Research Article

Positive Steady States of a Strongly Coupled Predator-Prey System with Holling-$(n + 1)$ Functional Response

Xiao-zhou Feng[1,2] and Zhi-guo Wang[2]

[1] College of Science, Xi'an Technological University, Xi'an 710032, China
[2] Institute of Mathematics, Shaanxi Normal University, Xi'an 710062, China

Correspondence should be addressed to Xiao-zhou Feng; flxzfxz8@163.com

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This paper discusses a predator-prey system with Holling-$(n + 1)$ functional response and the fractional type nonlinear diffusion term in a bounded domain under homogeneous Neumann boundary condition. The existence and nonexistence results concerning nonconstant positive steady states of the system were obtained. In particular, we prove that the positive constant solution $(\tilde{u}, \tilde{v})$ is asymptotically stable when the parameter k satisfies some conditions.

1. Introduction

In this paper, we are interested in the positive steady states of the strongly coupled predator-prey system with Holling-$(n + 1)$ functional response. The specific system is as follows:

$$-d_1 \Delta u = u \left(1 - \frac{u}{k} \right) - \frac{u^n v}{a + u^n} \quad \text{in } \Omega,$$

$$-d_2 \Delta \left(v + \frac{d_3 v}{1 + u} \right) = -bv + \frac{\sigma u^n v}{a + u^n} \quad \text{in } \Omega, \qquad (1)$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial \Omega,$$

where $1 \le n < +\infty$; Ω is a bounded domain in R^N with smooth boundary $\partial \Omega$; $\partial / \partial \nu$ is the outward directional derivative normal to $\partial \Omega$; u and v stand for the densities of the prey and predator; the given coefficients d_i ($i = 1, 2$), a, b, k, and σ are positive constants. The term $u^n / (a + u^n)$ is named Holling-$(n + 1)$ functional response [1, 2]. In the second equation, the fractional type nonlinear diffusion term $\Delta d_2 (d_3 v / (1 + u))$ models a situation in which the population pressure of the predator species weakens in high-density areas of the prey species. For more precise details, we can refer to [3, 4]. Paper [3] discusses a strongly coupled predator-prey system with nonmonotonic functional response, the

existence and nonexistence results concerning nonconstant positive steady states of the system were proved by degree theory. Paper [4] considers the positive steady states for a prey-predator model with some nonlinear diffusion terms, and the sufficient conditions for the existence of positive steady state solutions were obtained by bifurcation theory.

In recent years, there has been considerable interest in the dynamics of strongly coupled reaction-diffusion systems with cross-diffusion. We point out that most efforts have concentrated on the Lotka-Volterra competition system which was proposed first by Shigesada et al. [5]. Since their pioneering work, many authors have studied population models with cross-diffusion terms from various mathematical viewpoints, for example, the global existence of time-depending solutions [6–11], the stability analysis for steady states [12–14], and the steady state problems [15–21]. In this paper, we mainly consider the existence of solutions of (1). The research method refers to [3, 22, 23].

For convenience of the research, we write (1) as the following form:

$$-\Delta \Phi (U) = G (U), \quad U = \begin{bmatrix} u \\ v \end{bmatrix}, \qquad (2)$$

where

$$\Phi(U) = \begin{bmatrix} \phi_1(U) \\ \phi_2(U) \end{bmatrix} = \begin{bmatrix} d_1 u \\ d_2 v \left(1 + \dfrac{d_3}{1+u}\right) \end{bmatrix}, \qquad (3)$$

$$G(U) = \begin{bmatrix} g(u) - vp(u) \\ -bv + \sigma vp(u) \end{bmatrix},$$

$$g(u) = u\left(1 - \frac{u}{k}\right), \qquad p(u) = \frac{u^n}{a + u^n}, \quad n \in N^+. \quad (4)$$

The main work of this paper is to study the effects of the fractional type nonlinear diffusion pressures on the existence of nonconstant positive steady states of (1). Here, a positive solution means a smooth solution (u, v) with both u and v being positive. We will demonstrate that the cross-diffusion pressure d_3 may help forming more patterns. Obviously, for system (1), one notes that when $\sigma \leq b$, there holds $\sup_{s\geq 0}\{-b + \sigma s^n/(a + s^n)\} \leq 0$, so that the only nonnegative solutions to (1) are $U = (0,0)$ and $U = (k, 0)$. Consequently, (1) does not have any positive solution. On the other hand, when $\sigma > b$, the unique positive constant solution to (1) is $(\tilde{u}, \tilde{v})$; that is,

$$\tilde{u} = \sqrt[n]{\frac{ab}{\sigma - b}},$$

$$\tilde{v} = \frac{(1 - \tilde{u}/k)(a + \tilde{u}^n)}{\tilde{u}^{n-1}} = \frac{(k - \tilde{u})\sigma\tilde{u}}{kb}, \quad \text{if } k > \tilde{u}. \quad (5)$$

The organization of our paper is as follows. In Section 2, we establish a priori upper and lower bounds for positive solutions of (1). In Section 3, we use a degree theory to develop a general result to enable one to conclude the existence or nonexistence of nonconstant steady-state solutions or patterns as the index of positive constant steady states changes. In Section 4, we establish the existence of nonconstant positive solutions to (1) for a large range of diffusion and cross-diffusion coefficients. Meanwhile, we prove that the positive constant solution $(\tilde{u}, \tilde{v})$ is asymptotically stable for different ranges of parameters.

2. Upper and Lower Bounds for Positive Solutions

The main purpose of this section is to give a priori upper and lower positive bounds for positive solutions of (1). Firstly, we cite two known results.

Harnack Inequality (see [24]). Let $c \in C(\overline{\Omega})$ and $\omega \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a positive classical solution to $\Delta\omega(x) + c(x)\omega(x) = 0$ in Ω subject to the homogeneous Neumann boundary condition. Then, there exists a positive constant $C = C(\Omega, \|c\|_\infty)$ such that $\max_{\overline{\Omega}} \omega \leq C\min_{\overline{\Omega}} \omega$.

