

Research Article

Quantitative Global Estimates for Generalized Double Szasz-Mirakjan Operators

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We introduce the generalized double Szász-Mirakjan operators in this paper. We obtain several quantitative estimates for these operators. These estimates help us to determine some function classes \mathcal{S} (including some Lipschitz-type spaces) which provide uniform convergence on the whole domain $[0, \infty) \times [0, \infty)$.

1. Introduction

The well-known Szász-Mirakjan operators are defined on the space \mathcal{A}_1 as follows:

$$S_n(f; x) = e^{-nx} \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \frac{(nx)^k}{k!}, \quad (1)$$

where \mathcal{A}_1 is the set of all real functions on $[0, \infty)$ such that the right-hand side in (1) make sense for all $n > 0$ and $x \in [0, \infty)$. By modifying the Szász-Mirakjan operators as

$$D_n(f; x) = e^{-nu_n(x)} \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \frac{(nu_n(x))^k}{k!}, \quad (2)$$

where $\{u_n(x)\}$ is a sequence of real-valued, continuous functions defined on $[0, \infty)$ with $0 \leq u_n(x) < \infty$, it has been shown in [1] that if one let

$$u_n^*(x) := \frac{-1 + \sqrt{4n^2x^2 + 1}}{2n}, \quad n \in \mathbb{N}, \quad (3)$$

then the operators defined by

$$D_n^*(f; x) := S_n(f; u_n^*(x)) \quad (4)$$

preserve the test function $e_2(x) = x^2$ and provide a better error estimation than the operators $S_n(f; x)$ for all

$f \in C_B([0, \infty))$ and for each $x \in [0, \infty)$. Note that $C_B([0, \infty))$ denotes the space of all bounded and continuous functions on $[0, \infty)$. On the other hand, by letting

$$\nu_n(x) := x - \frac{1}{2n}; \quad n \in \mathbb{N}, \quad (5)$$

it has been shown in [2] that the operators defined by

$$V_n^*(f; x) := S_n(f; \nu_n(x)) \quad (6)$$

do not preserve the test functions $e_1(x) = x$ and $e_2(x) = x^2$ but provide the best error estimation among all the Szász-Mirakjan operators for all $f \in C_B([0, \infty))$ and for each $x \in [1/2, \infty)$. For the other linear positive operator families which preserve $e_2(x) = x^2$, we refer [3–9]. On the other hand, in [10, 11] the authors considered some operators preserving $e_1(x) = x$.

Favard was the first to introduce the double Szász-Mirakjan operators [12]:

$$S_n(f; x, y) = e^{-n(x+y)} \sum_{k,l=0}^{\infty} f\left(\frac{k}{n}, \frac{l}{n}\right) \frac{(nx)^k}{k!} \frac{(ny)^l}{l!}, \quad f \in \mathcal{A}_2, \quad (7)$$

where \mathcal{A}_2 is the set of all real functions on $[0, \infty) \times [0, \infty)$ such that the right-hand side in (7) has a meaning for all $n > 0$ and $x, y \in [0, \infty)$. Recently, Dirik and Demirci have

introduced and investigated different variants of the general double Szász-Mirakjan operators:

$$\begin{aligned} D_n(f; x, y) &: S_n(f; u_n(x), v_n(y)) \\ &= e^{-n(u_n(x)+v_n(y))} \\ &\quad \times \sum_{k,l=0}^{\infty} f\left(\frac{k}{n}, \frac{l}{n}\right) \times \frac{(nu_n(x))^k}{k!} \frac{(nv_n(y))^l}{l!}, \\ &\quad f \in \mathcal{A}_2. \end{aligned} \quad (8)$$

In [13], they considered the case of operators

$$\begin{aligned} u_n^{(1)}(x) &:= \frac{-1 + \sqrt{4n^2x^2 + 1}}{2n}, \\ v_n^{(1)}(y) &:= \frac{-1 + \sqrt{4n^2y^2 + 1}}{2n}, \\ n &\in \mathbb{N}, \end{aligned} \quad (9)$$

which preserve the test function $e_{2,0}(x, y) + e_{0,2}(x, y) := x^2 + y^2$ and provide a better error estimation than the operators $S_n(f; x, y)$ for all $f \in C_B([0, \infty) \times [0, \infty))$ and for each $x, y \in [0, \infty)$. On the other hand, in [14], they considered the case

$$\begin{aligned} u_n^{(2)}(x, \alpha) &:= \frac{-(n\alpha + 1) + \sqrt{4n^2(x^2 + \alpha x) + (n\alpha + 1)^2}}{2n}, \\ v_n^{(2)}(y, \beta) &:= \frac{-(n\beta + 1) + \sqrt{4n^2(y^2 + \beta y) + (n\beta + 1)^2}}{2n}, \\ n &\in \mathbb{N}, \alpha, \beta \in \mathbb{R}. \end{aligned} \quad (10)$$

Note that for this case, the operators $D_n(f; x, y)$ do not preserve any test function (i.e., $e_{0,0}(x, y) = 1$, $e_{1,0}(x, y) = x$, $e_{0,1}(x, y) = y$, and $e_{2,0}(x, y) + e_{0,2}(x, y) = x^2 + y^2$) but provide a better error estimation than the operators $S_n(f; x, y)$ for all $f \in C_B([0, \infty) \times [0, \infty))$ and $x, y \in [0, 1]$.

