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Research Article

Refinements of Generalized Aczél's Inequality and Bellman's Inequality and Their Applications

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We give some refinements of generalized Aczél's inequality and Bellman's inequality proposed by Tian. As applications, some refinements of integral type of generalized Aczél's inequality and Bellman's inequality are given.

1. Introduction

The famous Aczél's inequality [1] states that if a_i, b_i ($i = 1, 2, \ldots, n$) are real numbers such that $a_1^2 - \sum_{i=2}^n a_i^2 > 0$ and $b_1^2 - \sum_{i=2}^n b_i^2 > 0$, then

$$\left(a_1^2 - \sum_{i=2}^n a_i^2\right)\left(b_1^2 - \sum_{i=2}^n b_i^2\right) \le \left(a_1 b_1 - \sum_{i=2}^n a_i b_i\right)^2. \quad (1)$$

It is well known that Aczél's inequality plays an important role in the theory of functional equations in non-Euclidean geometry. In recent years, various attempts have been made by many authors to improve and generalize the Aczél's inequality (see [2–19] and references therein). We state here some improvements of Aczél's inequality.

One of the most important results in the references mentioned above is an exponential extension of (1), which is stated in the following theorem.

Theorem A. *Let p and q be real numbers such that $p, q \neq 0$, and $1/p + 1/q = 1$ and let a_i, b_i ($i = 1, 2, \ldots, n$) be positive numbers such that $a_1^p - \sum_{i=2}^n a_i^p > 0$ and $b_1^q - \sum_{i=2}^n b_i^q > 0$. If $p > 1$, then*

$$\left(a_1^p - \sum_{i=2}^n a_i^p\right)^{1/p}\left(b_1^q - \sum_{i=2}^n b_i^q\right)^{1/q} \le a_1 b_1 - \sum_{i=2}^n a_i b_i. \quad (2)$$

If $p < 1$ ($p \neq 0$), then the reverse inequality in (2) *holds.*

Remark 1. The case $p > 1$ of Theorem A was proved by Popoviciu [8]. The case $p < 1$ was given in [15] by Vasić and Pečarić.

Vasić and Pečarić [16] presented a further extension of inequality (1).

Theorem B. *Let $a_{rj} > 0$, $\lambda_j > 0$, $a_{1j}^{\lambda_j} - \sum_{r=2}^n a_{rj}^{\lambda_j} > 0$, $r = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, and let $\sum_{j=1}^m (1/\lambda_j) \ge 1$. Then*

$$\prod_{j=1}^m \left(a_{1j}^{\lambda_j} - \sum_{r=2}^n a_{rj}^{\lambda_j}\right)^{1/\lambda_j} \le \prod_{j=1}^m a_{1j} - \sum_{r=2}^n \prod_{j=1}^m a_{rj}. \quad (3)$$

In a recent paper [18], Wu and Debnath established an interesting generalization of Aczél's inequality [1] as follows.

Theorem C. *Let $a_{rj} > 0$, $\lambda_j > 0$, $a_{1j}^{\lambda_j} - \sum_{r=2}^n a_{rj}^{\lambda_j} > 0$, $r = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, and let $\rho = \min\{\sum_{j=1}^m (1/\lambda_j), 1\}$. Then*

$$\prod_{j=1}^m \left(a_{1j}^{\lambda_j} - \sum_{r=2}^n a_{rj}^{\lambda_j}\right)^{1/\lambda_j} \le n^{1-\rho} \prod_{j=1}^m a_{1j} - \sum_{r=2}^n \prod_{j=1}^m a_{rj}. \quad (4)$$

In 2012, Tian [10] presented the following reversed version of inequality (4).

