Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 721909, 6 pages
http://dx.doi.org/10.1155/2013/721909

Research Article

Positive Solutions for Third-Order Boundary-Value Problems with the Integral Boundary Conditions and Dependence on the First-Order Derivatives

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Received 8 April 2013; Revised 2 June 2013; Accepted 9 June 2013

Academic Editor: Wei-Shih Du

By using a fixed point theorem in a cone and the nonlocal third-order BVP's Green function, the existence of at least one positive solution for the third-order boundary-value problem with the integral boundary conditions $x'''(t) + f(t, x(t), x'(t)) = 0$, $t \in J$, $x(0) = 0$, $x''(0) = 0$, and $x(1) = \int_0^1 g(t)x(t)dt$ is considered, where f is a nonnegative continuous function, $J = [0, 1]$, and $g \in L[0, 1]$. The emphasis here is that f depends on the first-order derivatives.

1. Introduction

Third-order boundary-value problems for differential equation play a very important role in a variety of different areas of applied mathematics and physics. Recently, third-order boundary-value problems have been many scholars' research object. For example, heat conduction, chemical engineering, underground water flow, thermoelasticity, and plasma physics can produce boundary-value problems with integral boundary conditions [1–3]. For more information about the general theory of integral equations and their relation with boundary-value problems, we refer readers to the books of Corduneanu [4] and Agarwal and O'Regan [5].

Moreover, boundary-value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multipoint, and nonlocal boundary-value problems as special cases. Such kind of BVPs in Banach space has been studied by some researchers [6–8].

By the fixed point index theory in cones [9], Zhang et al. [10] investigated the multiplicity of positive solutions for a class of nonlinear boundary-value problems of fourth-order differential equations with integral boundary conditions in ordered Banach spaces. Feng et al. [11] investigated the existence and multiplicity of positive solutions for a class of nonlinear boundary-value problems of second-order differential equations with integral boundary conditions in ordered Banach spaces. Guo et al. [12] investigated the existence of positive solutions for the third-order boundary-value problems with integral boundary conditions and dependence on the second derivatives. In [13], by using the fixed point theorem of cone expansion and compression of norm type, Zhang and Ge proved the existence and multiplicity of symmetric positive solutions for the fourth-order boundary-value problems with integral boundary conditions. By using Krasnoselskii's fixed point theorem, Wang et al. [14] investigated the existence and nonexistence of positive solutions for a class of fourth-order nonlinear differential equation with integral boundary conditions

$$\begin{gathered}
x^{(4)}(t) = \omega(t) f\left(t, x(t), x''(t)\right), \quad 0 < t < 1, \\
x(0) = \int_0^1 h_1(s)\, x(s)\, ds, \\
x(1) = \int_0^1 k_1(s)\, x(s)\, ds, \\
x''(0) = \int_0^1 h_2(s)\, x''(s)\, ds, \\
x''(1) = \int_0^1 k_2(s)\, x''(s)\, ds,
\end{gathered} \tag{1}$$

where the arguments are based on Krasnoselskii's fixed point theorem for operators on a cone.

However, Zhao et al. [15] investigated the following third-order boundary-value problem with integral boundary conditions:

$$\begin{gathered} x'''(t) + f(t, x(t)) = \theta, \quad t \in J, \\ x(0) = \theta, \qquad x''(0) = \theta, \\ x(1) = \int_0^1 g(t)\, x(t)\, dt, \end{gathered} \tag{2}$$

under the assumptions

(1) $J = [0,1]$, and θ is the zero element of E,

(2) $f : C([0,1] \times P, P)$, and $g \in L[0,1]$ is nonnegative,

where P is a cone in the real Banach E.

All the above works were done under the assumption that the first-order derivative x' is not involved explicitly in the nonlinear term f. In this paper, we are concerned with the existence of positive solutions for the third-order boundary-value problem with the integral boundary conditions

$$\begin{gathered} x'''(t) + f\left(t, x(t), x'(t)\right) = 0, \quad t \in J, \\ x(0) = 0, \qquad x''(0) = 0, \\ x(1) = \int_0^1 g(t)\, x(t)\, dt. \end{gathered} \tag{3}$$

Throughout, we assume

(H_1) $J = [0,1]$, $f : [0,1] \times R^2 \to R^+$ is continuous, $g \in L[0,1]$, $g(t) \ge 0$, and $\sigma \in [0,1)$, where $\sigma = \int_0^1 s g(s) ds$.