Finally, we should note that, following the similar arguments as used in [2], the best error estimation among all the general double Szász-Mirakjan operators can be obtained from the case:

$$u_n^{(3)}(x) := x - \frac{1}{2n}, \quad v_n^{(3)}(y) := y - \frac{1}{2n}, \quad n \in \mathbb{N}, \quad (11)$$

for all $f \in C_B([0, \infty) \times [0, \infty))$ and $x, y \in [1/2, \infty)$.

For the operators $D_n(f; x, y)$ the following Lemma is straightforward.

Lemma 1. Let $\mathbf{x} = (x, y)$, $\mathbf{t} = (t, s)$, $e_{i,j}(\mathbf{x}) = x^i y^j$, $i, j = 0, 1, 2$, and $\psi_{\mathbf{x}}^2(\mathbf{t}) = \|\mathbf{t} - \mathbf{x}\|^2$. Then, for each $x, y \geq 0$ and $n > 1$, one has

- (a) $D_n(e_{0,0}; x, y) = 1$,
- (b) $D_n(e_{1,0}; x, y) = u_n(x)$, $D_n(e_{0,1}; x, y) = v_n(y)$,

$$(c) D_n(e_{2,0} + e_{0,2}; x, y) = u_n^2(x) + v_n^2(y) + ((u_n(x) + v_n(y))/n),$$

$$(d) D_n(\psi_{\mathbf{x}}^2(\mathbf{t}); x, y) = (u_n(x) - x)^2 + (v_n(y) - y)^2 + ((u_n(x) + v_n(y))/n).$$

2. Global Results

In this section we first introduce the following Lipschitz-type space:

$$\begin{aligned} \text{Lip}_M^*(\alpha) := \left\{ f \in C([0, \infty) \times [0, \infty)) : \right. \\ |f(\mathbf{t}) - f(\mathbf{x})| \leq M \frac{\|\mathbf{t} - \mathbf{x}\|^\alpha}{(\|\mathbf{t}\| + x + y)^{\alpha/2}}, \\ \left. t, s; x, y \in (0, \infty) \right\}, \end{aligned} \quad (12)$$

where $\mathbf{t} = (t, s)$, $\mathbf{x} = (x, y)$, M is any positive constant, and $0 < \alpha \leq 1$.

We should note that this space is the bivariate extension of Lipschitz-type space considered earlier by Szasz [15]. For the space $\text{Lip}_M^*(\alpha)$ with $0 < \alpha \leq 1$, we have the following approximation result.

Theorem 2. For any $f \in \text{Lip}_M^*(\alpha)$, $\alpha \in (0, 1]$ and for each $x, y \in (0, \infty)$, $n \in \mathbb{N}$, one has

$$\begin{aligned} &|D_n(f; x, y) - f(x, y)| \\ &\leq \frac{M}{(x + y)^{\alpha/2}} \left[(u_n(x) - x)^2 + (v_n(y) - y)^2 \right. \\ &\quad \left. + \frac{u_n(x) + v_n(y)}{n} \right]^{\alpha/2}. \end{aligned} \quad (13)$$

Proof. Take $\alpha = 1$. Then, for $f \in \text{Lip}_M^*(1)$ and for each $x, y \in (0, \infty)$, we get

$$\begin{aligned} &|D_n(f; x, y) - f(x, y)| \leq D_n(|f(t, s) - f(x, y)|; x, y) \\ &\leq MD_n\left(\frac{\|\mathbf{t} - \mathbf{x}\|}{(\|\mathbf{t}\| + x + y)^{1/2}}; x, y\right) \\ &\leq \frac{M}{(x + y)^{1/2}} D_n(\|\mathbf{t} - \mathbf{x}\|; x, y). \end{aligned} \quad (14)$$

Using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} & |D_n(f; x, y) - f(x, y)| \\ & \leq \frac{M}{(x+y)^{1/2}} \sqrt{D_n(\psi_x^2(t); x, y)} \\ & = \frac{M}{(x+y)^{1/2}} \\ & \quad \times \sqrt{(u_n(x)-x)^2 + (v_n(y)-y)^2 + \frac{u_n(x)+v_n(y)}{n}}. \end{aligned} \quad (15)$$