Theorem D. *Let* $a_{rj} > 0$, $\lambda_1 \neq 0$, $\lambda_j < 0$ $(j = 2, 3, \ldots, m)$, $a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j} > 0$, $r = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, *and let* $\tau = \max\{\sum_{j=1}^{m}(1/\lambda_j), 1\}$. *Then*

$$\prod_{j=1}^{m}\left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j}\right)^{1/\lambda_j} \geq n^{1-\tau}\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n}\prod_{j=1}^{m} a_{rj}. \quad (5)$$

Therefore, applying the above inequality, Tian gave the reversed version of inequality (3) as follows.

Theorem E. *Let* $a_{rj} > 0$, $\lambda_1 \neq 0$, $\lambda_j < 0$ $(j = 2, 3, \ldots, m)$, $\sum_{j=1}^{m}(1/\lambda_j) \leq 1$, $a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j} > 0$, *and* $r = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. *Then*

$$\prod_{j=1}^{m}\left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j}\right)^{1/\lambda_j} \geq \prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n}\prod_{j=1}^{m} a_{rj}. \quad (6)$$

Moreover, in [10], Tian obtained the following integral form of inequality (5).

Theorem F. *Let* $\lambda_1 > 0$, $\lambda_j < 0$ $(j = 2, 3, \ldots, m)$, $\sum_{j=1}^{m}\lambda_j = 1$, $P_j > 0$ $(j = 1, 2, \ldots, m)$, *and let* $f_j(x)$ $(j = 1, 2, \ldots, m)$ *be positive Riemann integrable functions on* $[a, b]$ *such that* $P_j^{\lambda_j} - \int_a^b f_j^{\lambda_j}(x)\mathrm{d}x > 0$. *Then*

$$\prod_{j=1}^{m}\left(P_j^{\lambda_j} - \int_a^b f_j^{\lambda_j}(x)\,\mathrm{d}x\right)^{1/\lambda_j} \geq \prod_{j=1}^{m} P_j - \int_a^b \prod_{j=1}^{m} f_j(x)\,\mathrm{d}x. \quad (7)$$

Bellman inequality [20] related with Aczél's inequality is stated as follows.

Theorem G. *Let* a_i, b_i $(i = 1, 2, \ldots, n)$ *be positive numbers such that* $a_1^p - \sum_{i=2}^{n} a_i^p > 0$ *and* $b_1^q - \sum_{i=2}^{n} b_i^q > 0$. *If* $p \geq 1$, *then*

$$\left[\left(a_1^p - \sum_{i=2}^{n} a_i^p\right)^{1/p} + \left(b_1^p - \sum_{i=2}^{n} b_i^p\right)^{1/p}\right]^p \leq (a_1 + b_1)^p - \sum_{i=2}^{n}(a_i + b_i)^p. \quad (8)$$

If $0 < p < 1$, *then the reverse inequality in* (8) *holds.*

Remark 2. The case $p > 1$ of Theorem G was proposed by Bellman [20]. The case $0 < p < 1$ was proved in [15] by Vasić and Pečarić.

The main purpose of this work is to give refinements of inequalities (5) and (8). As applications, some refinements of integral type of inequality (5) and (8) are given.

2. Refinements of Generalized Aczél's Inequality and Bellman's Inequality

Theorem 3. *Let* $a_{rj} > 0$, $\lambda_1 \neq 0$, $\lambda_j < 0$ $(j = 2, 3, \ldots, m)$, $a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j} > 0$, $r = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, *and let* $\tau = \max\{\sum_{j=1}^{m}(1/\lambda_j), 1\}$. *Then*

$$\prod_{j=1}^{m}\left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j}\right)^{1/\lambda_j} \geq T(s, a_{ij}) \geq n^{1-\tau}\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n}\prod_{j=1}^{m} a_{rj}, \quad (9)$$

where

$$T(s, a_{ij}) = (n + 1 - s)^{1-\tau}\prod_{j=1}^{m}\left(a_{1j}^{\lambda_j} - \sum_{r=2}^{s} a_{rj}^{\lambda_j}\right)^{1/\lambda_j} - \sum_{r=s+1}^{n}\prod_{j=1}^{m} a_{rj}. \quad (10)$$