To show the existence of positive solutions for (3), we define two positive continuous convex functionals. Then, by using the fixed point theorem [16] in a cone and the nonlocal third-order BVP's Green function, we give some new criteria for the existence of positive solutions for (3).

2. Preliminaries

Let $Y = C[0,1]$ be the Banach space equipped with the norm $\|x\|_0 = \max_{t\in[0,1]}|x(t)|$.

Lemma 1 (see [15]). *Suppose* (H_1) *holds. Then for any* $y(t) \in C[0,1]$, *the problem*

$$\begin{gathered} x'''(t) + y(t) = 0, \quad t \in J, \\ x(0) = 0, \qquad x''(0) = 0, \\ x(1) = \int_0^1 g(t)\, x(t)\, dt \end{gathered} \tag{4}$$

has a unique solution

$$x(t) = \int_0^1 H(t,s)\, y(s)\, ds, \tag{5}$$

where

$$\begin{gathered} H(t,s) = G(t,s) + \frac{t}{1-\sigma}\int_0^1 G(\tau,s)\, g(\tau)\, d\tau, \\ G(t,s) = \begin{cases} \frac{1}{2}t(1-s)^2 - \frac{1}{2}(t-s)^2, & 0 \le s \le t \le 1, \\ \frac{1}{2}t(1-s)^2, & 0 \le t \le s \le 1. \end{cases} \end{gathered} \tag{6}$$

Lemma 2 (see [15]). *For* $t, s \in [0,1]$, *one has* $0 \le G(t,s) \le \max_{0\le t,s\le 1} G(t,s) \le 1/8$.

Remark 3. When $t, s \in (0,1)$, it is easy to check that $G(t,s) > 0$.

In addition, for $0 \le s \le t \le 1$, the maximum of $G(t,s)$ occurs at $t = (1+s^2)/2$.

Lemma 4 (see [15]). *Choose* $\delta \in (0, 1/2)$ *and* $J_\delta = [\delta, 1-\delta]$; *then for all* $t \in J_\delta$, $v, s \in [0,1]$, *one has*

$$G(t,s) \ge \rho G(v,s), \tag{7}$$

where $\rho = 4\delta^2(1-\delta)$.

Remark 5. For $0 \le s \le t \le 1$, denote $G(t,s) = G_1(t,s)$. Notice that $G_1(t,s)$ is concave with respect to t; we have

$$\begin{aligned} \min_{t\in J_\delta, 0\le s\le t} G_1(t,s) &= \min\left\{G_1(\delta,s), G_1(1-\delta,s)\right\} \\ &= \frac{1}{2}\delta^2(1-\delta). \end{aligned} \tag{8}$$

Lemma 6 (see [15]). *Assume that* (H_1) *holds; then*

(i) $H(t,s) \le (1/2)\gamma$, $t \in [0,1]$,

(ii) $H(t,s) \ge \rho H(v,s)$, $t \in J_\delta$, $v, s \in [0,1]$,

where $\gamma = (1 + \int_0^1 (1-s) g(s) ds)/(1-\sigma)$.

Lemma 7. *If* $y \in C[0,1]$, $y(t) \ge 0$, *then the unique solution* $x(t)$ *of problem* (4) *satisfies*

$$\min_{t\in J_\delta} x(t) \ge \rho\|x\|_0. \tag{9}$$

Proof. By Lemmas 4 and 6 and (5), we get

$$\begin{aligned} \min_{t\in J_\delta} x(t) &= \min_{t\in J_\delta}\int_0^1 H(t,s)\, y(s)\, ds \\ &\ge \rho\int_0^1 H(v,s)\, y(s)\, ds \\ &\ge \rho x(v). \end{aligned} \tag{10}$$