Secondly let $0 < \alpha < 1$. Then, for $f \in \text{Lip}_M^*(\alpha)$ and for each $x, y \in (0, \infty)$, we have

$$\begin{aligned} & |D_n(f; x, y) - f(x, y)| \leq D_n(|f(t, s) - f(x, y)|; x, y) \\ & \leq MD_n\left(\frac{\|\mathbf{t}-\mathbf{x}\|^\alpha}{(\|\mathbf{t}\|+x+y)^{\alpha/2}}; x, y\right) \\ & \leq \frac{M}{(x+y)^{\alpha/2}} D_n(\|\mathbf{t}-\mathbf{x}\|^\alpha; x, y). \end{aligned} \quad (16)$$

Applying the Hölder inequality with $p = 2/\alpha$ and $q = 2/(2-\alpha)$, we have, for any $f \in \text{Lip}_M^*(\alpha)$,

$$\begin{aligned} & |D_n(f; x, y) - f(x, y)| \leq \frac{M}{(x+y)^{\alpha/2}} [D_n(\psi_x^2(t); x, y)]^{\alpha/2} \\ & = \frac{M}{(x+y)^{\alpha/2}} \\ & \quad \times \left[(u_n(x)-x)^2 + (v_n(y)-y)^2 \right. \\ & \quad \left. + \frac{u_n(x)+v_n(y)}{n} \right]^{\alpha/2}, \end{aligned} \quad (17)$$

Hence, the result. \square

The following lemma will be used in the rest of the paper.

Lemma 3. One has, for each $x, y > 0$,

$$\begin{aligned} & D_n\left(\sqrt{(\sqrt{t}-\sqrt{x})^2 + (\sqrt{s}-\sqrt{y})^2}; x, y\right) \\ & \leq \frac{1}{\sqrt{x}} \sqrt{(u_n(x)-x)^2 + \frac{u_n(x)}{n}} \\ & \quad + \frac{1}{\sqrt{y}} \sqrt{(v_n(y)-y)^2 + \frac{v_n(y)}{n}}. \end{aligned} \quad (18)$$

Proof. Using the fact that $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ ($a, b \geq 0$), we get

$$\begin{aligned} & D_n\left(\sqrt{(\sqrt{t}-\sqrt{x})^2 + (\sqrt{s}-\sqrt{y})^2}; x, y\right) \\ & = e^{-n(u_n(x)+v_n(y))} \sum_{k,l=0}^{\infty} \sqrt{\left(\sqrt{\frac{k}{n}} - \sqrt{x}\right)^2 + \left(\sqrt{\frac{l}{n}} - \sqrt{y}\right)^2} \\ & \quad \times \frac{(nu_n(x))^k}{k!} \frac{(nv_n(y))^l}{l!} \\ & \leq e^{-nu_n(x)} \sum_{k=0}^{\infty} \left| \sqrt{\frac{k}{n}} - \sqrt{x} \right| \frac{(nu_n(x))^k}{k!} \\ & \quad + e^{-nv_n(y)} \sum_{l=0}^{\infty} \left| \sqrt{\frac{l}{n}} - \sqrt{y} \right| \frac{(nv_n(y))^l}{l!} \\ & = e^{-nu_n(x)} \sum_{k=0}^{\infty} \frac{|k/n-x|}{\sqrt{k/n}+\sqrt{x}} \frac{(nu_n(x))^k}{k!} \\ & \quad + e^{-nv_n(y)} \sum_{l=0}^{\infty} \frac{|l/n-y|}{\sqrt{l/n}+\sqrt{y}} \frac{(nv_n(y))^l}{l!} \\ & \leq \frac{e^{-nu_n(x)}}{\sqrt{x}} \sum_{k=0}^{\infty} \left| \frac{k}{n} - x \right| \frac{(nu_n(x))^k}{k!} \\ & \quad + \frac{e^{-nv_n(y)}}{\sqrt{y}} \sum_{l=0}^{\infty} \left| \frac{l}{n} - y \right| \frac{(nv_n(y))^l}{l!}. \end{aligned} \quad (19)$$

Finally, applying the Cauchy-Schwarz inequality, we write

$$\begin{aligned} & D_n\left(\sqrt{(\sqrt{t}-\sqrt{x})^2 + (\sqrt{s}-\sqrt{y})^2}; x, y\right) \\ & \leq \frac{1}{\sqrt{x}} \sqrt{e^{-nu_n(x)} \sum_{k=0}^{\infty} \left(\frac{k}{n} - x \right)^2 \frac{(nu_n(x))^k}{k!}} \\ & \quad + \frac{1}{\sqrt{y}} \sqrt{e^{-nv_n(y)} \sum_{l=0}^{\infty} \left(\frac{l}{n} - y \right)^2 \frac{(nv_n(y))^l}{l!}} \\ & = \frac{1}{\sqrt{x}} \sqrt{(u_n(x)-x)^2 + \frac{u_n(x)}{n}} \\ & \quad + \frac{1}{\sqrt{y}} \sqrt{(v_n(y)-y)^2 + \frac{v_n(y)}{n}}. \end{aligned} \quad (20)$$

