Research Article

Generalized Minimax Programming with Nondifferentiable (G, β)-Invexity

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We consider the generalized minimax programming problem (P) in which functions are locally Lipschitz (G, β)-invex. Not only G-sufficient but also G-necessary optimality conditions are established for problem (P). With G-necessary optimality conditions and (G, β)-invexity on hand, we construct dual problem (DI) for the primal one (P) and prove duality results between problems (P) and (DI). These results extend several known results to a wider class of programs.

1. Introduction

Convexity plays a central role in many aspects of mathematical programming including analysis of stability, sufficient optimality conditions, and duality. Based on convexity assumptions, nonlinear programming problems can be solved efficiently. There have been many attempts to weaken the convexity assumptions in order to treat many practical problems. Therefore, many concepts of generalized convex functions have been introduced and applied to mathematical programming problems in the literature [1]. One of these concepts, invexity, was introduced by Hanson in [2]. Hanson has shown that invexity has a common property in mathematical programming with convexity that Karush-Kuhn-Tucker conditions are sufficient for global optimality of nonlinear programming under the invexity assumptions. Ben-Israel and Mond [3] introduced the concept of preinvex functions which is a special case of invexity.

Recently, Antczak extended further invexity to G-invexity [4] for scalar differentiable functions and introduced new necessary optimality conditions for differentiable mathematical programming problem. Antczak also applied the introduced G-invexity notion to develop sufficient optimality conditions and new duality results for differentiable mathematical programming problems. Furthermore, in the natural way, Antczak's definition of G-invexity was also extended to the case of differentiable vector-valued functions. In [5], Antczak defined vector G-invex (G-incave) functions with respect to η and applied this vector G-invexity to develop optimality conditions for differentiable multiobjective programming problems with both inequality and equality constraints. He also established the so-called G-Karush-Kuhn-Tucker necessary optimality conditions for differentiable vector optimization problems under the Kuhn-Tucker constraint qualification [5]. With this vector G-invexity concept, Antczak proved new duality results for nonlinear differentiable multiobjective programming problems [6]. A number of new vector duality problems such as G-Mond-Weir, G-Wolfe, and G-mixed dual vector problems to the primal one were also defined in [6].

In the last few years, many concepts of generalized convexity, which include (p, r)-invexity [7], (F, ρ)-convexity [8], (F, α, ρ, d)-convexity [9], (C, α, ρ, d)-convexity [10], (ϕ, ρ)-invexity [11], V-r-invexity [12], and their extensions, have been introduced and applied to different mathematical programming problems. In particular, they have also been applied to deal with minimax programming; see [13–17] for details. However, we have not found a paper which deals with generalized minimax programming problem (P) under G-invexity or its generalizations assumptions.

Note that the function $G \circ f$ may not be differentiable even if the function G is differentiable. Yuan et al. [18] introduced the (G_f, β_f)-invexity concept for the locally Lipschitz

function f. This (G_f, β_f)-invexity extended Antczak's G-invexity concept to the nonsmooth case. In this paper, we deal with nondifferentiable generalized minimax programming problem (P) with the vector (G, β)-invexity proposed in [18]. Here, the generalized minimax programming problem (P) is presented as follows:

$$\min \quad \sup_{y \in Y} \phi(x, y) \tag{P}$$

$$\text{subject to} \quad g_j(x) \leq 0, \quad j = 1, \ldots, m,$$

where Y is a compact subset of $\mathbb{R}^p$, $\phi(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$, and $g_j(\cdot) : \mathbb{R}^n \to \mathbb{R}$ $(j \in M)$. Let E_P be the set of feasible solutions of problem (P); in other words, $E_p = \{x \in \mathbb{R}^n \mid g_j(x) \leq 0, j \in M\}$. For convenience, let us define the following sets for every $x \in E$:

$$J(x) = \left\{ j \in M \mid g_j(x) = 0 \right\},$$

$$Y(x) = \left\{ y \in Y \mid \phi(x, y) = \sup_{z \in Y} \phi(x, z) \right\}. \tag{1}$$

The rest of the paper is organized as follows. In Section 2, we present concepts in regards to nondifferentiable vector (G, β)-invexity. In Section 3, we present not only G-sufficient but also G-necessary optimality conditions for problem (P). When the G-necessary optimality conditions and the (G, β)-invexity concept are utilized, dual problem (DI) is formulated for the primal one (P) and duality results between them are presented in Section 4.