Maximum Principle (see [25]). Suppose that $g \in C(\overline{\Omega} \times R^1)$. If $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies

$$\Delta w(x) + g(x, w(x)) \geq 0 \quad \text{in } \Omega,$$

$$\frac{\partial w}{\partial \nu} \leq 0 \quad \text{on } \partial\Omega, \quad (6)$$

and $w(x_0) = \max_{\overline{\Omega}} w$, then $g(x_0, w(x_0)) \geq 0$.

Theorem 1. *Let* $P = (\sigma, a, k, b) \in (0, \infty)^4$ *and* $d, \hat{d} > 0$ *be fixed constants. Assume that* $d_1, d_2 \geq d$ *and* $0 \leq d_3 \leq \hat{d}$ *there exists a positive constant* $C = C(P, d, \hat{d}, \Omega)$, *such that any positive solution* (u, v) *of* (1) *satisfies*

$$\max_{\overline{\Omega}} u(x) + \max_{\overline{\Omega}} v(x) + \frac{\max_{\overline{\Omega}} u(x)}{\min_{\overline{\Omega}} u(x)} + \frac{\max_{\overline{\Omega}} v(x)}{\min_{\overline{\Omega}} v(x)} \leq C. \quad (7)$$

Proof. Let $\phi_1(U)$ and $\phi_2(U)$ be defined as in (3), and denote that

$$g_1(U) = u\left(1 - \frac{u}{k} - \frac{u^{n-1}v}{a + u^n}\right),$$

$$g_2(U) = v\left(\frac{\sigma u^n}{a + u^n} - b\right), \quad (8)$$

$$G(U) = (g_1(U), g_2(U))^T.$$

Then, (1) becomes

$$-\Delta\Phi(U) = G(U) \quad \text{in } \Omega, \qquad \frac{\partial U}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \quad (9)$$

For the first equation of (9), by the Maximum Principle, we have

$$\max_{\overline{\Omega}} u(x) \leq \frac{1}{d_1}\max_{\overline{\Omega}} \phi_1(U) < k. \quad (10)$$

The function $\phi_2(U)$ satisfies

$$\Delta\phi_2(U) + c(x)\phi_2(U) = 0 \quad \text{in } \Omega,$$

$$\frac{\partial\phi_2(U)}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \quad (11)$$

where $c(x) = g_2(U)/\phi_2(U) \in C(\overline{\Omega})$. It is easy to verify that the norm

$$\|c\|_\infty = \left\| \frac{v(\sigma u^n/(a + u^n) - b)}{d_2(v + d_3 v/(1+u))} \right\|_\infty \leq \frac{\sigma k^n/(a + k^n) + b}{d_2}$$

$$\leq \frac{\sigma k^n + b(a + k^n)}{d(a + k^n)}; \quad (12)$$

then, $\|c\|_\infty$ is bounded by a constant depending only on P and d ($d_2 \geq d$). By the Harnack inequality, we have

$$\max_{\overline{\Omega}} \phi_2\left(U\left(x\right)\right) \leq C_1 \min_{\overline{\Omega}} \phi_2\left(U\left(x\right)\right), \tag{13}$$

where $C_1 = C_1(\Omega, \|c\|_\infty) = C_1(P, d, \Omega)$. It follows that

$$\frac{\max_{\overline{\Omega}} v}{\min_{\overline{\Omega}} v} \leq \frac{\max_{\overline{\Omega}} \phi_2\left(U\left(x\right)\right)}{\min_{\overline{\Omega}} \phi_2\left(U\left(x\right)\right)} \times \frac{\max_{\overline{\Omega}} \left\{1 + d_3(1+u)^{-1}\right\}}{\min_{\overline{\Omega}} \left\{1 + d_3(1+u)^{-1}\right\}}$$

$$\leq C_1\left(1 + d_3\right) \leq C_1\left(1 + \widehat{d}\right). \tag{14}$$

Integrating the equations of (1) over Ω by parts and making use of the boundary conditions, we have

$$\sigma \int_\Omega u\left(1 - \frac{u}{k}\right) dx = b \int_\Omega v\, dx. \tag{15}$$

For any $f \in L^1(\Omega)$, we denote that $\overline{f} = (1/|\Omega|) \int_\Omega f(x) dx$. Then, from (15) and (10), we have

$$\overline{v} = \frac{\sigma}{|\Omega|\, b} \int_\Omega u\left(1 - \frac{u}{k}\right) dx < \frac{\sigma}{b} k. \tag{16}$$

Along with (14), we have

$$\max_{\overline{\Omega}} v \leq C_1\left(1 + \widehat{d}\right) \frac{\sigma}{b} k. \tag{17}$$

Similarly, consider the equation of ϕ_1 as follows:

$$\|c\|_\infty = \frac{1}{d_1}\left\|1 - \frac{u}{k} - \frac{u^{n-1}v}{a+u^n}\right\|_\infty$$

$$\leq \frac{1}{d_1}\left[1 + \frac{1}{k}\|u(x)\|_\infty + m\|v(x)\|_\infty\right], \tag{18}$$

where $m = [(n-1)a]^{(n-1)/n}/na$. Then, $\|c\|_\infty$ is bounded by a constant depending only on P and d ($d_1 \geq d$). By the Harnack Inequality, there exists a positive constant $C_2 = C_2(P, d, \widehat{d}, \Omega)$ such that $\max_{\overline{\Omega}} \phi_1(U(x)) \leq C_2 \min_{\overline{\Omega}} \phi_1(U(x))$. Then,

$$\frac{\max_{\overline{\Omega}} u}{\min_{\overline{\Omega}} u} = \frac{\max_{\overline{\Omega}} \phi_1\left(U\left(x\right)\right)}{\min_{\overline{\Omega}} \phi_1\left(U\left(x\right)\right)} \leq C_2. \tag{19}$$

Thus, along with (10), (14), and (17), we can complete the proof. □

Theorem 2. *Assume that $\sigma k^n \neq b(a+k^n)$. Let d, $\widehat{d} > 0$ be fixed constants. There exists a constant $\underline{C} = \underline{C}(d, \widehat{d}, P, \Omega) > 0$, such that any positive solution (u, v) of (1) satisfies*

$$\min_{\overline{\Omega}} u\left(x\right) > \underline{C}, \qquad \min_{\overline{\Omega}} v\left(x\right) > \underline{C}, \tag{20}$$

provided that $d_1, d_2 \geq d$ and $d_3 \leq \widehat{d}$.

