem 4. *If $|z|\le 1$, the probability generating function (PGF) of steady-state queue length at random slots is given by*

$$
\begin{aligned}
L(z) = \Big\{1+\Big\{c_0'\frac{rz}{1-rz}&+\frac{p\bar{\theta}\bar{\mu}_v(1-\xi)z}{\omega(1-\xi z)}\\
&+\frac{p\theta\bar{\theta}\bar{\mu}_v(p+\bar{p}\xi)(1-\xi)z}{\omega\varpi(1-\xi z)}\Big\}\Big\}\pi_{0,0},
\end{aligned}
\tag{28}
$$

and the average queue length is

$$
E(L)=\left\{\frac{rc_0'}{(1-r)^2}+\frac{[\varpi+\theta(p+\bar{p}\xi)]p\bar{\theta}\bar{\mu}_v}{\omega\varpi(1-\xi)}\right\}\pi_{0,0}. \tag{29}
$$

Proof. In the steady state, the queue length L at random slots has marginal distribution as

$$
\begin{aligned}
P\{L=0\} &= \pi_{0,0},\\
P\{L=k\} &= \pi_{k,0}+\pi_{k,1}\\
&= c_0'r^k+\frac{p\bar{\theta}\bar{\mu}_v(1-\xi)\xi^{k-1}}{\omega}\pi_{0,0}\\
&\quad +\frac{p\theta\bar{\theta}\bar{\mu}_v(p+\bar{p}\xi)(1-\xi)\xi^{k-1}\pi_{0,0}}{\omega\varpi},\quad (k\ge 1).
\end{aligned}
\tag{30}
$$

Using $L(z)=P\{L=0\}+\sum_{k=1}^{\infty}P\{L=k\}z^k$, we can obtain (28) easily; furthermore, taking derivation to $L(z)$ and let $z=1$, we can get (29). □

4. The Waiting Time Distribution

Let the random variable T_q be the total waiting time of an arriving customer in the queue, N_w represents the number of the customers in the system. Assume that an arriving customer finds i customers in the system, the conditional distribution law that he waits for k slots is subject to $w_i(k)=P\{T_q=k/N_w=i\}$, $i=0,1,2,\ldots$, $k=0,1,2,\ldots$, and PGF is $W_i(z)=\sum_{k=0}^{\infty}w_i(k)z^k$. In the steady state, the waiting time with finite mean w_q has PGF $w_q(z)=\sum_{i=0}^{\infty}\pi_{il}W_i(z)$, $l=0,1$.

Theorem 5. *In the steady state, the PGF of waiting time for an arriving customer is given by*

$$
\begin{aligned}
w_q(z) = \pi_{0,0} &+ \frac{p\theta\bar{\theta}\bar{\mu}_v(p+\bar{p}\xi)(1-\xi^a)q(z)}{\omega\varpi}\pi_{0,0}\\
&+\frac{p\bar{\mu}_v(1-\xi^a)\tilde{q}(\bar{\theta}z)}{\omega}\pi_{0,0}\\
&+\frac{p\theta\bar{\theta}\bar{\mu}_v(p+\bar{p}\xi)\{1-q(z)\xi^a-[1-q(z)]\xi^{a-1}\}\xi^a q(z)}{\omega\varpi[1-q(z)\xi^a]}\pi_{0,0}\\
&+\frac{\{1-\tilde{q}(\bar{\theta}z)\xi^a-[1-\tilde{q}(\bar{\theta}z)]\xi^{a-1}\}p\bar{\mu}_v\tilde{q}(\bar{\theta}z)\xi^a}{\omega[1-\tilde{q}(\bar{\theta}z)\xi^a]}\pi_{0,0}\\
&+c_0'\left\{\frac{r-r^{2a}}{1-r}+\frac{q(z)r^{2a}(1-r^a)}{(1-r)[1-q(z)r^a]}\right\}q(z)\\
&+\frac{p\theta\bar{\mu}_v q(z)(1-\xi^a)\xi^{a-1}}{\omega\left(1-\bar{\mu}_v z\bar{\theta}\right)[1-\xi^a q(z)]\left[1-\xi^a\tilde{q}(z\bar{\theta})\right]}\pi_{0,0},
\end{aligned}
\tag{31}
$$