Proof

Case (I). When $\sum_{j=1}^{m}(1/\lambda_j) < 1$, then $\tau = 1$. On the one hand, we split the left-hand side of inequality (9) as follows:

$$\begin{aligned}\prod_{j=1}^{m}\left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j}\right)^{1/\lambda_j} &= \prod_{j=1}^{m}\left(a_{1j}^{\lambda_j} - \sum_{r=2}^{s} a_{rj}^{\lambda_j} - \sum_{r=s+1}^{n} a_{rj}^{\lambda_j}\right)^{1/\lambda_j} \\ &= \prod_{j=1}^{m}\left(A_j^{\lambda_j} - \sum_{r=s+1}^{n} a_{rj}^{\lambda_j}\right)^{1/\lambda_j},\end{aligned} \quad (11)$$

where

$$A_j = \left(a_{1j}^{\lambda_j} - \sum_{r=2}^{s} a_{rj}^{\lambda_j}\right)^{1/\lambda_j}, \quad j = 1, 2, \ldots, m. \quad (12)$$

From this hypothesis, it is immediate to obtain the inequality

$$A_j^{\lambda_j} \geq \sum_{r=s+1}^{n} a_{rj}^{\lambda_j}. \quad (13)$$

Thus, by using inequality (6), we have

$$\prod_{j=1}^{m}\left(A_j^{\lambda_j} - \sum_{r=s+1}^{n} a_{rj}^{\lambda_j}\right)^{1/\lambda_j} \geq \prod_{j=1}^{m} A_j - \sum_{r=s+1}^{n}\prod_{j=1}^{m} a_{rj}. \quad (14)$$

On the other hand, by using inequality (6) again, we obtain the inequality

$$\prod_{j=1}^{m} A_j = \prod_{j=1}^{m}\left(a_{1j}^{\lambda_j} - \sum_{r=2}^{s} a_{rj}^{\lambda_j}\right)^{1/\lambda_j} \geq \prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{s}\prod_{j=1}^{m} a_{rj}. \quad (15)$$

Combining inequalities (11), (14), and (15) we can get inequality (9).

Case (II). When $\sum_{j=1}^{m}(1/\lambda_j) \geq 1$, using Case (I) with $\sum_{j=1}^{m+1}(1/\lambda_j) = 1$, we have

$$\prod_{j=1}^{m+1}\left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j}\right)^{1/\lambda_j} \geq \prod_{j=1}^{m+1}\left(a_{1j}^{\lambda_j} - \sum_{r=2}^{s} a_{rj}^{\lambda_j}\right)^{1/\lambda_j} - \sum_{r=s+1}^{n}\prod_{j=1}^{m+1} a_{rj}, \quad (16)$$

where $a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j} > 0$, $a_{rj} > 0$, $\lambda_1 \neq 0$, $\lambda_j < 0$ ($j = 2, 3, \ldots, m+1$).

Hence, taking $a_{1(m+1)}^{\lambda_{m+1}} = 1$, $a_{2(m+1)}^{\lambda_{m+1}} = a_{3(m+1)}^{\lambda_{m+1}} = \cdots = a_{n(m+1)}^{\lambda_{m+1}} = 1/n$ into (16), we obtain

$$\begin{aligned}\left(\frac{1}{n}\right)^{1-1/\lambda_1-1/\lambda_2-\cdots-1/\lambda_m} &\prod_{j=1}^{m}\left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j}\right)^{1/\lambda_j} \\ &\geq \left(\frac{n+1-s}{n}\right)^{1-1/\lambda_1-1/\lambda_2-\cdots-1/\lambda_m} \\ &\quad \times \prod_{j=1}^{m}\left(a_{1j}^{\lambda_j} - \sum_{r=2}^{s} a_{rj}^{\lambda_j}\right)^{1/\lambda_j} \\ &\quad - \left(\frac{1}{n}\right)^{1-1/\lambda_1-1/\lambda_2-\cdots-1/\lambda_m} \sum_{r=s+1}^{n}\prod_{j=1}^{m} a_{rj},\end{aligned} \tag{17}$$