Proof. Since $\int_\Omega g_2(U) dx = 0$, there exists $x_1 \in \Omega$, such that $g_2(U(x_1)) = 0$; that is,

$$\sigma u^n\left(x_1\right) = ba + bu^n\left(x_1\right). \tag{21}$$

It follows that $\max_{\overline{\Omega}} u \geq \sqrt[n]{ab/\sigma}$. Consequently, by $\max_{\overline{\Omega}} u / \min_{\overline{\Omega}} u \leq C$ in Theorem 1, we have

$$\min_{\overline{\Omega}} u \geq \frac{1}{C} \max_{\overline{\Omega}} u \geq \frac{\sqrt[n]{ab/\sigma}}{C} := C_3 = C_3\left(d, \widehat{d}, P, \Omega\right), \tag{22}$$

where $d_3 \leq \widehat{d}$ and $d_1, d_2 \geq d$.

In the following, we mainly prove that $\min_{\overline{\Omega}} v(x) > \underline{C}$ as $\sigma k^n \neq b(a+k^n)$. Now, we suppose that claim is not true; then, there exists a sequence $(d_{1,i}, d_{2,i}, d_{3,i}) \rightarrow (d_1, d_2, d_3)$ with $d_{1,i}, d_{2,i} \geq d$ and $d_{3,i} \leq \widehat{d}$. And the positive solution (u_i, v_i) of (1) corresponding to $(d_{1,i}, d_{2,i}, d_{3,i})$ is such that

$$u_i \longrightarrow u, \quad \min_{\overline{\Omega}} v_i\left(x\right) \longrightarrow 0, \quad \text{as } i \longrightarrow \infty. \tag{23}$$

By (14), we have $\max_{\overline{\Omega}} v_i(x)/\min_{\overline{\Omega}} v_i(x) \leq C_1(1+\widehat{d})$, and then, $\max_{\overline{\Omega}} v_i(x) \rightarrow 0$; furthermore, $v_i(x) \rightarrow 0$ uniformly holds as $i \rightarrow \infty$. Then, the first equation of (1) becomes the following:

$$-\Delta u_i = \frac{1}{d_{1,i}}\left(1 - \frac{u_i}{k} - \frac{u_i^{n-1}v_i}{a+u_i^n}\right) u_i \quad \text{in } \Omega,$$

$$\frac{\partial u_i}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{24}$$

According to (7), there exists $C = C(d, \widehat{d}, P, \Omega)$ such that $\|u_i\|_\infty \leq C$, $\|(1 - u_i/k) - (u_i^{n-1}v_i/(a+u_i^n))/d_{1,i}(1+d_{3,i}v_i)\|_\infty \leq C$, where C is a positive constant which does not depend on i. For each i given in problem (24), it follows from L^p estimate that $\|u_i\|_{W^{2,p}(\Omega)} \leq C^2 |\Omega|^{1/p}$, where $p > 1$. Let $p > N$; then, by Sobolev Imbedding Theorems, we get

$$\|u_i\|_{C^{1,\alpha}(\Omega)} \leq C^* \|u_i\|_{W^{2,p}(\Omega)} \leq C^* C^2 |\Omega|^{1/p}, \tag{25}$$

where $\alpha \in (0, 1)$. We choose $\alpha < 1 - N/p$ such that Imbedding is compact, and along with elliptic equation regularity theory, there exists the subsequence of $\{u_i\}$, which is still denoted by $\{u_i\}$, and exists u such that $u_i \rightarrow u$ uniformly holds in $C^{2,\alpha}(\Omega)$. On the other hand, $v_i \rightarrow 0$, and when $d_{1,i} \rightarrow d_1 \in [d, \infty)$, the limit of (24) becomes the following problem:

$$-\Delta u = \frac{1}{d_1} u\left(1 - \frac{u}{k}\right) \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{26}$$

Applying Maximum Principle to problem (26) and noting that $\min_{\overline{\Omega}} u_i \geq C_3$, we have $u = k$.

However, when $d_{1,i} \rightarrow \infty$, the limit of (24) becomes the following problem:

$$-\Delta u = 0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{27}$$

which implies that $u = \widehat{C}$ for some nonnegative constant $\widehat{C}$. Since $\int_\Omega u_i(1 - u_i/k - u_i^{n-1}v_i/(a+u_i^n)) dx = 0$, by letting $i \rightarrow \infty$

and noting that $\min_{\overline{\Omega}} u_i \geq C_3$, $v_i \to 0$, and $u_i \to \widehat{C}$, we also have $u = k$.

By a similar argument as that in (24), for the second equation of (1), we can prove that there exists a subsequence in $C^{2,\alpha}(\Omega)$, such that $v_i(x)/\max_{\overline{\Omega}} v_i(x) \to v_0$. Since $v_i/\max_{\overline{\Omega}} v_i(x) \geq \min_{\overline{\Omega}} v_i(x)/\max_{\overline{\Omega}} v_i(x) \geq 1/C(1+\widehat{d})$, then $v_0 \geq 1/C(1+\widehat{d})$. Dividing the second equation of (1) by $\max_{\overline{\Omega}} v_i(x)$, and integrating over Ω, we have

$$\int_{\Omega} \frac{v_i}{\max_{\overline{\Omega}} v_i(x)} \left(\frac{\sigma u_i^n}{a + u_i^n} - b \right) dx = 0. \qquad (28)$$

Let $i \to \infty$, and note that $v_i/\max_{\overline{\Omega}} v_i(x) \to v_0$ and $u_i \to k$; then, $\sigma k^n/(a + k^n) = b$. This contradiction to the assumption completes the proof. □

3. A Result on Degree Theory

In this section, we obtain nonexistence of nonconstant positive solutions to (1) as $d_3 = 0$. Meanwhile, by degree theory, a general result to establish the existence of nonconstant positive solutions to (1) in the next section is proved.