and the average waiting time as

$$
\begin{aligned}
w_q = \frac{1}{\mu_b}\Bigg\{&\frac{\left(1-r^a+r^{2a-1}\right)r}{(1-r^a)(1-r)}c_0'\\
&+\frac{p\theta\bar{\theta}\bar{\mu}_v(p+\bar{p}\xi)\left(1-\xi^a+\xi^{2a-1}\right)}{\omega\varpi(1-\xi^a)}\pi_{0,0}\\
&+\frac{p\theta\bar{\mu}_v\xi^{a-1}}{\omega\left(1-\bar{\mu}_v\bar{\theta}-\mu_v\bar{\theta}\xi^a\right)(1-\xi^a)}\pi_{0,0}\Bigg\}\\
&+\frac{p\mu_v\bar{\mu}_v\bar{\theta}}{\omega}\\
&\times\Big\{\Big(\Big\{\left[1+\xi^{a-1}+\theta\left(\xi^{a-1}-\xi^{2a-1}\right)-2\xi^a\right]\\
&\qquad\times\left(1-\bar{\mu}_v\bar{\theta}\right)-\mu_v\bar{\theta}\left(\xi^{a-1}-\xi^a\right)\xi^a\Big\}\xi^a\Big)\\
&\quad\times\left(\left(1-\bar{\mu}_v\bar{\theta}\right)\left(1-\bar{\mu}_v\bar{\theta}-\mu_v\bar{\theta}\xi^a\right)^2\right)^{-1}\\
&\quad+(1-\xi^a)\left(\left(1-\bar{\mu}_v\bar{\theta}\right)^2\right)^{-1}\Big\}\pi_{0,0}\\
&+\frac{p\theta\bar{\theta}\bar{\mu}_v^2\xi^{a-1}}{\omega\left(1-\bar{\mu}_v\bar{\theta}\right)\left(1-\bar{\mu}_v\bar{\theta}-\mu_v\bar{\theta}\xi^a\right)}\pi_{0,0}.
\end{aligned}
\tag{32}
$$

Proof. Firstly, we define $\lfloor x\rfloor$ as a greatest integer function (floor), which returns the greatest integer less than or equal to a real number x. An arriving customer may observe the system in any one of the following two cases.

Case 1. When $T_q = 0$, this case has no customers in the system and the server is on vacation, that is,

$$w_0(0) = P\left\{\frac{T_q = 0}{N_w = 0}\right\} = \pi_{0,0}. \tag{33}$$

Case 2. When $T_q = m$, $(m \geq 1)$, there are two cases as follows.

(1) The server is on a normal busy period and i customers in the system, meantime an arriving customer cannot go into the queue being served immediately.

Under this condition, an arriving customer has to wait for one period of service for $1 \leq i \leq a$ and $\lfloor i/a \rfloor$ periods of service for $i > a$. Each period of service S_{b_i} $(i = 1, 2, \ldots)$ is independent and geometrically distributed with p.m.f. $P\{S_{b_i} = k\} = \mu_b \overline{\mu}_b^{k-1}$, $k \geq 1$, $\overline{\mu}_b = 1 - \mu_b$, which has PGF as $\mu_b z/(1 - \overline{\mu}_b z)$.

We have

$$w_i(m) = \begin{cases} P\left\{\dfrac{s_{b_1} = m}{N_w = i}\right\}, & i \leq a, \\ P\left\{s_{b_1} + s_{b_2} + \cdots + s_{b\lfloor i/a \rfloor} = \dfrac{m}{N_w = i}\right\}, & i > a. \end{cases} \tag{34}$$

Hence,

$$\begin{aligned} W_i(z) &= \sum_{m=1}^{\infty} w_i(m) z^m = \frac{\mu_b z}{1 - \overline{\mu}_b z}, \quad 1 \leq i \leq a, \\ W_i(z) &= \sum_{m=1}^{\infty} w_i(m) z^m = \left(\frac{\mu_b z}{1 - \overline{\mu}_b z}\right)^{\lfloor i/a \rfloor}, \quad i > a. \end{aligned} \tag{35}$$