For $v \in [0,1]$, we have

$$\min_{t\in J_\delta} x(t) \ge \rho x(v). \tag{11}$$

So,

$$\min_{t\in J_\delta} x(t) \ge \rho \max_{v\in[0,1]} x(v) = \rho \max_{v\in[0,1]} |x(v)| = \rho\|x\|_0. \tag{12}$$

The proof is completed. □

Let X be a Banach space and $K \subset X$ a cone. Suppose $\alpha, \beta : X \to R^+$ are two continuous convex functionals satisfying $\alpha(\lambda x) = |\lambda|\alpha(x)$, $\beta(\lambda x) = |\lambda|\beta(x)$, for $x \in X$, $\lambda \in R$, $\|x\| \leq M \max\{\alpha(x), \beta(x)\}$, for $x \in X$, and $\alpha(x) \leq \alpha(y)$ for $x, y \in K$, $x \leq y$, where $M > 0$ is a constant.

Theorem 8 (see [16]). *Let $r_2 > r_1 > 0$, $L > 0$ be constants and*

$$\Omega_i = \{x \in X : \alpha(x) < r_i, \beta(x) < L\}, \quad i = 1, 2, \tag{13}$$

two bounded open sets in X. Set

$$D_i = \{x \in X : \alpha(x) = r_i\}, \quad i = 1, 2. \tag{14}$$

Assume $T : K \to K$ is a completely continuous operator satisfying

(A_1) *$\alpha(Tx) < r_1$, $x \in D_1 \cap K$; $\alpha(Tx) > r_2$, $x \in D_2 \cap K$;*

(A_2) *$\beta(Tx) < L$, $x \in K$;*

(A_3) *there is a $p \in (\Omega_2 \cap K) \setminus \{0\}$ such that $\alpha(p) \neq 0$ and $\alpha(x + \lambda p) \geq \alpha(x)$, for all $x \in K$ and $\lambda \geq 0$.*

Then T has at least one fixed point in $(\Omega_2 \setminus \overline{\Omega}_1) \cap K$.

3. Main Results

Let $X = C^1[0,1]$ be the Banach space equipped with the norm $\|x\| = \max_{t\in[0,1]}|x(t)| + \max_{t\in[0,1]}|x'(t)|$, and $K = \{x \in X : x(t) \geq 0, \min_{t\in J_\delta} x(t) \geq \rho\|x\|_0\}$ is a cone in X.

Define two continuous convex functionals $\alpha(x) = \max_{t\in[0,1]}|x(t)|$ and $\beta(x) = \max_{t\in[0,1]}|x'(t)|$, for each $x \in X$; then $\|x\| \leq 2 \max\{\alpha(x), \beta(x)\}$ and $\alpha(\lambda x) = |\lambda|\alpha(x)$, $\beta(\lambda x) = |\lambda|\beta(x)$, for $x \in X$, $\lambda \in R$; $\alpha(x) \leq \alpha(y)$ for $x, y \in K$, $x \leq y$.

In the following, we denote

$$\begin{gathered}\eta_0 = \frac{1}{8} + \int_0^1 \left[\frac{1}{1-\sigma}\int_0^1 G(\tau, s) g(\tau) d\tau\right] ds, \\ \eta_1 = \max_{v\in[0,1]} \int_\delta^{1-\delta} H(v, s) ds, \\ \eta_2 = \frac{2}{3} + \int_0^1 \left[\frac{1}{1-\sigma}\int_0^1 G(\tau, s) g(\tau) d\tau\right] ds.\end{gathered} \tag{15}$$

We will suppose that there are $L > b > \rho b > c > 0$ such that $f(t, x, y)$ satisfies the following growth conditions:

(H_2) $f(t, x, y) < c/\eta_0$, for $(t, x, y) \in [0,1] \times [0, c] \times [-L, L]$,

(H_3) $f(t, x, y) \geq b/\rho\eta_1$, for $(t, x, y) \in [\delta, 1-\delta] \times [\rho b, b] \times [-L, L]$,

(H_4) $f(t, x, y) < L/\eta_2$, for $(t, x, y) \in [0,1] \times [0, b] \times [-L, L]$.