2. Notations and Preliminaries

In this section, we provide some definitions and results that we will use in the sequel. The following convention for equalities and inequalities will be used throughout the paper. For any $x = (x_1, x_2, \ldots, x_n)^T$, $y = (y_1, y_2, \ldots, y_n)^T$, we define the following:

$$x > y \text{ if and only if } x_i > y_i, \quad \text{for } i = 1, 2, \ldots, n,$$

$$x \geqq y \text{ if and only if } x_i \geq y_i, \quad \text{for } i = 1, 2, \ldots, n,$$

$$x \geqslant y \text{ if and only if } x_i \geq y_i, \quad \text{for } i = 1, 2, \ldots, n, \text{ but } x \neq y,$$

$$x \not> y \text{ is the negation of } x > y,$$

$$x \not\geqslant y \text{ is the negation of } x \geqslant y. \tag{2}$$

Let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x \geqq 0\}$, $\dot{\mathbb{R}}^n_+ = \{x \in \mathbb{R}^n \mid x > 0\}$ and X be a subset of $\mathbb{R}^n$. For our convenience, denote $Q := \{1, 2, \ldots, q\}$, $Q^* := \{1, 2, \ldots, q^*\}$, $K := \{1, 2, \ldots, k\}$, and $M := \{1, 2, \ldots, m\}$. Further, we recall some definitions and a lemma.

Definition 1 (see [19]). Let $d \in \mathbb{R}^n$, X be a nonempty set of $\mathbb{R}^n$ and $f : X \to \mathbb{R}$. If

$$f^0(x; d) := \lim_{\substack{y \to x \\ \mu \downarrow 0}} \sup \frac{1}{\mu} \left(f(y + \mu d) - f(y) \right) \tag{3}$$

exists, then $f^0(x; d)$ is called the Clarke derivative of f at x in the direction d. If this limit superior exists for all $d \in \mathbb{R}^n$, then f is called the Clarke differentiable at x. The set

$$\partial f(x) = \left\{ \zeta \mid f^0(x; d) \geq \langle \zeta, d \rangle, \forall d \in \mathbb{R}^n \right\} \tag{4}$$

is called the Clarke subdifferential of f at x.

Note that if a given function f is locally Lipschitz, then the Clarke subdifferential $\partial f(x)$ exists.

Lemma 2 (see [18]). *Let ψ be a real-valued Lipschitz continuous function defined on X and denote the image of X under ψ by $I_\psi(X)$; let $\varphi : I_\psi(X) \to \mathbb{R}$ be a differentiable function such that $\varphi'(\gamma)$ is continuous on $I_\psi(X)$ and $\varphi'(\gamma) \geq 0$ for each $\gamma \in I_\psi(X)$. Then the chain rule*

$$(\varphi \circ \psi)^0(x, d) = \varphi'(\psi(x)) \psi^0(x, d), \tag{5}$$

holds for each $d \in \mathbb{R}^n$. Therefore,

$$\partial(\varphi \circ \psi)(x) = \varphi'(\psi(x)) \partial(\psi)(x). \tag{6}$$

Definition 3. Let $f = (f_1, \ldots, f_k)$ be a vector-valued locally Lipschitz function defined on a nonempty set $X \subset \mathbb{R}^n$. Consider the functions $\eta : X \times X \to \mathbb{R}^n$, $G_{f_i} : I_{f_i}(X) \to \mathbb{R}$, and $\beta_i^f : X \times X \to \mathbb{R}_+$ for $i \in K$. Moreover, G_{f_i} is strictly increasing on its domain $I_{f_i}(X)$ for each $i \in K$. If

$$G_{f_i} \circ f_i(x) - G_{f_i} \circ f_i(u)$$

$$\geq (>) \beta_i^f(x, u) G'_{f_i}(f_i(u)) \langle \zeta_i, \eta(x, u) \rangle, \quad \forall \zeta_i \in \partial f_i(u), \tag{7}$$