Denote $\mathbf{d} = (d_1, d_2, d_3)$ and $P = (\sigma, a, k, b)$. We will fix $P \in (0, \infty)^4$ and take $\mathbf{d} \in (0, \infty)^2 \times [0, \infty)$ as bifurcation parameters, the dependence of P will often be suppressed. Define

$$X = \left\{ U \in \left[C^2\left(\overline{\Omega}\right) \right]^2 \mid \frac{\partial U}{\partial \nu} = 0 \ \text{on} \ \partial\Omega \right\},$$

$$X^+ = \left\{ (u, v) \in X \mid u > 0, \ v > 0 \ \text{on} \ \overline{\Omega} \right\}, \qquad (29)$$

$$B(C) = \left\{ (u, v) \in X \mid \frac{1}{C} < u, \ v < C \ \text{on} \ \overline{\Omega} \right\}.$$

Since

$$\Phi_U(U) = \begin{bmatrix} d_1 & 0 \\ -\dfrac{d_2 d_3 v}{(1+u)^2} & d_2 \left[1 + \dfrac{d_3}{(1+u)} \right] \end{bmatrix} \qquad (30)$$

and $\det \Phi_U(U)$ is positive for all nonnegative U, Φ_U^{-1} exists. Hence, U is a positive solution to (1) if and only if

$$F(\mathbf{d}; U) := U - (I - \Delta)^{-1}$$

$$\times \left\{ \Phi_U(U)^{-1} \left[G(U) + \nabla U \Phi_{UU} \nabla U \right] + U \right\} \qquad (31)$$

$$= 0 \quad \text{in} \ X^+,$$

where $(I - \Delta)^{-1}$ is the inverse of $I - \Delta$ in X. As $F(\mathbf{d}; \cdot)$ is a compact perturbation of the identity operator I, the Leray-Schauder $\deg(F(\mathbf{d}; \cdot), 0, B)$ is well defined if $F(\mathbf{d}; U) \neq 0$ for all $U \in \partial B$.

By Theorems 1 and 2, there exists a positive constant $C_0(\mathbf{d}, P)$ such that if $F(\mathbf{d}; U) = 0$ in X^+, then $U \in B(C_0)$. Note that $C_0(\mathbf{d}, P)$ can be taken as a continuous function for $\sigma \neq ba/k^n + b$. By the invariance property of the Leray-Schauder degree, we then conclude that $\deg(F(\mathbf{d}; \cdot), 0, B(C))$

does not depend on $\mathbf{d}$ if $C > C_0(\mathbf{d}, P)$, and it also does not depend on P, if P changes continuously in $(0, \infty)^4$ without touching the surface $\sigma = ba/k^n + b$.

For the case $\sigma > ba/k^n + b$, by degree invariance, we need only consider a special $\mathbf{d}$; say $\mathbf{d} = (d, d, 0)$ with large d. For this we can use the following nonexistence result. To compare the existence regions of (1) with and without cross-diffusion, we give a nonexistence result stronger than what is needed here.

Theorem 3. Let $(d_2, P) \in (0, \infty)^5$ be given. Then, there exists a positive constant $\widehat{d}_1(d_2, P)$ such that when $\mathbf{d} = (d_1, d_2, 0)$ and $d_1 \geq \widehat{d}_1(d_2, P)$, (1) does not have any nonconstant positive solution.

Proof. First, by (7), there exists a positive constant $C = C(P, \widehat{d}_2)$ such that a classical solution to (1) satisfies $u(x)$, $v(x) \leq C$ provided that $d_2 \geq \widehat{d}_2$. Now, we write $\overline{f}$ as average of f over Ω, where

$$f(u, v) = u \left(1 - \frac{u}{k} - \frac{u^{n-1} v}{a + u^n} \right), \qquad g(u, v) = \frac{\sigma u^n v}{a + u^n}. \qquad (32)$$

Multiplying the equation for u of (1) by $(u - \overline{u})$ and integrating over Ω by parts, we have

$$d_1 \int_{\Omega} \nabla |u - \overline{u}|^2 dx = \int_{\Omega} f(u, v)(u - \overline{u}) dx$$

$$= \int_{\Omega} \left\{ f(u, v) - f(\overline{u}, \overline{v}) \right\} (u - \overline{u}) dx$$

$$= \int_{\Omega} \left\{ f_u(\xi, \eta)(u - \overline{u})^2 \qquad (33) \right.$$

$$\left. + f_v(\xi, \eta)(u - \overline{u})(v - \overline{v}) \right\} dx$$

$$\leq \int_{\Omega} \left\{ (C_1 u - \overline{u})^2 + \varepsilon (v - \overline{v})^2 \right\} dx$$

for some positive constant $C_1 = C_1(P, \widehat{d}_2)$ and $\varepsilon = \varepsilon(P, \widehat{d}_2) \ll 1$, where ξ and η lie between u and $\overline{u}$, v and $\overline{v}$, respectively. Similarly, we have

$$d_2 \int_{\Omega} \nabla |v - \overline{v}|^2 dx = \int_{\Omega} [-bv + g(u, v)](v - \overline{v}) dx$$

$$= \int_{\Omega} [-b(v - \overline{v}) + g(u, v) - g(\overline{u}, \overline{v})]$$

$$\times (v - \overline{v}) dx$$

$$= \int_{\Omega} \left\{ -b(v - \overline{v})^2 + g_v(\xi, \eta)(v - \overline{v})^2 \right.$$

$$\left. + g_u(\xi, \eta)(u - \overline{u})(v - \overline{v}) \right\} dx$$

$$\leq \int_{\Omega} \left\{ (\sigma - b)(v - \overline{v})^2 + \varepsilon_1 (v - \overline{v})^2 \right.$$

$$\left. + C_2(u - \overline{u})^2 \right\} dx$$

$$(34)$$

for some positive constant $C_2 = C_2(P, \widehat{d}_2)$ and $\varepsilon_1 = \varepsilon_1(P, \widehat{d}_2) \ll 1$, where ξ and η are the same as in (33). Adding (33) and (34), we obtain

$$\int_\Omega \left\{ d_1 \nabla |u - \overline{u}|^2 + d_2 \nabla |v - \overline{v}|^2 \right\} dx$$

$$\leq \int_\Omega \left\{ (C_1 + C_2)(u - \overline{u})^2 + (\sigma - b + \varepsilon + \varepsilon_1)(v - \overline{v})^2 \right\} dx. \tag{35}$$