Let

$$\begin{gathered} f^*(t, x, y) = \begin{cases} f(t, x, y), (t, x, y) \in [0,1] \times [0, b] \times (-\infty, \infty), \\ f(t, b, y), (t, x, y) \in [0,1] \times (b, \infty) \times (-\infty, \infty), \end{cases} \\ f_1(t, x, y) = \begin{cases} f^*(t, x, y), (t, x, y) \in [0,1] \times [0, \infty) \times [-L, L], \\ f^*(t, x, -L), (t, x, y) \in [0,1] \times [0, \infty) \times (-\infty, -L], \\ f^*(t, x, L), (t, x, y) \in [0,1] \times [0, \infty) \times [L, \infty). \end{cases}\end{gathered} \tag{16}$$

We denote

$$\begin{gathered}(Tx)(t) = \int_0^1 H(t, s) f_1(s, x, x') ds, \\ (Tx)'(t) = \int_0^1 \frac{\partial H(t, s)}{\partial t} f_1(s, x, x') ds.\end{gathered} \tag{17}$$

Lemma 9. *Suppose (H_1) holds. Then $T : K \to K$ is completely continuous.*

Proof. For $x \in K$, by Lemmas 2 and 4, it is obviously that $Tx \geq 0$.

By Lemma 7, we have

$$\min_{t\in J_\delta} Tx(t) \geq \rho\|Tx\|_0. \tag{18}$$

So, we can get $T(K) \subset K$.

In the following, we will show that $T : K \to K$ is completely continuous.

At first we show that $T : K \to K$ is continuous.

Let $x_n, x^* \in K$, it satisfies $\|x_n - x^*\| \to 0$, $(n \to \infty)$, and then there is a constant $M_0 > 0$, such that $\max_{t\in[0,1]}\{|x_n(t)|, |x^*(t)|, |x_n'(t)|, |x^{*'}(t)|\} \leq M_0$; then

$$\begin{aligned}|(Tx_n)(t) - (Tx^*)(t)| &= \left|\int_0^1 H(t, s) f_1(s, x_n, x_n') ds - \int_0^1 H(t, s) f_1(s, x^*, x^{*'}) ds\right| \\ &\leq \int_0^1 H(t, s)\left|f_1(s, x_n, x_n') - f_1(s, x^*, x^{*'})\right| ds,\end{aligned}$$

$$|(Tx_n)'(t) - (Tx^*)'(t)| = \left|\int_0^1 \frac{\partial H(t, s)}{\partial t} f_1(s, x, x_n') ds - \int_0^1 \frac{\partial H(t, s)}{\partial t} f_1(s, x^*, x^{*'}) ds\right|$$

$$\begin{aligned}
&\le \int_0^1 \left|\frac{\partial H(t,s)}{\partial t}\right| \left|f_1\left(s,x,x_n'\right) - f_1\left(s,x^*,x^{*'}\right)\right| ds \\
&< \int_0^1 \left[\frac{1}{2}(1-s)^2 + (1-s)\right] \\
&\quad \times \left|f_1\left(s,x,x_n'\right) - f_1\left(s,x^*,x^{*'}\right)\right| ds \\
&\quad + \int_0^1 \left[\frac{1}{1-\sigma}\int_0^1 G(\tau,s)\, g(\tau)\, d\tau\right] \\
&\qquad \times \left|f_1\left(s,x,x_n'\right) - f_1\left(s,x^*,x^{*'}\right)\right| ds.
\end{aligned} \tag{19}$$

By f which is uniformly continuous on $[0,1] \times [-M_0, M_0] \times [-M_0, M_0]$, we get

$$\left\|Tx_n - Tx^*\right\| \longrightarrow 0, \quad (n \longrightarrow \infty). \tag{20}$$

Next we show that $T : K \to K$ is compact.