that is,

$$\begin{aligned}\prod_{j=1}^{m}\left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j}\right)^{1/\lambda_j} &\geq (n+1-s)^{1-1/\lambda_1-1/\lambda_2-\cdots-1/\lambda_m} \\ &\quad \times \prod_{j=1}^{m}\left(a_{1j}^{\lambda_j} - \sum_{r=2}^{s} a_{rj}^{\lambda_j}\right)^{1/\lambda_j} \\ &\quad - \sum_{r=s+1}^{n}\prod_{j=1}^{m} a_{rj}.\end{aligned} \tag{18}$$

Therefore, repeating the foregoing arguments, we get

$$\begin{aligned}(n+1-s)^{1-1/\lambda_1-1/\lambda_2-\cdots-1/\lambda_m} &\prod_{j=1}^{m}\left(a_{1j}^{\lambda_j} - \sum_{r=2}^{s} a_{rj}^{\lambda_j}\right)^{1/\lambda_j} \\ - \sum_{r=s+1}^{n}\prod_{j=1}^{m} a_{rj} &\geq n^{1-\tau}\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n}\prod_{j=1}^{m} a_{rj}.\end{aligned} \tag{19}$$

Combining inequalities (18) and (19) leads to inequality (9) immediately. The proof of Theorem 3 is completed. □

If we set $\sum_{j=1}^{m}(1/\lambda_j) \leq 1$, then from Theorem 3, we obtain the following refinement of inequality (6).

Corollary 4. *Let $a_{rj} > 0$, $\lambda_1 \neq 0$, $\lambda_j < 0$ ($j = 2, 3, \ldots, m$), $\sum_{j=1}^{m}(1/\lambda_j) \leq 1$, $a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j} > 0$, $r = 1, 2, \ldots, n$, and $j = 1, 2, \ldots, m$. Then*

$$\begin{aligned}\prod_{j=1}^{m}\left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j}\right)^{1/\lambda_j} &\geq \prod_{j=1}^{m}\left(a_{1j}^{\lambda_j} - \sum_{r=2}^{s} a_{rj}^{\lambda_j}\right)^{1/\lambda_j} - \sum_{r=s+1}^{n}\prod_{j=1}^{m} a_{rj} \\ &\geq \prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n}\prod_{j=1}^{m} a_{rj}.\end{aligned} \tag{20}$$

Putting $m = 2$, $\lambda_1 = p \neq 0$, $\lambda_2 = q < 0$, $a_{r1} = a_r$, and $a_{r2} = b_r$ ($r = 1, 2, \ldots, n$) in Theorem 3, we obtain the refinement and generalization of Theorem A for $p < 1$.

Corollary 5. *Let $a_r > 0$, $b_r > 0$ ($r = 1, 2, \ldots, n$), $a_1^p - \sum_{r=2}^{n} a_r^p > 0$, $b_1^q - \sum_{r=2}^{n} b_r^q > 0$, $p \neq 0$, $q < 0$, and $\rho = \max\{1/p + 1/q, 1\}$. Then, the following inequality holds:*

$$\begin{aligned}\left(a_1^p - \sum_{r=2}^{n} a_r^p\right)^{1/p}\left(b_1^q - \sum_{r=2}^{n} b_r^q\right)^{1/q} &\geq (n+1-s)^{1-\rho}\left(a_1^p - \sum_{r=2}^{s} a_r^p\right)^{1/p} \\ &\quad \times \left(b_1^q - \sum_{r=2}^{s} b_r^q\right)^{1/q} - \sum_{r=s+1}^{n} a_r b_r \\ &\geq n^{1-\rho} a_1 b_1 - \sum_{r=2}^{n} a_r b_r.\end{aligned} \tag{21}$$

Based on the mathematical induction, it is easy to see that the following generalized Bellman's inequality is true.