It follows from the Poincaré Inequality that

$$\mu_1 \int_\Omega \left\{ d_1(u - \overline{u})^2 + d_2(v - \overline{v})^2 \right\} dx$$

$$\leq \int_\Omega \left\{ (C_1 + C_2)(u - \overline{u})^2 + (\sigma - b + \varepsilon + \varepsilon_1)(v - \overline{v})^2 \right\} dx. \tag{36}$$

Since we can choose $d_2 \geq \widehat{d}_2$ such that $\mu_1 d_2 > \sigma - b$, we may also choose ε and ε_1 sufficiently small such that $\mu_1 d_2 > \sigma - b + \varepsilon + \varepsilon_1$. Consequently, by (36),

$$\mu_1 d_1 \int_\Omega (u - \overline{u})^2 dx \leq (C_1 + C_2) \int_\Omega (u - \overline{u})^2 dx, \tag{37}$$

which implies that $u = \overline{u} = $ constant, and, hence, $v = \overline{v} = $ constant if $d_1 \geq \widehat{d}_1(d_2, P) := (C_1(P, \widehat{d}_2) + C_2(P, \widehat{d}_2))/\mu_1$. Thus, we complete the proof of the theorem. □

In the following, we only calculate $\deg(F, 0, B)$ when all solutions to $F = 0$ are positive constant solutions in $B(C)$.

Let $0 = \mu_1 < \mu_2 < \mu_3 < \cdots$ be the eigenvalues of the operator $-\Delta$ in Ω with zero flux boundary condition, $E(\mu_i)$ be the eigenspace corresponding to μ_i in $C^1(\overline{\Omega})$, ϕ_{ij}, $j = 1, 2, 3, \ldots$, dim $E(\mu_i)$ an orthonormal basis of $E(\mu_i)$, and $X_{ij} = \{C\phi_{ij} \mid C \in R^2\}$. Then, $X = \oplus_{i=1}^\infty X_i$, where $X_i = \oplus_{j=1}^{\dim E(\mu_i)} X_{ij}$. This decomposed method is similar to that of [22].

Let U^* be a positive root to $G(U) = 0$. We can calculate

$$D_U F(\mathbf{d}; U^*) = I - (I - \Delta)^{-1}$$

$$\times \left\{ \Phi_U^{-1}(U^*) G_U(U^*) + I \right\} \quad \text{in } L(X, X), \tag{38}$$

where $L(X, X)$ is a linear mapping from X to itself.

Denote that $H(\mathbf{d}, U^*; \mu) := \det[\mu I - \Phi_U(U^*)^{-1} G_U(U^*)]$.

Lemma 4. *Let U^* be a positive root to $G(\cdot) = 0$, and assume that $H(\mathbf{d}, U^*; \mu_i) \neq 0$ for all i. Then,*

$$index\left(F(\mathbf{d}; \cdot), U^*\right) = (-1)^r, \quad r = \sum_{i \geq 1, H(\mathbf{d}, \mu_i) < 0} 1. \tag{39}$$

Proof. If $D_U F$ is invertible, then the index of F at U^* is defined as $index(F(\mathbf{d}; \cdot), U^*) = (-1)^r$, where r is the number of eigenvalues of $D_U F$ with negative real parts. The $\deg(F(\mathbf{d}; \cdot), 0, B)$ is then equal to summation of the indexes over all solutions to $F = 0$ in B, provided that $F \neq 0$ on

∂B. Since for each integer $i \geq 1$, X_i is invariant under $D_U(F(\mathbf{d}; U^*))$, and λ is an eigenvalue of $D_U F$ on X_i if and only if λ is an eigenvalue of following matrix:

$$I - \frac{1}{1 + \mu_i} \left[\Phi_U(U^*)^{-1} G_U(U^*) + I \right]$$

$$= \frac{1}{1 + \mu_i} \left[\mu_i I - \Phi_U(U^*)^{-1} G_U(U^*) \right]. \tag{40}$$

Since $H(d, U^*; \mu) \neq 0$, $D_U F(\mathbf{d}; U^*)$ is invertible. Therefore, the number of eigenvalues with negative real parts of $D_U F(\mathbf{d}; U^*)$ on X_i is odd if and only if $H(\mathbf{d}, U^*; \mu_i) < 0$, and therefore

$$index\left(F(\mathbf{d}; \cdot), U^*\right) = (-1)^r, \quad r = \sum_{i \geq 1, H(\mathbf{d}, \mu_i) < 0} 1. \tag{41}$$

□

Theorem 5. *Assume that $P = (\sigma, a, k, b) \in (0, \infty)^4$ with $\sigma k^n \neq b(a + k^n)$. Then, for every $\mathbf{d} \in (0, \infty)^2 \times [0, \infty)$, there exists a constant $C_0(\mathbf{d}, P)$ such that for every $C > C_0(\mathbf{d}, P)$,*

$$\deg\left(F(\mathbf{d}; \cdot), 0, B(C)\right) = \begin{cases} 0, & \text{if } \sigma < \dfrac{ba}{k^n} + b, \\[2mm] 1, & \text{if } \sigma < \dfrac{ba}{k^n} + b. \end{cases} \tag{42}$$