Let $B \subset K$ be bounded; then there is $M > 0$, such that $\|x\| \le M$. For $x \in B$, we have

$$\begin{aligned}
|(Tx)(t)| &= \left|\int_0^1 H(t,s)\, f_1\left(s,x,x'\right) ds\right| \\
&\le \int_0^1 \frac{1}{2}\gamma f_1\left(s,x,x'\right) ds \\
&= \frac{1}{2}\int_0^1 \frac{1 + \int_0^1 (1-s)\, g(s)\, ds}{1-\sigma}\, ds \times C^*,
\end{aligned} \tag{21}$$

where $C^* = \max\{|f_1(t,x,x')|; t \in [0,1], x \in B\}$.

Consider

$$\begin{aligned}
&\left|(Tx)'(t)\right| \\
&\quad = \left|\int_0^1 \frac{\partial H(t,s)}{\partial t} f_1\left(s,x,x'\right) ds\right| \\
&\quad = \left|\int_0^1 \left[\frac{\partial G(t,s)}{\partial t} + \frac{1}{1-\sigma}\int_0^1 G(\tau,s)\, g(\tau)\, d\tau\right]\right. \\
&\qquad \left. \times f_1\left(s,x,x'\right) ds\right| \\
&\quad < \left|\int_0^1 \left[\frac{1}{2}(1-s)^2 + (1-s)\right] ds\right. \\
&\qquad \left. + \int_0^1 \left[\frac{1}{1-\sigma}\int_0^1 G(\tau,s)\, g(\tau)\, d\tau\right] ds\right| \times C^* \\
&\quad = \left|\frac{2}{3} + \int_0^1 \left[\frac{1}{1-\sigma}\int_0^1 G(\tau,s)\, g(\tau)\, d\tau\right] ds\right| \times C^*.
\end{aligned} \tag{22}$$

It is clear that $T(B)$ is a bounded set in K, because $H(t,s)$ is uniformly continuous on $[0,1] \times [0,1]$, for $\varepsilon > 0$, there exists $\delta \in (0,\varepsilon)$, such that $|H(t_1,s) - H(t_2,s)| < \varepsilon$, and for t_1, $t_2 \in [0,1]$, $|t_1 - t_2| < \delta$.

For $x \in B$, we have

$$\begin{aligned}
&\left|(Tx)(t_1) - (Tx)(t_2)\right| \\
&\quad = \left|\int_0^1 H(t_1,s)\, f_1\left(s,x,x'\right) ds - \int_0^1 H(t_2,s)\, f_1\left(s,x,x'\right) ds\right| \\
&\quad \le \int_0^1 \left|H(t_1,s) - H(t_2,s)\right| ds \times C^* \le \varepsilon C^*, \\
&\left|(Tx)'(t_1) - (Tx)'(t_2)\right| \\
&\quad = \left|\int_0^1 \left.\frac{\partial H(t,s)}{\partial t}\right|_{t=t_1} f_1\left(s,x,x'\right) ds - \int_0^1 \left.\frac{\partial H(t,s)}{\partial t}\right|_{t=t_2} f_1\left(s,x,x'\right) ds\right| \\
&\quad = \left|\int_0^1 \left.\frac{\partial G(t,s)}{\partial t}\right|_{t=t_1} f_1\left(s,x,x'\right) ds - \int_0^1 \left.\frac{\partial G(t,s)}{\partial t}\right|_{t=t_2} f_1\left(s,x,x'\right) ds\right| \\
&\quad = \left|\int_0^{t_1} (s - t_1)\, f_1\left(s,x,x'\right) ds + \int_0^{t_2} (t_2 - s)\, f_1\left(s,x,x'\right) ds\right| \\
&\quad \le \frac{1}{2}\left|(t_1 - t_2)(t_1 + t_2)\right| \times C^* \le \varepsilon C^*.
\end{aligned} \tag{23}$$

Therefore $T(B)$ is equicontinuous. Using the Arzela-Ascoli theorem, a standard proof yields $T : K \to K$ which is completely continuous. □