Using Lemma 1, we get the result. \square

Recall that, for all $f \in C_B([0, \infty) \times [0, \infty))$, the modulus of f denoted by $\omega(f; \delta)$ is defined as

$$\begin{aligned} \omega(f; \delta) := \sup & \left\{ |f(t, s) - f(x, y)| : \right. \\ & \sqrt{(t-x)^2 + (s-y)^2} < \delta, \\ & \left. (t, s), (x, y) \in [0, \infty) \times [0, \infty) \right\}. \end{aligned} \quad (21)$$

Theorem 4. Let $f^*(x, y) = f(x^2, y^2)$. Then one has, for each $x, y > 0$,

$$|D_n(f; x, y) - f(x, y)| \leq 2\omega(f^*; \delta_n(x, y)), \quad (22)$$

where

$$\begin{aligned} \delta_n(x, y) := & \frac{1}{\sqrt{x}} \sqrt{(u_n(x) - x)^2 + \frac{u_n(x)}{n}} \\ & + \frac{1}{\sqrt{y}} \sqrt{(v_n(y) - y)^2 + \frac{v_n(y)}{n}}. \end{aligned} \quad (23)$$

Proof. We directly have

$$\begin{aligned} & |D_n(f; x, y) - f(x, y)| \\ & \leq D_n(|f(t, s) - f(x, y)|; x, y) \\ & = D_n(|f^*(\sqrt{t}, \sqrt{s}) - f^*(\sqrt{x}, \sqrt{y})|; x, y) \\ & \leq D_n\left(\omega(f^*; \sqrt{(\sqrt{t} - \sqrt{x})^2 + (\sqrt{s} - \sqrt{y})^2}); x, y\right) \\ & = e^{-n(u_n(x) + v_n(y))} \\ & \times \sum_{k,l=0}^{\infty} \omega\left(f^*; \sqrt{\left(\sqrt{\frac{k}{n}} - \sqrt{x}\right)^2 + \left(\sqrt{\frac{l}{n}} - \sqrt{y}\right)^2}; x, y\right) \\ & \times \frac{(nu_n(x))^k}{k!} \frac{(nv_n(y))^l}{l!}. \end{aligned} \quad (24)$$

Therefore,

$$\begin{aligned} & |D_n(f; x, y) - f(x, y)| \\ & = e^{-n(u_n(x) + v_n(y))} \sum_{k,l=0}^{\infty} \frac{(nu_n(x))^k}{k!} \frac{(nv_n(y))^l}{l!} \end{aligned}$$

$$\begin{aligned} & \times \omega\left(f^*; \frac{\sqrt{(\sqrt{k/n} - \sqrt{x})^2 + (\sqrt{l/n} - \sqrt{y})^2}}{D_n\left(\sqrt{(\sqrt{t} - \sqrt{x})^2 + (\sqrt{s} - \sqrt{y})^2}; x, y\right)}\right. \\ & \left. \times D_n\left(\left((\sqrt{t} - \sqrt{x})^2 + (\sqrt{s} - \sqrt{y})^2\right)^{1/2}; x, y\right); x, y\right). \end{aligned} \quad (25)$$

Because of the fact that

$$\omega(f; \lambda\delta) \leq (1 + \lambda)\omega(f; \delta), \quad (26)$$

we have

$$\begin{aligned} & |D_n(f; x, y) - f(x, y)| \\ & \leq \omega\left(f^*; D_n\left(\sqrt{(\sqrt{t} - \sqrt{x})^2 + (\sqrt{s} - \sqrt{y})^2}; x, y\right)\right) \\ & \times e^{-n(u_n(x) + v_n(y))} \\ & \times \sum_{k,l=0}^{\infty} \left[1 + \frac{\sqrt{(\sqrt{k/n} - \sqrt{x})^2 + (\sqrt{l/n} - \sqrt{y})^2}}{D_n\left(\sqrt{(\sqrt{t} - \sqrt{x})^2 + (\sqrt{s} - \sqrt{y})^2}; x, y\right)} \right] \\ & \times \frac{(nu_n(x))^k}{k!} \frac{(nv_n(y))^l}{l!}, \end{aligned} \quad (27)$$

and hence

$$\begin{aligned} & |D_n(f; x, y) - f(x, y)| \\ & \leq 2\omega\left(f^*; D_n\left(\sqrt{(\sqrt{t} - \sqrt{x})^2 + (\sqrt{s} - \sqrt{y})^2}; x, y\right)\right). \end{aligned} \quad (28)$$

Finally, using Lemma 3, the proof is completed. \square

Theorem 5. Let $f^*(x, y) = f(x^2, y^2)$. Let

$$\begin{aligned} f^* \in Lip_M(\alpha) := & \{f^* \in C_B([0, \infty) \times [0, \infty)) : \\ & |f^*(\mathbf{t}) - f^*(\mathbf{x})| \leq M\|\mathbf{t} - \mathbf{x}\|^\alpha; \\ & t, s; x, y \in (0, \infty)\}, \end{aligned} \quad (29)$$

where $\mathbf{t} = (t, s)$, $\mathbf{x} = (x, y)$, M is any positive constant, and $0 < \alpha \leq 1$. Then

$$|D_n(f; x, y) - f(x, y)| \leq M\delta_n^\alpha(x, y), \quad (30)$$

where $\delta_n(x, y)$ is the same as in Theorem 4.