Let $q(z) = u_b z/(1 - \overline{\mu}_b z)$, the PGF of the waiting time is given by

$$\begin{aligned} &\frac{p\theta\overline{\theta}\overline{\mu}_v (p + \overline{p}\xi)(1 - \xi^a) q(z)}{\omega\varpi} \pi_{0,0} \\ &\quad + c_0' \left\{\frac{r - r^{2a}}{1 - r} + \frac{q(z) r^{2a}(1 - r^a)}{(1 - r)[1 - q(z) r^a]}\right\} q(z) \\ &\quad + \frac{p\theta\overline{\theta}\overline{\mu}_v (p + \overline{p}\xi)\{1 - q(z)\xi^a - [1 - q(z)]\xi^{a-1}\}\xi^a q(z)}{\omega\varpi[1 - q(z)\xi^a]} \pi_{0,0}. \end{aligned} \tag{36}$$

(2) An arriving customer finds the server is on vacation.

Let s_{v_j} be the jth length of the period of service with service rate μ_v and let $s_v^{(j)}$ be the sum of the length of j periods of service with service rate μ_v, $j = 1, 2, 3, \ldots$, where $s_v^{(0)} = 0$. Each period of service S_{v_i} $(i = 1, 2, \ldots)$ is mutually independent and geometrically distributed with p.m.f. $P\{S_{v_i} = k\} = \mu_v \overline{\mu}_v^{k-1}$, $k \geq 1$, $\overline{\mu}_v = 1 - \mu_v$. There are two cases in this condition.

(A) A vacation is going on whereas all of the arrived customers have been served. Then, an arriving customer has to wait for one period of service for $i \leq a$ and $\lfloor i/a \rfloor$ periods of service for $i > a$. We have

$$\begin{aligned} w_i(m) &= P\left\{\frac{s_{v_1} = m}{N_w = i}; V \geq m\right\}, \quad 1 \leq i \leq a, \\ W_i(z) &= \sum_{m=1}^{\infty} w_i(m) z^m = \frac{\mu_v z}{1 - \overline{\theta}\overline{\mu}_v z}, \quad 1 \leq i \leq a, \\ w_i(m) &= P\left\{s_{v_1} + s_{v_2} + \cdots + s_{v\lfloor i/a \rfloor} = \frac{m}{N_w = i}; V \geq m\right\}, \quad i > a, \\ W_i(z) &= \sum_{m=1}^{\infty} w_i(m) z^m = \frac{1}{\overline{\theta}}\left(\frac{\mu_v \overline{\theta} z}{1 - \overline{\mu}_v \overline{\theta} z}\right)^{\lfloor i/a \rfloor}, \quad i > a. \end{aligned} \tag{37}$$

Let $\tilde{q}(z) = \mu_v z/(1 - \overline{\mu}_v z)$. The PGF of the waiting time is given by

$$\begin{aligned} &\frac{p\overline{\mu}_v (1 - \xi^a)}{\omega} \tilde{q}(\overline{\theta} z) \pi_{0,0} \\ &\quad + \frac{\{1 - \tilde{q}(\overline{\theta} z)\xi^a - [1 - \tilde{q}(\overline{\theta} z)]\xi^{a-1}\} p\overline{\mu}_v \tilde{q}(\overline{\theta} z)\xi^a}{\omega[1 - \tilde{q}(\overline{\theta} z)\xi^a]} \pi_{0,0}. \end{aligned} \tag{38}$$