Theorem 10. *Suppose* (H_1)–(H_4) *hold. Then BVP* (3) *has at least one positive solution* $x(t)$ *satisfying*

$$c < \alpha(x) < b, \qquad \beta(x) < L. \tag{24}$$

Proof. Take

$$\begin{aligned}
\Omega_1 &= \left\{x \in X : |x(t)| < c, \left|x(t)'\right| < L\right\}, \\
\Omega_2 &= \left\{x \in X : |x(t)| < b, \left|x(t)'\right| < L\right\},
\end{aligned} \tag{25}$$

two bounded open sets in X, and

$$D_1 = \{x \in X : \alpha(x) = c\}, \qquad D_2 = \{x \in X : \alpha(x) = b\}. \tag{26}$$

By Lemma 9, $T : K \to K$ is completely continuous, and there is a $p \in (\Omega_2 \cap K) \setminus \{0\}$ such that $\alpha(p) \ne 0$ and $\alpha(x + \lambda p) \ge \alpha(x)$ for all $u \in K$ and $\lambda \ge 0$.

By (H_2), for $x \in D_1 \cap K$ and $\alpha(x) = c$, we get

$$\begin{aligned}\alpha(Tx) &= \max_{t\in[0,1]}\left|\int_0^1 H(t,s) f_1(s,x,x')\,ds\right| \\ &= \max_{t\in[0,1]}\left|\int_0^1 \left[G(t,s) + \frac{t}{1-\sigma}\int_0^1 G(\tau,s) g(\tau)\,d\tau\right] \right. \\ &\qquad \left. \times f_1(s,x,x')\,ds\right| \\ &< \int_0^1 \left[\max_{t\in[0,1]} G(t,s) + \frac{t}{1-\sigma}\int_0^1 G(\tau,s) g(\tau)\,d\tau\right] \\ &\qquad \times f_1(s,x,x')\,ds \\ &< \left[\int_0^1 \frac{1}{8}\,ds + \int_0^1 \left(\frac{t}{1-\sigma}\int_0^1 G(\tau,s) g(\tau)\,d\tau\right) ds\right] \\ &\qquad \times \frac{c}{\eta_0} \\ &= \left[\frac{1}{8} + \int_0^1 \left(\frac{t}{1-\sigma}\int_0^1 G(\tau,s) g(\tau)\,d\tau\right) ds\right] \\ &\qquad \times \frac{c}{\eta_0} = c.\end{aligned} \tag{27}$$

By Lemma 7, for $x \in D_2 \cap K$ and $\alpha(x) = b$, there is $x(t) \geq \rho\alpha(x) = \rho b$, $t \in J_\delta$.

So, by (H_3), we get

$$\begin{aligned}\alpha(Tx) &= \max_{t\in[0,1]}\left|\int_0^1 H(t,s) f_1(s,x,x')\,ds\right| \\ &> \int_\delta^{1-\delta} H(t,s) f_1(s,x,x')\,ds \\ &> \rho\int_\delta^{1-\delta} H(v,s)\,ds \times \frac{b}{\rho\eta_1}.\end{aligned} \tag{28}$$

For $v \in [0,1]$, we have

$$\alpha(Tx) > \rho\int_\delta^{1-\delta} H(v,s)\,ds \times \frac{b}{\rho\eta_1}. \tag{29}$$

So,

$$\alpha(Tx) > \rho\max_{v\in[0,1]}\int_\delta^{1-\delta} H(v,s)\,ds \times \frac{b}{\rho\eta_1} = b. \tag{30}$$