Proof. We directly have

$$\begin{aligned}
& |D_n(f; x, y) - f(x, y)| \\
& \leq D_n(|f(t, s) - f(x, y)|; x, y) \\
& = D_n(|f^*(\sqrt{t}, \sqrt{s}) - f^*(\sqrt{x}, \sqrt{y})|; x, y) \\
& \leq MD_n\left(\left((\sqrt{t} - \sqrt{x})^2 + (\sqrt{s} - \sqrt{y})^2\right)^{\alpha/2}; x, y\right) \\
& = Me^{-n(u_n(x)+v_n(y))} \\
& \quad \times \sum_{k,l=0}^{\infty} \left(\left(\sqrt{\frac{k}{n}} - \sqrt{x} \right)^2 + \left(\sqrt{\frac{l}{n}} - \sqrt{y} \right)^2 \right)^{\alpha/2} \\
& \quad \times \frac{(nu_n(x))^k}{k!} \frac{(nv_n(y))^l}{l!}. \tag{31}
\end{aligned}$$

Applying the Hölder inequality with $p = 1/\alpha$ and $q = 1/(1-\alpha)$, we have

$$\begin{aligned}
& |D_n(f; x, y) - f(x, y)| \\
& \leq M \left[D_n\left(\sqrt{(\sqrt{t} - \sqrt{x})^2 + (\sqrt{s} - \sqrt{y})^2}; x, y\right) \right]^{\alpha}. \tag{32}
\end{aligned}$$

Using Lemma 3, we get the result. \square

3. Concluding Remarks

In this section we show that taking $u_n(x) = x$ and $v_n(y) = y$ or $u_n(x) = u_n^{(i)}(x)$ and $v_n(y) = v_n^{(i)}(y)$, $i = 1, 3$, in Theorems 2, 4, and 5 gives global results. Also we present the results obtained by Theorems 2, 4, and 5 for $u_n(x) = u_n^{(2)}(x)$ and $v_n(y) = v_n^{(2)}(y)$.

Corollary 6. For any $f \in Lip_M^*(\alpha)$, $\alpha \in (0, 1]$ and for all $x, y \in (0, \infty)$, $n \in \mathbb{N}$, one has

$$|D_n(f; x, y) - f(x, y)| \leq \frac{M}{n^{\alpha/2}}, \tag{33}$$

uniformly as $n \rightarrow \infty$, for the following pairs of $u_n(x)$ and $v_n(x)$:

- (i) $u_n(x) = x$ and $v_n(y) = y$,
- (ii) $u_n(x) = (-1 + \sqrt{4n^2x^2 + 1})/2n$ and $v_n(y) = (-1 + \sqrt{4n^2y^2 + 1})/2n$,
- (iii) $u_n(x) = x - 1/2n$ and $v_n(y) = y - 1/2n$.

Proof. (i) Taking $u_n(x) = x$ and $v_n(y) = y$ in (13), we directly have

$$|D_n(f; x, y) - f(x, y)| \leq \frac{M}{n^{\alpha/2}}. \tag{34}$$

(ii) Taking $u_n(x) = (-1 + \sqrt{4n^2x^2 + 1})/2n$ and $v_n(y) = (-1 + \sqrt{4n^2y^2 + 1})/2n$ in (13) gives

$$\begin{aligned}
& |D_n(f; x, y) - f(x, y)| \\
& = \frac{M}{(x+y)^{\alpha/2}} \left[\frac{1}{n} \left(x + y - x\sqrt{4n^2x^2 + 1} \right. \right. \\
& \quad \left. \left. - y\sqrt{4n^2y^2 + 1} + 2nx^2 + 2ny^2 \right) \right]^{\alpha/2} \\
& \leq \frac{M}{(x+y)^{\alpha/2}} \left[\frac{1}{n} \left(x + y - x\sqrt{4n^2x^2} \right. \right. \\
& \quad \left. \left. - y\sqrt{4n^2y^2} + 2nx^2 + 2ny^2 \right) \right]^{\alpha/2} \\
& = \frac{M}{n^{\alpha/2}}. \tag{35}
\end{aligned}$$