(B) If a vacation is over and j $(j = 0,\ 1 \leq i \leq a;\ 1 \leq j < \lfloor i/a \rfloor,\ i > a)$ periods of service ended, the service rate is converted to μ_b from μ_v and the normal busy period begins. The waiting time of an arriving customer should be equal to the sum of the server's vacation times and one period of service for $1 \leq i \leq a$ and $\lfloor i/a \rfloor - j$ periods of service for $i > a$. The service rate of $\lfloor i/a \rfloor - j$ periods of service is μ_b. We have

$$\begin{aligned} w_i(m) &= P\left\{\frac{T_q = m}{N = i};\ V < s_{v_1}\right\}, \quad 1 \leq i \leq a, \\ w_i(m) &= \sum_{j=0}^{\lfloor i/a \rfloor - 1} P\left\{\frac{T_q = m}{N = i};\ s_v^{(j)} \leq V < s_v^{(j+1)}\right\}, \quad i > a. \end{aligned} \tag{39}$$

Hence,

$$\begin{aligned} &w_i(m) \\ &\quad = \sum_{j=0}^{\lfloor i/a \rfloor - 1} P\left\{\frac{T_q = m}{N = i};\ s_v^{(j)} \leq V < s_v^{(j+1)}\right\} \\ &\quad = \sum_{j=0}^{\lfloor i/a \rfloor - 1} P\left\{V + s_{b_1} + s_{b_2} + \cdots + s_{b\lfloor i/a \rfloor - j} = m;\ s_v^{(j)} \leq V < s_v^{(j+1)}\right\} \end{aligned}$$

Table 1: Queue length distributions at random slots for $a = 1$, $p = 0.3$, $\mu_v = 0.4$, $\mu_b = 0.6$, and $\theta = 0.7$.

n	$\pi_{n,0}$	$\pi_{n,1}$	n	$\widetilde{\pi}_{n,0}$	$\widetilde{\pi}_{n,1}$
0	0.5580	0	0	0.3906	0
1	0.0363	0.3128	1	0.0603	0.3539
2	0.0022	0.0679	2	0.0037	0.1501
3	$1.39E-04$	0.0128	3	$2.32E-04$	0.0299
4	$8.64E-06$	0.0023	4	$1.44E-05$	0.0055
5	$5.35E-07$	$4.17E-04$	5	$8.90E-07$	$9.92E-04$
6	$3.32E-08$	$7.43E-05$	6	$5.51E-08$	$1.77E-04$
7	$2.05E-09$	$1.32E-05$	7	$3.42E-09$	$3.16E-05$
8	$1.27E-10$	$2.35E-06$	8	$2.12E-10$	$5.62E-06$
9	$7.89E-12$	$4.19E-07$	9	$1.31E-11$	$1.00E-06$
10	$4.89E-13$	$7.45E-08$	10	$8.13E-13$	$1.78E-07$
11	$3.03E-14$	$1.32E-08$	11	$5.04E-14$	$3.16E-08$
12	$1.88E-15$	$2.36E-09$	12	$3.12E-15$	$5.62E-09$
13	$1.16E-16$	$4.19E-10$	13	$1.93E-16$	$1.00E-09$
14	$7.21E-18$	$7.45E-11$	14	$1.20E-17$	$1.78E-10$
15	$4.47E-19$	$1.33E-11$	15	$7.43E-19$	$3.16E-11$
Sum	1		Sum	1	

$E(L) = 0.2741$, $E(w_q) = 1.2783$.

Table 2: Queue length distributions at random slots for $a = 4$, $p = 0.3$, $\mu_v = 0.4$, $\mu_b = 0.6$, and $\theta = 0.7$.

n	$\pi_{n,0}$	$\pi_{n,1}$	n	$\widetilde{\pi}_{n,0}$	$\widetilde{\pi}_{n,1}$
0	0.6092	0	0	0.4265	0
1	0.0394	0.263	1	0.0654	0.3313
2	0.0024	0.0623	2	0.0041	0.1319
3	$1.51E-04$	0.012	3	$2.51E-04$	0.0277
4	$9.37E-06$	0.0022	4	$1.56E-05$	0.0052
5	$5.80E-07$	$3.95E-04$	5	$9.65E-07$	$9.39E-04$
6	$3.60E-08$	$7.05E-05$	6	$5.98E-08$	$1.68E-04$
7	$2.23E-09$	$1.26E-05$	7	$3.70E-09$	$2.99E-05$
8	$1.38E-10$	$2.23E-06$	8	$2.30E-10$	$5.33E-06$
9	$8.56E-12$	$3.97E-07$	9	$1.42E-11$	$9.48E-07$
10	$5.30E-13$	$7.07E-08$	10	$8.81E-13$	$1.69E-07$
11	$3.29E-14$	$1.26E-08$	11	$5.46E-14$	$3.00E-08$
12	$2.04E-15$	$2.24E-09$	12	$3.38E-15$	$5.33E-09$
13	$1.26E-16$	$3.98E-10$	13	$2.10E-16$	$9.49E-10$
14	$7.82E-18$	$7.07E-11$	14	$1.30E-17$	$1.69E-10$
15	$4.84E-19$	$1.26E-11$	15	$8.05E-19$	$3.00E-11$
Sum	1		Sum	1	