Theorem 6. *Let $a_{rj} > 0$, $a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j} > 0$, $r = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, and let $0 < p < 1$. Then*

$$\left[\prod_{j=1}^{m}\left(a_{1j}^{p} - \sum_{r=2}^{n} a_{rj}^{p}\right)^{1/p}\right]^p \geq \left(\sum_{j=1}^{m} a_{1j}\right)^p - \sum_{r=2}^{n}\left(\sum_{j=1}^{m} a_{rj}\right)^p. \tag{22}$$

Next, we give a refinement of generalized Bellman's inequality (22) as follows.

Theorem 7. *Let $a_{rj} > 0$, $a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j} > 0$, $r = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, and let $0 < p < 1$. Then*

$$\begin{aligned}\left[\prod_{j=1}^{m}\left(a_{1j}^{p} - \sum_{r=2}^{n} a_{rj}^{p}\right)^{1/p}\right]^p &\geq \left[\prod_{j=1}^{m}\left(a_{1j}^{p} - \sum_{r=2}^{s} a_{rj}^{p}\right)^{1/p}\right]^p \\ &\quad - \sum_{r=s+1}^{n}\left(\sum_{j=1}^{m} a_{rj}\right)^p \\ &\geq \left(\sum_{j=1}^{m} a_{1j}\right)^p - \sum_{r=2}^{n}\left(\sum_{j=1}^{m} a_{rj}\right)^p.\end{aligned} \tag{23}$$

Proof. The proof of Theorem 7 is similar to the one of Theorem 3. Applying generalized Bellman's inequality (22) twice, we can deduce the inequality (23). □

3. Application

In this section, we show two applications of the inequalities newly obtained in Section 2.

Firstly, we give an improvement of inequality (7) by using Theorem 3.

Theorem 8. *Let $B_j > 0$ $(j = 1, 2, \ldots, m)$, let $\lambda_1 > 0$, $\lambda_j < 0$ $(j = 2, 3, \ldots, m)$, $\sum_{j=1}^{m} \lambda_j = 1$, and let $f_j(x)$ $(j = 1, 2, \ldots, m)$ be positive integrable functions defined on $[a, b]$ with $B_j^{\lambda_j} - \int_a^b f_j^{\lambda_j}(x)\mathrm{d}x > 0$. Then, for any $t \in [a, b)$, one has*

$$\prod_{j=1}^{m}\left(B_j^{\lambda_j} - \int_a^b f_j^{\lambda_j}(x)\,\mathrm{d}x\right)^{1/\lambda_j} \geq \prod_{j=1}^{m}\left(B_j^{\lambda_j} - \int_a^t f_j^{\lambda_j}(x)\,\mathrm{d}x\right)^{1/\lambda_j} - \int_t^b \prod_{j=1}^{m} f_j(x)\,\mathrm{d}x \geq \prod_{j=1}^{m} B_j - \int_a^b \prod_{j=1}^{m} f_j(x)\,\mathrm{d}x. \tag{24}$$

Proof. We need to prove only the left side of inequality (24). The proof of the right side of inequality (24) is similar. For any positive integers n and l, we choose an equidistant partition of $[a, t]$ and $[t, b]$, respectively, as

$$\begin{gathered} a < a + \frac{c-a}{n} < \cdots < a + \frac{c-a}{n}k < \cdots < a + \frac{c-a}{n}(n-1) < c, \\ c < c + \frac{b-c}{l} < \cdots < c + \frac{b-c}{l}i < \cdots < c + \frac{b-c}{l}(l-1) < b, \\ x_k = a + \frac{c-a}{n}k, \quad \Delta x_k = \frac{c-a}{n}, \quad k = 1, 2, \ldots, n, \\ x_i = c + \frac{b-c}{l}i, \quad \Delta x_i = \frac{b-c}{l}, \quad i = 1, 2, \ldots, n. \end{gathered} \tag{25}$$