(iii) Taking $u_n(x) = x - 1/2n$ and $v_n(y) = y - 1/2n$ in (13), we have

$$\begin{aligned}
& |D_n(f; x, y) - f(x, y)| \\
& = \frac{M}{(x+y)^{\alpha/2}} \left[\left(-\frac{1}{2n} \right)^2 + \left(-\frac{1}{2n} \right)^2 + \frac{x+y-1/n}{n} \right]^{\alpha/2} \\
& \leq \frac{M}{(x+y)^{\alpha/2}} \left[\frac{1}{2n^2} (2nx + 2ny - 1) \right]^{\alpha/2} \\
& = \frac{M}{(x+y)^{\alpha/2}} \left[\frac{1}{n} (x+y) \right]^{\alpha/2} \\
& = \frac{M}{n^{\alpha/2}}. \tag{36}
\end{aligned}$$

\square

Corollary 7. Let $f^*(x, y) = f(x^2, y^2)$. Then one has

$$|D_n(f; x, y) - f(x, y)| \leq 2\omega\left(f^*; \frac{2}{\sqrt{n}}\right), \tag{37}$$

uniformly as $n \rightarrow \infty$, for the following pairs of $u_n(x)$ and $v_n(x)$:

- (i) $u_n(x) = x$ and $v_n(y) = y$,
- (ii) $u_n(x) = (-1 + \sqrt{4n^2x^2 + 1})/2n$ and $v_n(y) = (-1 + \sqrt{4n^2y^2 + 1})/2n$,
- (iii) $u_n(x) = x - 1/2n$ and $v_n(y) = y - 1/2n$.

Proof. (i) Taking $u_n(x) = x$ and $v_n(y) = y$ in (23), we directly have

$$|D_n(f; x, y) - f(x, y)| \leq 2\omega\left(f^*; \frac{2}{\sqrt{n}}\right). \tag{38}$$

(ii) Taking $u_n(x) = (-1 + \sqrt{4n^2x^2 + 1})/2n$ and $v_n(y) = (-1 + \sqrt{4n^2y^2 + 1})/2n$ in (23) gives

$$|D_n(f; x, y) - f(x, y)| \leq 2\omega(f^*; \delta_n(x, y)), \quad (39)$$

where

$$\delta_n(x, y)$$

$$\begin{aligned} &:= \frac{1}{\sqrt{x}\sqrt{y}} \left(\sqrt{y} \sqrt{\frac{1}{n} (x - x\sqrt{4n^2x^2 + 1} + 2nx^2)} \right. \\ &\quad \left. + \sqrt{x} \sqrt{\frac{1}{n} (y - y\sqrt{4n^2y^2 + 1} + 2ny^2)} \right) \\ &\leq \frac{1}{\sqrt{x}\sqrt{y}} \left(\sqrt{y} \sqrt{\frac{1}{n} x} + \sqrt{x} \sqrt{\frac{1}{n} y} \right) \\ &= \frac{2}{\sqrt{n}}. \end{aligned} \quad (40)$$

(iii) Taking $u_n(x) = x - 1/2n$ and $v_n(y) = y - 1/2n$ in (23), we have

$$|D_n(f; x, y) - f(x, y)| \leq 2\omega(f^*; \delta_n(x, y)), \quad (41)$$

where

$$\begin{aligned} \delta_n(x, y) &:= \frac{1}{\sqrt{x}} \sqrt{\left(\frac{1}{2n}\right)^2 + \frac{x}{n} - \frac{1}{2n^2}} \\ &\quad + \frac{1}{\sqrt{y}} \sqrt{\left(\frac{1}{2n}\right)^2 + \frac{y}{n} - \frac{1}{2n^2}} \\ &= \frac{1}{2\sqrt{x}\sqrt{y}} \left(\sqrt{x} \sqrt{\frac{1}{n^2} (4ny - 1)} \right. \\ &\quad \left. + \sqrt{y} \sqrt{\frac{1}{n^2} (4nx - 1)} \right) \\ &\leq \frac{1}{\sqrt{x}\sqrt{y}} \left(\sqrt{x} \sqrt{\frac{1}{n} y} + \sqrt{y} \sqrt{\frac{1}{n} x} \right) \\ &= \frac{2}{\sqrt{n}}. \end{aligned} \quad (42)$$

□

Corollary 8. Let $f^*(x, y) = f(x^2, y^2)$, and let

$$f^* \in Lip_M(\alpha) := \{f^* \in C_B([0, \infty) \times [0, \infty)) :$$

$$|f^*(\mathbf{t}) - f^*(\mathbf{x})| \leq M\|\mathbf{t} - \mathbf{x}\|^\alpha; \quad (43)$$

$$t, s; x, y \in (0, \infty) \},$$

where $\mathbf{t} = (t, s)$, $\mathbf{x} = (x, y)$, M is any positive constant, and $0 < \alpha \leq 1$. Then

$$|D_n(f; x, y) - f(x, y)| \leq M \left(\frac{4}{n} \right)^{\alpha/2}, \quad (44)$$

uniformly as $n \rightarrow \infty$, for the following pairs of $u_n(x)$ and $v_n(y)$:

$$(i) u_n(x) = x \text{ and } v_n(y) = y,$$

$$(ii) u_n(x) = (-1 + \sqrt{4n^2x^2 + 1})/2n \text{ and } v_n(y) = (-1 + \sqrt{4n^2y^2 + 1})/2n,$$

$$(iii) u_n(x) = x - 1/2n \text{ and } v_n(y) = y - 1/2n.$$

Proof. (i) Taking $u_n(x) = x$ and $v_n(y) = y$ in (23), we directly have

$$|D_n(f; x, y) - f(x, y)| \leq M \left(\frac{4}{n} \right)^{\alpha/2}. \quad (45)$$