$E(L) = 0.2597$, $E(w_q) = 0.8425$.

$$\begin{aligned} &= \sum_{j=0}^{\lfloor i/a \rfloor - 1} \sum_{u=1}^{m-\lfloor i/a \rfloor + j} P\{V = u\} \\ &\quad \times P\left\{s_{b_1} + s_{b_2} + \cdots + s_{b\lfloor i/a \rfloor - j} = m - u\right\} \\ &\quad \times P\left\{s_v^{(j)} \le u < s_v^{(j+1)}\right\}, \quad i > a. \end{aligned} \tag{40}$$

The PGF of the waiting time can be given by

$$\frac{p\theta\overline{\mu}_v q(z)\left(1 - \xi^a\right)\xi^{a-1}}{\omega\left(1 - \overline{\mu}_v z\overline{\theta}\right)\left[1 - \xi^a q(z)\right]\left[1 - \xi^a \widetilde{q}\left(z\overline{\theta}\right)\right]}\pi_{0,0}. \tag{41}$$

Adding (33)–(41), we can get (31), using $(dw_q(z)/dz)\mid_{z=1}$, we can obtain (32). ☐

Table 3: Queue length distributions at random slots for $a = 5$, $p = 0.3$, $\mu_v = 0.4$, $\mu_b = 0.6$, and $\theta = 0.7$.

n	$\pi_{n,0}$	$\pi_{n,1}$	n	$\tilde{\pi}_{n,0}$	$\tilde{\pi}_{n,1}$
0	0.6095	0	0	0.4266	0
1	0.0394	0.2627	1	0.0655	0.3312
2	0.0024	0.0622	2	0.0041	0.1319
3	$1.51E-04$	0.012	3	$2.51E-04$	0.0277
4	$9.37E-06$	0.0022	4	$1.56E-05$	0.0052
5	$5.81E-07$	$3.95E-04$	5	$9.65E-07$	$9.39E-04$
6	$3.60E-08$	$7.05E-05$	6	$5.98E-08$	$1.68E-04$
7	$2.23E-09$	$1.25E-05$	7	$3.71E-09$	$2.99E-05$
8	$1.38E-10$	$2.23E-06$	8	$2.30E-10$	$5.33E-06$
9	$8.56E-12$	$3.97E-07$	9	$1.42E-11$	$9.48E-07$
10	$5.30E-13$	$7.06E-08$	10	$8.82E-13$	$1.69E-07$
11	$3.29E-14$	$1.26E-08$	11	$5.46E-14$	$3.00E-08$
12	$2.04E-15$	$2.23E-09$	12	$3.39E-15$	$5.33E-09$
13	$1.26E-16$	$3.97E-10$	13	$2.10E-16$	$9.48E-10$
14	$7.82E-18$	$7.07E-11$	14	$1.30E-17$	$1.69E-10$
15	$4.85E-19$	$1.26E-11$	15	$8.06E-19$	$3.00E-11$
Sum	1		Sum	1	

$E(L) = 0.2597$, $E(w_q) = 0.8421$.