Noting that $B_j^{\lambda_j} - \int_a^b f_j^{\lambda_j}(x)\mathrm{d}x = B_j^{\lambda_j} - (\int_a^c f_j^{\lambda_j}(x)\mathrm{d}x + \int_c^b f_j^{\lambda_j}(x)\mathrm{d}x) > 0$ $(j = 1, 2, \ldots, m)$, we have

$$B_j^{\lambda_j} - \left\{\lim_{n\to\infty}\sum_{k=1}^{n} f_j^{\lambda_j}\left[a + \frac{k(c-a)}{n}\right]\frac{c-a}{n} + \lim_{l\to\infty}\sum_{i=1}^{l} f_j^{\lambda_j}\left[c + \frac{i(b-c)}{l}\right]\frac{b-c}{l}\right\} > 0, \quad (j = 1, 2, \ldots, m). \tag{26}$$

Consequently, there exists a positive integer N, such that

$$B_j^{\lambda_j} - \left[\sum_{k=1}^{n} f_j^{\lambda_j}\left(a + \frac{k(c-a)}{n}\right)\frac{c-a}{n} + \sum_{i=1}^{l} f_j^{\lambda_j}\left(c + \frac{i(b-c)}{l}\right)\frac{b-c}{l}\right] > 0 \tag{27}$$

for all n, $l > N$ and $j = 1, 2, \ldots, m$.

By using Theorem 3, for any n, $l > N$, the following inequality holds:

$$\begin{aligned} &\prod_{j=1}^{m}\left\{B_j^{\lambda_j} - \left[\sum_{k=1}^{n} f_j^{\lambda_j}\left(a + \frac{k(c-a)}{n}\right)\frac{c-a}{n} + \sum_{i=1}^{l} f_j^{\lambda_j}\left(c + \frac{i(b-c)}{l}\right)\frac{b-c}{l}\right]\right\}^{1/\lambda_j} \\ &\geq \prod_{j=1}^{m}\left[B_j^{\lambda_j} - \sum_{k=1}^{n} f_j^{\lambda_j}\left(a + \frac{k(c-a)}{n}\right)\frac{c-a}{n}\right]^{1/\lambda_j} - \sum_{i=1}^{l}\left[\prod_{j=1}^{m} f_j\left(c + \frac{i(b-c)}{l}\right)\right]\left(\frac{b-c}{l}\right)^{\sum_{j=1}^{m}(1/\lambda_j)}. \end{aligned} \tag{28}$$

Since

$$\sum_{j=1}^{m}\frac{1}{\lambda_j} = 1, \tag{29}$$

we have

$$\begin{aligned} &\prod_{j=1}^{m}\left\{B_j^{\lambda_j} - \left[\sum_{k=1}^{n} f_j^{\lambda_j}\left(a + \frac{k(c-a)}{n}\right)\frac{c-a}{n} + \sum_{i=1}^{l} f_j^{\lambda_j}\left(c + \frac{i(b-c)}{l}\right)\frac{b-c}{l}\right]\right\}^{1/\lambda_j} \\ &\geq \prod_{j=1}^{m}\left[B_j^{\lambda_j} - \sum_{k=1}^{n} f_j^{\lambda_j}\left(a + \frac{k(c-a)}{n}\right)\frac{c-a}{n}\right]^{1/\lambda_j} - \sum_{i=1}^{l}\left[\prod_{j=1}^{m} f_j\left(c + \frac{i(b-c)}{l}\right)\right]\left(\frac{b-c}{l}\right). \end{aligned} \tag{30}$$

Noting that $f_j(x)$ $(j = 1, 2, \ldots, m)$ are positive Riemann integrable functions on $[a, b]$, we know that $\prod_{j=1}^{m} f_j(x)$ and $f_j^{\lambda_j}(x)$ are also integrable on $[a, b]$. Letting $n \to \infty$ on both sides of inequality (30), we get the left side of inequality (24). The proof of Theorem 8 is completed. □

We give here a direct consequence from Theorem 8. Putting $m = 2$, $\lambda_1 = p$, $\lambda_2 = q$, $B_1 = a_1$, $B_2 = b_1$, $f_1 = f$, and $f_2 = g$ in (24), we obtain a special important case as follows.