By (H_4), for $x \in K$, we have

$$\begin{aligned}\beta(Tx) &= \max_{t\in[0,1]}\left|\int_0^1 \frac{\partial H(t,s)}{\partial t} f_1(s,x,x')\,ds\right| \\ &< \left|\int_0^1 \left[\frac{\partial G(t,s)}{\partial t} + \frac{1}{1-\sigma}\int_0^1 G(\tau,s) g(\tau)\,d\tau\right] \right. \\ &\qquad \left. \times f_1(s,x,x')\,ds\right. \\ &= \left|\int_0^t \left(\frac{1}{2}(1-s)^2 - (t-s)\right) f_1(s,x,x')\,ds \right. \\ &\qquad + \int_t^1 \frac{1}{2}(1-s)^2 f_1(s,x,x')\,ds \\ &\qquad + \int_0^1 \left[\frac{1}{1-\sigma}\int_0^1 G(\tau,s) g(\tau)\,d\tau\right] \\ &\qquad \left. \times f_1(s,x,x')\,ds\right| \\ &< \left[\left|\int_0^1 \left(\frac{1}{2}(1-s)^2 + (1-s)\right) ds \right. \right. \\ &\qquad \left. + \int_0^1 \left(\frac{1}{1-\sigma}\int_0^1 G(\tau,s) g(\tau)\,d\tau\right) ds\right] \times \frac{L}{\eta_2} \\ &= \left[\frac{2}{3} + \int_0^1 \left(\frac{1}{1-\sigma}\int_0^1 G(\tau,s) g(\tau)\,d\tau\right) ds\right] \\ &\qquad \times \frac{L}{\eta_2} = L.\end{aligned} \tag{31}$$

Theorem 8 implies there is $x \in (\Omega_2 \setminus \overline{\Omega}_1) \cap K$ such that $x = Tx$. So, $x(t)$ is a positive solution for BVP (3) satisfying

$$c < \alpha(x) < b, \qquad \beta(x) < L. \tag{32}$$

Thus, Theorem 10 is completed. □

4. Example

Example 1. Consider the following boundary-value problem

$$\begin{gathered} x'''(t) + f(t, x(t), x'(t)) = 0, \quad 0 < t < 1, \\ x(0) = 0, \qquad x''(0) = 0, \\ x(1) = \int_0^1 x(t)\,dt, \end{gathered} \tag{33}$$

where

$$f(t,x,y) = \begin{cases} \frac{t}{3}x + 2x + |\cos y|, \\ (t,x,y) \in [0,1] \times [0,0.5] \times [-3667, 3667], \\ \frac{109t}{3}(x-0.5) + 25742(x-0.5) + \frac{t}{6} + 1 + |\cos y|, \\ (t,x,y) \in [0,1] \times [0.5,0.6] \times [-3667, 3667], \\ \frac{t}{3}(11-x) + 222(x+11) + |\cos y|, \\ (t,x,y) \in [0,1] \times [0.6,11] \times [-3667, 3667]. \end{cases} \tag{34}$$

In this problem, we know that $g(t) = 1$; then we can get $\sigma = \int_0^1 sg(s)ds = 1/2$. Choose $\delta = 1/8 \in (0, 1/2)$; then $\rho = 4\delta^2(1-\delta) = 7/128$.

Furthermore, we obtain

$$\eta_0 = \frac{5}{24}, \qquad \rho\eta_1 = \frac{35}{8192}, \qquad \eta_2 = \frac{3}{4}. \tag{35}$$

If we take $c = 0.5$, $b = 11$, and $L = 3667$, then we get $\rho b \approx 0.601 > 0.6$:

$$f(t, x, y) = \frac{t}{3}x + 2x + |\cos y| \leq 2.17 < \frac{c}{\eta_0} \approx 2.4, \tag{36}$$

for $(t, x, y) \in [0, 1] \times [0, 0.5] \times [-3667, 3667]$,

$$\begin{aligned} f(t, x, y) = {} & \frac{t}{3}(11 - x) + 222(x + 11) \\ & + |\cos y| \geq 2575.2 > \frac{b}{\rho\eta_1} \approx 2574.1, \end{aligned} \tag{37}$$

for $(t, x, y) \in [\delta, 1 - \delta] \times [\delta b, 11] \times [-3667, 3667]$,

$$f(t, x, y) \leq 4888.8 < \frac{L}{\eta_2} \approx 4889.3, \tag{38}$$

for $(t, x, y) \in [0, 1] \times [0, 11] \times [-3667, 3667]$.

Then all the conditions of Theorem 10 are satisfied. Therefore, by Theorem 10 we know that boundary-value problem (33) has at least one positive solution $x(t)$ satisfying

$$0.5 < \alpha(x) < 11, \qquad \beta(x) < 3667. \tag{39}$$

Acknowledgments

The project is supported by the Natural Science Foundation of China (10971045) and the Natural Science Foundation of Hebei Province (A2013208147).

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