Proof. Since (1) does not have any positive solution in $B(C)$, when $\sigma \leq b$, we then conclude that

$$\deg\left(F(\mathbf{d}; \cdot), 0, B(C)\right) = 0,$$

$$\forall P \in \left\{ (\sigma, a, k, b) \in (0, \infty)^4 \mid \sigma < \frac{ba}{k^n} + b \right\}, \tag{43}$$

$$C > C_0(\mathbf{d}, P).$$

Next, to complete the proof of Theorem 5. We need only to calculate the degree for the case $\sigma > ba/k^n + b$. In this case, by Theorem 3, all positive solutions to $F = 0$ are the unique positive constant solution to $G(\cdot) = 0$, denoted by $\widetilde{U}$, when $\mathbf{d} = (d, d, 0)$, $\Phi_U(\widetilde{U}) = dI$ so that $H(\mathbf{d}, \widetilde{U}; \mu) = (1/d) \det(d\mu I - G_U(\widetilde{U}))$, $\det(G_U(\widetilde{U})) = nb^2(\sigma - b)\widetilde{v}/\sigma^2 \widetilde{u} > 0$. This implies that when d is sufficiently large, $H(\mathbf{d}, \widetilde{U}; \mu_i) > 0$ for all $i = 1, 2, 3, \ldots$, and thus $r = 0$. It follows from Lemma 4 that $\deg(F(\mathbf{d}; \cdot), 0, B(C)) = 1$. This completes the proof. □

Remark 6. The change of degree when σ passes the borderline $\sigma = ba/k^n + b$ is due to the appearance (disappearance) of a positive constant steady-state bifurcating from $U \equiv (k, 0)$.

4. Existence of Nonconstant Positive Solutions

In this section, we establish the existence of positive nonconstant solutions for (1). In particular, we show that for certain ranges of parameters where (1) does not have any positive nonconstant steady state, our model can still produce patterns. The idea is as follows. First we calculate the index of $F(\mathbf{d}; \cdot)$ at positive constant steady states. Suppose that the

sum of all these indices is not equal to the degree stated in Theorem 5. Then, $F(\mathbf{d}; \cdot) = 0$ in $B(C)$ for $C = C_0(\mathbf{d}, P)$ must have a nonconstant positive solution, which also solves (1).

In the following, we always assume that $\sigma > b$, and (1) has a unique positive constant solution $(\tilde{u}, \tilde{v})$, when $\widetilde{U} = (\tilde{u}, \tilde{v})$ is a positive solution to $G(U) = 0$. We can get the following results at $\widetilde{U}$ by simply calculating

$$
G_U\left(\widetilde{U}\right) = \begin{bmatrix} \dfrac{1}{\sigma k}\{(n(\sigma - b) - 2\sigma)\tilde{u} - k[n(\sigma - b) - \sigma]\} & -\dfrac{b}{\sigma} \\[2mm] \dfrac{nb(\sigma - b)\tilde{v}}{\sigma\tilde{u}} & 0 \end{bmatrix}
$$

$$
=: \begin{bmatrix} g_{11} & -\dfrac{b}{\sigma} \\[2mm] g_{21} & 0 \end{bmatrix},
$$

$$
\Phi_U\left(\widetilde{U}\right) = \begin{bmatrix} d_1 & 0 \\[2mm] -\dfrac{d_2 d_3 \tilde{v}}{(1 + \tilde{u})^2} & d_2\left[1 + \dfrac{d_3}{(1 + \tilde{u})}\right] \end{bmatrix}
$$

$$
=: \begin{bmatrix} \phi_{11} & 0 \\ \phi_{21} & \phi_{22} \end{bmatrix},
$$

$$
H\left(\mathbf{d}, \widetilde{U}; \mu\right) \equiv \det\left(\mu I - \Phi_U^{-1}\left(\widetilde{U}\right) G_U\left(\widetilde{U}\right)\right)
$$

$$
= \mu^2 + \mu\frac{|\phi_{21}|\,b/\sigma - \phi_{22}g_{11}}{\phi_{11}\phi_{22}} + \frac{g_{21}b/\sigma}{\phi_{11}\phi_{22}}. \tag{44}
$$

To calculate the roots of $H(\mathbf{d}, \widetilde{U}; \cdot) = 0$, we will restrict our attention to large $|\mathbf{d}|$. Note that

$$
\lim_{d_i \to \infty} H\left(\mathbf{d}, \widetilde{U}; \mu\right) = \mu^2 - \Lambda_i \mu, \quad \forall i = 2, 3, \tag{45}
$$

where

$$
\Lambda_2 = \Lambda_2\left(d_1, d_3; U\right) = \left(-d_3 bk\tilde{v} + (1 + \tilde{u} + d_3)(1 + \tilde{u})\right.
$$

$$
\times \{(n(\sigma - b) - 2\sigma)\tilde{u} - k(n(\sigma - b) - \sigma)\}\bigr)
$$

$$
\times \left(\sigma k d_1 (1 + \tilde{u} + d_3)(1 + \tilde{u})\right)^{-1},
$$

$$
\Lambda_3 = \Lambda_3\left(d_1; U\right)
$$

$$
= \frac{-bk\tilde{v} + \{(n(\sigma - b) - 2\sigma)\tilde{u} - k(n(\sigma - b) - \sigma)\}(1 + \tilde{u})}{\sigma k d_1 (1 + \tilde{u})}. \tag{46}
$$

The sign of the trace $\text{tr}(G_U) = g_{11}$ is determined by $\gamma = (n(\sigma - b) - 2\sigma)\tilde{u} - k(n(\sigma - b) - \sigma)$ and the sign of the determinant $\det(G_U) = nb^2(\sigma - b)v/\sigma^2 u > 0$.

Hence, we will discuss separately the following cases:

(i) $n(\sigma - b) - 2\sigma > 0 (n > 2\sigma/(\sigma - b))$, obviously $k > \tilde{u} > ((n(\sigma - b) - 2\sigma)/(n(\sigma - b) - \sigma))\tilde{u}$, then $\gamma < 0$;

(ii) $n(\sigma - b) - \sigma < 0 (n < \sigma/(\sigma - b))$:

(iia) if $\tilde{u} < k < ((2\sigma - n(\sigma - b))/(\sigma - n(\sigma - b)))\tilde{u}$, then $\gamma < 0$;

(iib) if $k > ((2\sigma - n(\sigma - b))/(\sigma - n(\sigma - b)))\tilde{u}$, then $\gamma > 0$.