Table 4: Queue length distributions at random slots for $a = 10$, $p = 0.3$, $\mu_v = 0.4$, $\mu_b = 0.6$, and $\theta = 0.7$.

n	$\pi_{n,0}$	$\pi_{n,1}$	n	$\tilde{\pi}_{n,0}$	$\tilde{\pi}_{n,1}$
0	0.6095	0	0	0.4267	0
1	0.0394	0.2627	1	0.0655	0.3312
2	0.0024	0.0622	2	0.0041	0.1318
3	$1.51E-04$	0.012	3	$2.51E-04$	0.0277
4	$9.37E-06$	0.0022	4	$1.56E-05$	0.0052
5	$5.81E-07$	$3.95E-04$	5	$9.65E-07$	$9.38E-04$
6	$3.60E-08$	$7.05E-05$	6	$5.98E-08$	$1.68E-04$
7	$2.23E-09$	$1.25E-05$	7	$3.71E-09$	$2.99E-05$
8	$1.38E-10$	$2.23E-06$	8	$2.30E-10$	$5.33E-06$
9	$8.56E-12$	$3.97E-07$	9	$1.42E-11$	$9.48E-07$
10	$5.30E-13$	$7.06E-08$	10	$8.82E-13$	$1.69E-07$
11	$3.29E-14$	$1.26E-08$	11	$5.46E-14$	$3.00E-08$
12	$2.04E-15$	$2.23E-09$	12	$3.39E-15$	$5.33E-09$
13	$1.26E-16$	$3.97E-10$	13	$2.10E-16$	$9.48E-10$
14	$7.82E-18$	$7.07E-11$	14	$1.30E-17$	$1.69E-10$
15	$4.85E-19$	$1.26E-11$	15	$8.06E-19$	$3.00E-11$
Sum	1		Sum	1	

$E(L) = 0.2597$, $E(w_q) = 0.8421$.

5. Outside Observer's Observation Epoch Distributions

For an early arrive system, since an outside observer's observation epoch falls in the time interval after a potential arrival and before a potential batch departure, let, $\tilde{\pi}_{n,0}$, $\tilde{\pi}_{n,1}$ be n $(n \geq 0)$ customers in the system and the server is on vacation (including the servicing customers), n customers in the system and the server is in regular busy period (including the servicing customers). Through observing the relationship between random slot t and the outside observer's observation epoch $(*)$, we have

$$\begin{gathered}\tilde{\pi}_{0,0} = \overline{p}\pi_{0,0}, \\ \tilde{\pi}_{n,0} = \overline{p}\overline{\theta}\pi_{n,0} + p\overline{\theta}\pi_{n-1,0}, \quad (n \geq 1), \\ \tilde{\pi}_{1,1} = \overline{p}\pi_{1,1} + p\theta\pi_{0,0} + \overline{p}\theta\pi_{1,0}, \\ \tilde{\pi}_{n,1} = \overline{p}\pi_{n,1} + p\pi_{n-1,1} + \overline{p}\theta\pi_{n,0} + p\theta\pi_{n-1,0}, \quad (n \geq 2).\end{gathered} \tag{42}$$

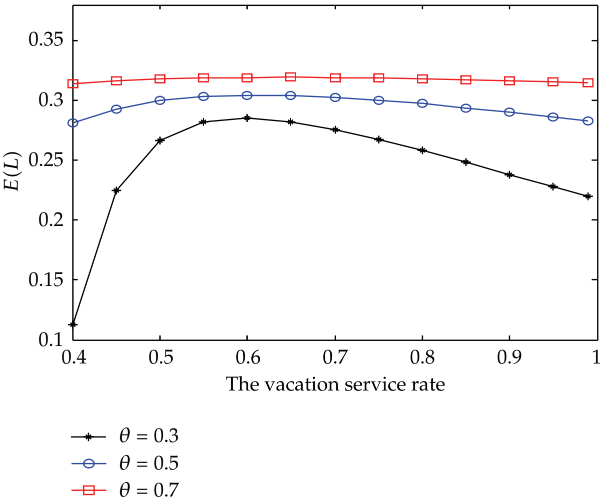

FIGURE 2: Effect of μ_v on the average queue length.