Corollary 9. *Let p and q be real numbers such that $p > 0$, $q < 0$, and $(1/p) + (1/q) = 1$, let a_1, $b_1 > 0$, and let f, g be positive integrable functions defined on $[a, b]$ with $a_1^p - \int_a^b f^p(x)\mathrm{d}x > 0$ and $b_1^q - \int_a^b g^q(x)\mathrm{d}x > 0$. Then, for any $t \in [a, b)$, one has*

$$\begin{aligned}\left(a_1^p - \int_a^b f^p(x)\,\mathrm{d}x\right)^{1/p} & \left(b_1^q - \int_a^b g^q(x)\,\mathrm{d}x\right)^{1/q} \\ & \geq \left(a_1^p - \int_a^c f^p(x)\,\mathrm{d}x\right)^{1/p}\left(b_1^q - \int_a^c g^q(x)\,\mathrm{d}x\right)^{1/q} \\ & \geq a_1 b_1 - \int_a^b f(x)\, g(x)\,\mathrm{d}x.\end{aligned} \tag{31}$$

Finally, we present a refinement of integral type of generalized Bellman's inequality.

Theorem 10. *Let $B_j > 0$ $(j = 1, 2, \ldots, m)$, let $0 < p < 1$, and let $f_j(x)$ $(j = 1, 2, \ldots, m)$ be positive integrable functions defined on $[a, b]$ with $B_j^{\lambda_j} - \int_a^b f_j^{\lambda_j}(x)\mathrm{d}x > 0$. Then, for any $t \in [a, b)$, one has*

$$\begin{aligned}\left[\sum_{j=1}^m \left(B_j^p - \int_a^b f_j^p(x)\,\mathrm{d}x\right)^{1/p}\right]^p & \\ \geq \left[\sum_{j=1}^m \left(B_j^p - \int_a^t f_j^p(x)\,\mathrm{d}x\right)^{1/p}\right]^p & - \int_t^b \left(\sum_{j=1}^m f_j(x)\right)^p \mathrm{d}x \\ \geq \left(\sum_{j=1}^m B_j\right)^p - \int_a^b \left(\sum_{j=1}^m f_j(x)\right)^p \mathrm{d}x. &\end{aligned} \tag{32}$$

Proof. The proof of Theorem 10 is similar to the proof of Theorem 8. □

A special case to the last theorem is as follows.

Corollary 11. *Let $0 < p < 1$, let $a_1, b_1 > 0$, and let f, g be positive integrable functions defined on $[a, b]$ with $a_1^p - \int_a^b f^p(x)\mathrm{d}x > 0$ and $b_1^q - \int_a^b g^q(x)\mathrm{d}x > 0$. Then, for any $t \in [a, b)$, one has*

$$\begin{aligned}\left[\left(a_1^p - \int_a^b f^p(x)\,\mathrm{d}x\right)^{1/p} + \left(b_1^p - \int_a^b g^p(x)\,\mathrm{d}x\right)^{1/p}\right]^p & \\ \geq \left[\left(a_1^p - \int_a^t f^p(x)\,\mathrm{d}x\right)^{1/p} + \left(b_1^p - \int_a^t g^p(x)\,\mathrm{d}x\right)^{1/p}\right]^p & \\ - \int_t^b (f(x) + g(x))^p \mathrm{d}x & \\ \geq (a_1 + b_1)^p - \int_a^b (f(x) + g(x))^p \mathrm{d}x. &\end{aligned} \tag{33}$$

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