4.1. The Case $n(\sigma - b) - 2\sigma > 0$ $(n > 2\sigma/(\sigma - b))$.
In this subsection, we consider local stability of the constant steady state $U \equiv \widetilde{U}$ for evolution dynamics

$$
U_t = \Delta\Phi(U) + G(U) \quad \text{in } \Omega \times (0, \infty),
$$

$$
\frac{\partial U}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{47}
$$

where $\widetilde{U} = (\tilde{u}, \tilde{v})$ is a positive constant solution to $G(\cdot) = 0$ by (5).

Theorem 7. *Let* $k > \tilde{u} > ((n(\sigma - b) - 2\sigma)/(n(\sigma - b) - \sigma))\tilde{u}$ $(n > 2\sigma/(\sigma - b))$. *Then the positive constant solution* $U(x, t) \equiv (\tilde{u}, \tilde{v})$ *is asymptotically stable with respect to the dynamics* (47). *Consequently, in a small neighborhood of* $\widetilde{U}$, (1) *does not have any nonconstant positive solution.*

Proof. The linearization of (47) at $\widetilde{U}$ takes the form

$$
V_t = \Phi_U\left(\widetilde{U}\right)\Delta V + G_U\left(\widetilde{U}\right)V \quad \text{in } \Omega \times (0, \infty),
$$

$$
\frac{\partial V}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{48}
$$

Denote that the corresponding linear operator $L := \Phi_U(\widetilde{U})\Delta + G_U(\widetilde{U})$. For each eigen-pair (μ_i, ϕ_i) $(i = 1, 2, 3, \ldots)$ of $-\Delta$ on Ω with no flux boundary condition, $V = Ce^{\lambda t}\phi$ is a solution to (48) if and only if (λ, C) is an eigen-pair of the matrix $A_i := -\mu_i\Phi_U(\widetilde{U}) + G_U(\widetilde{U})$. X_i is invariant under the operator L. From (44), we have

$$
g_{11} < 0 \quad \left(k > \frac{n(\sigma - b) - 2\sigma}{n(\sigma - b) - \sigma}\tilde{u}\right), \quad g_{21} > 0 \quad (\sigma > b). \tag{49}
$$

Then

$$
\det A_i := \phi_{11}\phi_{22}\mu_i^2 - \left(\frac{b}{\sigma}\phi_{21} + g_{11}\phi_{22}\right)\mu_i + \frac{b}{\sigma}g_{21} > 0,
$$

$$
\text{tr } A_i := -(\phi_{11} + \phi_{22})\mu_i + g_{11} < 0. \tag{50}
$$

Hence, we conclude that the two eigenvalues λ_i^+ and λ_i^- of A_i have negative real parts. More specifically, we have the following:

(i) For $i = 1$, $\mu_1 = 0$, if $g_{11}^2 - 4g_{21}b/\sigma \leq 0$, then $\text{Re } \lambda_i^\pm = (1/2)g_{11} < 0$; if $g_{11}^2 - 4g_{21}b/\sigma > 0$, then

$$
\text{Re } \lambda_i^+ = \frac{1}{2}\left[g_{11} + \sqrt{g_{11}^2 - \frac{4g_{21}b}{\sigma}}\right] \leq 0,
$$

$$
\text{Re } \lambda_i^- = \frac{1}{2}\left[g_{11} - \sqrt{g_{11}^2 - \frac{4g_{21}b}{\sigma}}\right] < 0. \tag{51}
$$

(ii) For $i \geq 2$, as μ_i is increasing with respect to i and $\mu_i \to \infty$ as $i \to \infty$, it follows that if $(\operatorname{tr} A_i)^2 - 4 \det A_i \leq 0$, then

$$
\begin{aligned}
\operatorname{Re} \lambda_i^{\pm} &= \frac{1}{2} \operatorname{tr} A_i = \frac{1}{2} \left[-(\phi_{11} + \phi_{22}) \mu_i + g_{11} \right] \\
&\leq \frac{1}{2} \left[-(\phi_{11} + \phi_{22}) \mu_2 + g_{11} \right] < 0;
\end{aligned}
\tag{52}
$$

if $(\operatorname{tr} A_i)^2 - 4 \det A_i > 0$, since $\det A_i > 0$ and $\operatorname{tr} A_i < 0$, then

$$
\begin{aligned}
\operatorname{Re} \lambda_i^- &= \frac{1}{2} \left(\operatorname{tr} A_i - \sqrt{(\operatorname{tr} A_i)^2 - 4 \det A_i} \right) \\
&\leq \frac{1}{2} \operatorname{tr} A_i \leq \frac{1}{2} \left[-(\phi_{11} + \phi_{22}) \mu_2 + g_{11} \right] < 0; \\
\operatorname{Re} \lambda_i^+ &= \frac{1}{2} \left(\operatorname{tr} A_i + \sqrt{(\operatorname{tr} A_i)^2 - 4 \det A_i} \right) \\
&= \frac{2 \det A_i}{\operatorname{tr} A_i - \sqrt{(\operatorname{tr} A_i)^2 - 4 \det A_i}} \leq \frac{\det A_i}{\operatorname{tr} A_i} < -\widetilde{\varepsilon}.
\end{aligned}
\tag{53}
$$

for some positive constant $\widetilde{\varepsilon}$ that does not depend on i. The previous arguments show that there exists a positive constant ε, which does not depend on i, such that $\operatorname{Re} \lambda_i < -\varepsilon$, $\forall i$. Consequently, the spectrum of L, which consists of eigenvalues, lies in $\{\operatorname{Re}(\lambda) < -\varepsilon\}$. It then follows from Theorem 5.1.1 of [26, page 98] that the constant steady-state $U(x, t) \equiv \widetilde{U}$ is asymptotically stable to (47). □