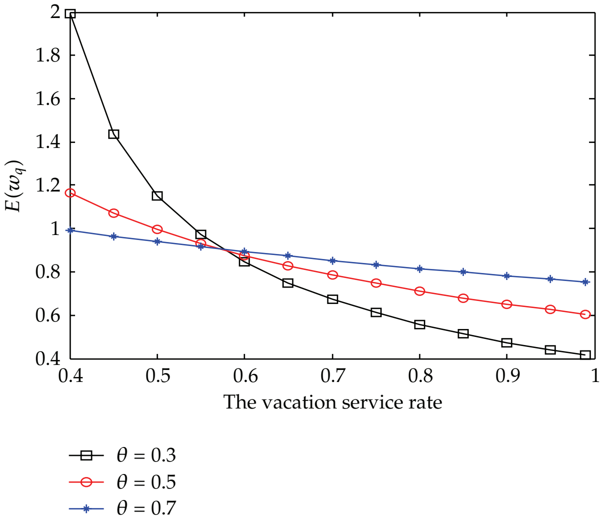

FIGURE 3: Effect of μ_v on the average waiting time.

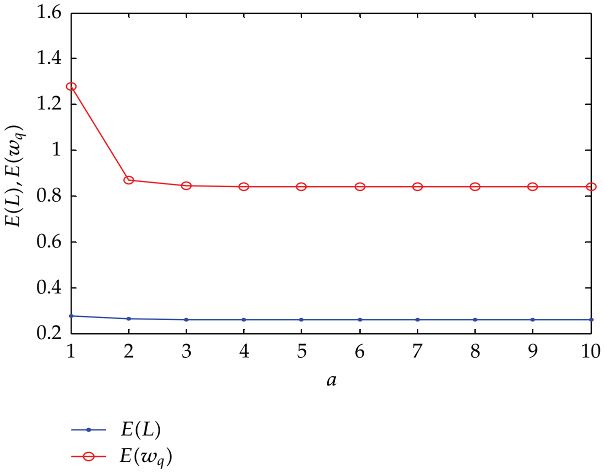

FIGURE 4: Effect of a on the average queue length and the average waiting time.

6. Numerical Results and the Sensitivity Analysis of this System

In this section, we present some numerical results in self-explanatory tables and graphs for queue length distributions at random slots and all the numerical results have been obtained using the results derived in this paper.

We observe that $\pi_{n,0}$ and $\tilde{\pi}_{n,0}$ monotonically increase whereas $\pi_{n,1}$ and $\tilde{\pi}_{n,1}$ monotonically decrease as a increases in Tables 1–4. This situation continues until a is equal to some constant; all data will tend to be a steady state. The above description is consistent with actual situation. In the meantime, $E(L)$ and $E(w_q)$ monotonically decease as a increases. In Figures 2 and 3, fixing $a = 10$, $p = 0.3$, $\mu_b = 0.5$, and $\theta = 0.3, 0.5, 0.7$, we have plotted the effect of various vacation service rates on the average queue length and the average waiting time, respectively. We observe that the average queue length and the average waiting time decrease as the vacation service rate increases. In Figure 4, fixing $p = 0.3$, $\mu_v = 0.4$, $\mu_b = 0.6$, and $\theta = 0.7$, the steady-state average queue length equals 0.2597 from $a = 4$ on and the steady-state average waiting time equals 0.8421 from $a = 5$ on. They do not change as the batch size increases.

7. Conclusions

A $Geom/Geom^{[a]}/1/MWV$ queueing system has been investigated. Assume that the server takes a working vacation after emptying the system in regular busy period. By using embedded Markov chain approach and the method of non-homogeneous and homogeneous difference operator, the number of customers of the whole system at random slots has been discussed. This is different from general batch service queue literatures (excluding customers being served). The waiting time for an arriving customer and numerical results are obtained. In the future, further study such as $Geom^{[a]}/Geom^{[b]}/1/MWV$ queue will be the research topic using similar idea and method.

Acknowledgments

The authors wish to thank the referee for his careful reading of the paper and for his helpful suggestion. Meantime, this work is supported by the National Natural Science Foundation of China (no. 71171138), and the Talent Introduction Foundation of Sichuan University of Science & Engineering (2012RC23).

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