(ii) Taking $u_n(x) = (-1 + \sqrt{4n^2x^2 + 1})/2n$ and $v_n(y) = (-1 + \sqrt{4n^2y^2 + 1})/2n$ in (23) gives

$$|D_n(f; x, y) - f(x, y)| \leq M \delta_n^{\alpha/2}(x, y), \quad (46)$$

where

$$\delta_n(x, y) := \delta_n(x, y)$$

$$:= \frac{1}{\sqrt{x}\sqrt{y}}$$

$$\times \left(\sqrt{y} \sqrt{\frac{1}{n} (x - x\sqrt{4n^2x^2 + 1} + 2nx^2)} \right.$$

$$\left. + \sqrt{x} \sqrt{\frac{1}{n} (y - y\sqrt{4n^2y^2 + 1} + 2ny^2)} \right)$$

$$\leq \frac{1}{\sqrt{x}\sqrt{y}} \left(\sqrt{y} \sqrt{\frac{1}{n} x} + \sqrt{x} \sqrt{\frac{1}{n} y} \right)$$

$$= \frac{2}{\sqrt{n}}.$$

□

(iii) Taking $u_n(x) = x - 1/2n$ and $v_n(y) = y - 1/2n$ in (23), we have

$$|D_n(f; x, y) - f(x, y)| \leq M \delta_n^{\alpha/2}(x, y), \quad (48)$$

where

$$\begin{aligned}
\delta_n(x, y) &:= \frac{1}{\sqrt{x}} \sqrt{\left(\frac{1}{2n}\right)^2 + \frac{x}{n} - \frac{1}{2n^2}} \\
&\quad + \frac{1}{\sqrt{y}} \sqrt{\left(\frac{1}{2n}\right)^2 + \frac{y}{n} - \frac{1}{2n^2}} \\
&= \frac{1}{2\sqrt{x}\sqrt{y}} \left(\sqrt{x} \sqrt{\frac{1}{n^2} (4ny - 1)} \right. \\
&\quad \left. + \sqrt{y} \sqrt{\frac{1}{n^2} (4nx - 1)} \right) \\
&\leq \frac{1}{\sqrt{x}\sqrt{y}} \left(\sqrt{x} \sqrt{\frac{1}{n} y} + \sqrt{y} \sqrt{\frac{1}{n} x} \right) \\
&= \frac{2}{\sqrt{n}}.
\end{aligned} \tag{49}$$

□

Remark 9. Corollaries 7 and 8 conclude that f is a real continuous and bounded function on $[0, \infty) \times [0, \infty)$ and if $f^*(x, y) = f(x^2, y^2)$ is uniformly continuous on $[0, \infty) \times [0, \infty)$, then $D_n(f)$ converges uniformly to f as $n \rightarrow \infty$. Note that the one variable version of Corollary 7 was given in [16].

Corollary 10. Take

$$\begin{aligned}
u_n(x) &= u_n^{(2)}(x, \gamma) \\
&= \frac{-(ny + 1) + \sqrt{4n^2(x^2 + \gamma x) + (ny + 1)^2}}{2n}, \\
v_n(y) &= v_n^{(2)}(y, \beta) \\
&= \frac{-(n\beta + 1) + \sqrt{4n^2(y^2 + \beta y) + (n\beta + 1)^2}}{2n},
\end{aligned} \tag{50}$$

where $\alpha, \beta \in \mathbb{R}$. Then

(i) for any $f \in Lip_M^*(\alpha)$, $\alpha \in (0, 1]$ and for each $x, y \in (0, \infty)$, $n \in \mathbb{N}$, one has

$$\begin{aligned}
|D_n(f; x, y) - f(x, y)| &\leq \frac{M}{[2n(x + y)]^{\alpha/2}} [\delta(x, \gamma) + \delta(y, \beta)]^{\alpha/2},
\end{aligned} \tag{51}$$

where

$$\begin{aligned}
\delta(x, \gamma) &= \left[(2x + \gamma) \right. \\
&\quad \left. \times \left(ny - \sqrt{n^2(2x + \gamma)^2 + 2ny + 1} + 2nx + 1 \right) \right],
\end{aligned} \tag{52}$$