4.2. The Case $n(\sigma - b) - \sigma < 0$ $(n < \sigma/(\sigma - b))$

4.2.1. $(\gamma < 0) \widetilde{u} < k < ((2\sigma - n(\sigma - b))/(\sigma - n(\sigma - b)))\widetilde{u}$

Theorem 8. *Let $\widetilde{u} < k < ((2\sigma - n(\sigma - b))/(\sigma - n(\sigma - b)))\widetilde{u}$. Then the positive constant solution $U(x, t) \equiv (\widetilde{u}, \widetilde{v})$ is asymptotically stable with respect to the dynamics (47).*

Proof. Similar to the proof of Theorem 7. □

4.2.2. $(\gamma > 0)$ $k > ((2\sigma - n(\sigma - b))/(\sigma - n(\sigma - b)))\widetilde{u}$.
In this case, $\widetilde{U} = (\widetilde{u}, \widetilde{v})$ is the only positive constant solution to $G(\cdot) = 0$. By fixing the diffusion coefficients d_1 (for prey) and using the diffusion coefficients d_2 and d_3 (for predator) as bifurcation parameters, we will show that (1) can create nonconstant positive solutions. We want to emphasize that it is caused by the presence of cross-diffusion which has a more complex role than that of the diffusion coefficients d_1 and d_2.

Theorem 9 (existence with suitable d_2 and d_3). *Assume that $\sigma > ba/k^n + b$ and $k > ((2\sigma - n(\sigma - b))/(\sigma - n(\sigma - b)))\widetilde{u}$, define $\Lambda_2(d_1, d_3; U)$ and $\Lambda_3(d_1; U)$ as (46).*
(i) Suppose that d_1 and d_3 are given such that $\Lambda_2(d_1, d_3; \widetilde{U}) \in (\mu_j, \mu_{j+1})$ for some positive even integer j. There exists a positive constant D_2 such that if $d_2 \geq D_2$, then (1) has at least one nonconstant positive solution.
(ii) Suppose that d_1 is given such that $\Lambda_3(d_1; U) \in (\mu_j, \mu_{j+1})$ for some positive even integer j. Then, for any given $d_2 > 0$, there exists a positive constant D_3 such that if $d_3 \geq D_3$, (1) has at least one nonconstant positive solution.

Proof. Denote by $\mu^{\pm}(\mathbf{d}, \widetilde{U})$, with $\operatorname{Re}(\mu^-) \leq \operatorname{Re}(\mu^+)$, the two roots to $H(\mathbf{d}, \widetilde{U}; \mu) = 0$, and then

$$
\begin{aligned}
\mu^+ \left(\mathbf{d}, \widetilde{U} \right) \mu^- \left(\mathbf{d}, \widetilde{U} \right) &= \left[\det \left(\Phi_U \left(\widetilde{U} \right) \right) \right]^{-1} \det \left(G_U \left(\widetilde{U} \right) \right) \\
&= \frac{g_{21} b/\sigma}{\phi_{11} \phi_{22}} > 0.
\end{aligned}
\tag{54}
$$

From (45), we see that

$$
\begin{aligned}
\lim_{d_2 \to \infty} \mu^- \left(\mathbf{d}, \widetilde{U} \right) &= 0, \qquad \lim_{d_2 \to \infty} \mu^+ \left(\mathbf{d}, \widetilde{U} \right) = \Lambda_2 \left(d_1, d_3; \widetilde{U} \right), \\
\lim_{d_3 \to \infty} \mu^- \left(\mathbf{d}, \widetilde{U} \right) &= 0, \qquad \lim_{d_3 \to \infty} \mu^+ \left(\mathbf{d}, \widetilde{U} \right) = \Lambda_3 \left(d_1; \widetilde{U} \right).
\end{aligned}
\tag{55}
$$

Suppose that $\Lambda_2(d_1, d_3; \widetilde{U}) \in (\mu_j, \mu_{j+1})$ for some positive even integer j; then, there exists a positive constant $D_2 \gg 1$ such that $d_2 \gg D_2$, and we have

$$
0 = \mu_1 < \mu^- \left(\mathbf{d}, \widetilde{U} \right) < \mu_2, \qquad \mu^+ \left(\mathbf{d}, \widetilde{U} \right) \in \left(\mu_j, \mu_{j+1} \right). \tag{56}
$$

Hence, if $d_2 \gg 1$, $H(\mathbf{d}, \widetilde{U}; \mu_i) < 0$ is equivalent to $i \in \{2, 3, \ldots, j\}$, since j is even. It follows from Lemma 4 that

$$
\operatorname{index} \left(F \left(\mathbf{d}; \cdot \right), \widetilde{U} \right) = (-1)^{j-1} = -1. \tag{57}
$$

Consequently, $F(\mathbf{d}; U) = 0$ has at least one nonconstant positive solution that is different from the constant function $U = \widetilde{U}$. Otherwise, the degree of $F = 0$ in $B(C)$ would be -1 for all large enough C, which would contradict Theorem 5. This proves, the first assertion of the theorem, and the second assertion is similarly proved. □

Remark 10. For $\Lambda_3(d_1; U)$ to be positive, it is necessary and sufficient to have

$$
k > \frac{(k+1) \sigma \widetilde{u}}{(1 + \widetilde{u}) [\sigma - n (\sigma - b)]} + \widetilde{u}. \tag{58}
$$

When this inequality holds, $\Lambda_2(d_1, d_3; \widetilde{U})$ is also positive provided that d_3 is large, and we then can adjust to d_1 and make the assumptions in (i) or (ii) of theorem hold.

Remark 11. In fact, if $\sigma/(\sigma - b) < n < 2\sigma/(\sigma - b)$, then $\gamma < 0$. This case is similar to case (i). Furthermore, the positive constant solution $U(x, t) \equiv (\widetilde{u}, \widetilde{v})$ is also asymptotically stable with respect to the dynamics (47).

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