(ii) let $f^*(x, y) = f(x^2, y^2)$. Then one has for each $x, y > 0$

$$|D_n(f; x, y) - f(x, y)| \leq 2\omega(f^*; \delta(x, \gamma) + \delta(y, \beta)), \tag{53}$$

where

$$\begin{aligned}
\delta(x, \gamma) &= \frac{1}{\sqrt{2x}} \\
&\quad \times \left(\frac{1}{n} (2x + \gamma) \left(ny - \sqrt{n^2(2x + \gamma)^2 + 2ny + 1} \right. \right. \\
&\quad \left. \left. + 2nx + 1 \right) \right)^{1/2},
\end{aligned} \tag{54}$$

(iii) let $f^*(x, y) = f(x^2, y^2)$, and let

$$f^* \in Lip_M(\alpha) := \{f^* \in C_B([0, \infty) \times [0, \infty)) :$$

$$\begin{aligned}
|f^*(\mathbf{t}) - f^*(\mathbf{x})| &\leq M \|\mathbf{t} - \mathbf{x}\|^\alpha; \\
t, s; x, y \in (0, \infty) \}
\end{aligned} \tag{55}$$

where $\mathbf{t} = (t, s)$, $\mathbf{x} = (x, y)$, M is any positive constant, and $0 < \alpha \leq 1$. Then

$$|D_n(f; x, y) - f(x, y)| \leq M [\delta(x, \gamma) + \delta(y, \beta)]^{\alpha/2}, \tag{56}$$

where $\delta(x, \gamma)$ is the same given in Corollary 10(ii).

It should be mentioned that, for $\alpha = 0$ and $\beta = 0$, $u_n^{(2)}(x, 0) = (-1 + \sqrt{4n^2x^2 + 1})/2n$ and $v_n^{(2)}(y, 0) = (-1 + \sqrt{4n^2y^2 + 1})/2n$. Therefore, Corollary 10(i), Corollary 10(ii), and Corollary 10(iii) reduce to Corollary 6(ii), Corollary 7(ii), and Corollary 8(ii), respectively.

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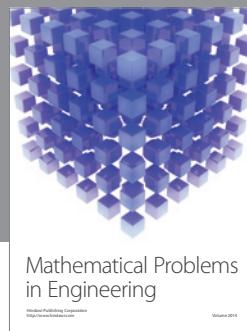
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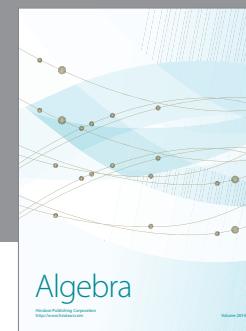
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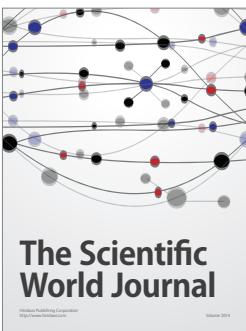
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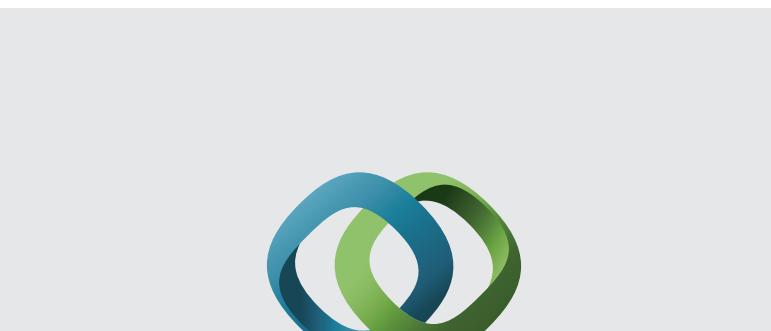
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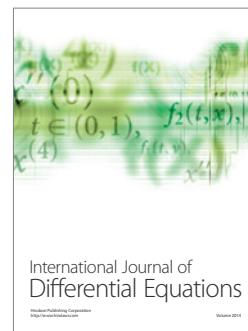


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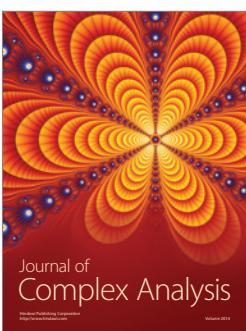
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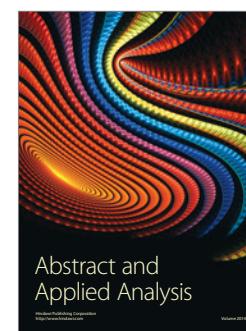
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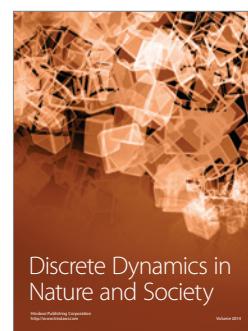
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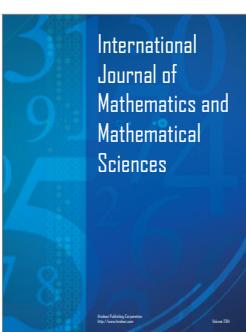
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