holds for all $x \in X$ $(x \neq u)$ and $i \in K$, then f is said to be a (strictly) nondifferentiable vector (G_f, β_f)-invex at u on X (with respect to η) (or shortly, (G_f, β_f)-invex at u on X), where $G_f = (G_{f_1}, \ldots, G_{f_k})$ and $\beta := (\beta_1^f, \beta_2^f, \ldots, \beta_k^f)$. If f is a (strictly) nondifferentiable vector (G_f, β_f)-invex at u on X (with respect to η) for all $u \in X$, then f is a (strictly) nondifferential vector (G_f, β_f)-invex on X (with respect to η).

3. Optimality Conditions

In this section, we firstly establish the G-necessary optimality conditions for problem (P) involving functions which are locally Lipschitz with respect to the variable x. For this purpose, we will need some additional assumptions with respect to problem (P).

Condition 4. Assume the following: (a) the set Y is compact;
(b) $\phi(x, y)$ and $\partial \phi_x(x, y)$ are upper semicontinuous at (x, y);
(c) $\phi(x, y)$ is locally Lipschitz in x and this Lipschitz continuity is uniform for y in Y;
(d) $\phi(x, y)$ is regular in x; that is, $\phi_x^\circ(x, y; \cdot) = \phi_x'(x, y; \cdot)$;
(e) $g_j, j \in M$ are regular and locally Lipschitz at x_0.

Condition 5. For each $\eta = (\eta_1, \ldots, \eta_m) \in \mathbb{R}^m$ satisfying the conditions

$$\eta_j = 0, \quad \forall j \in M \setminus J(x^*),$$
$$\eta_j \geq 0, \quad \forall j \in J(x^*), \tag{8}$$

the following implication holds:

$$z_j^* \in \partial g_j(x^*) \quad (\forall j \in M),$$
$$\sum_{j=1}^{m} \eta_j z_j^* = 0 \Longrightarrow \eta_j = 0, \quad j \in M. \tag{9}$$

We will also use the following auxiliary programming problem (G-P):

$$\min \quad \sup_{y \in Y} G_\phi \circ \phi(x, y)$$

$$\text{s.t.} \quad G_g \circ g(x)$$
$$:= \left(G_{g_1} \circ g_1(x), G_{g_2} \circ g_2(x), \ldots, G_{g_m} \circ g_m(x) \right)$$
$$\leqq G_g(0), \tag{G-P}$$

where $G_g(0) := (G_{g_1}(0), G_{g_2}(0), \ldots, G_{g_m}(0))$. We denote by $E_{G\text{-}P} = \{x \in \mathbb{R}^n \mid G_g \circ g(x) \leqq G_g(0)\}$, $J'(\overline{x}) := \{j \in M : G_{g_j} \circ g_j(\overline{x}) = G_{g_j}(0)\}$. If function G_{g_j} is strictly increasing on $I_{g_j}(X)$ for each $j \in M$, then $E_P = E_{G\text{-}P}$ and $J(\overline{x}) = J'(\overline{x})$. So, we represent the set of all feasible solutions and the set of constraint active indices for either (P) or (G-P) by the notations E and $J(\overline{x})$, respectively.

The following necessary optimality conditions are presented in [20].

Theorem 6 (necessary optimality conditions). *Let x^* be an optimal solution of (P). One also assumes that Conditions 4 and 5 hold. Then there exist positive integer q^* and vectors $y_i \in Y(x^*)$ together with scalars λ_i^* ($i \in Q^*$) and μ_j^* ($j \in M$) such that*

$$0 \in \sum_{i=1}^{q^*} \lambda_i^* \partial_x \phi(x^*, y_i) + \sum_{j=1}^{m} \mu_j^* \partial g_j(x^*),$$
$$\mu_j^* g_j(x^*) = 0, \quad \mu_j^* \geq 0, \quad j \in M, \tag{10}$$
$$\sum_{i=1}^{q^*} \lambda_i^* = 1, \quad \lambda_i^* > 0, \quad i \in Q^*.$$

Furthermore, if α is the number of nonzero λ_i^* and β is the number of nonzero μ_j^*, then

$$1 \leq \alpha + \beta \leq n + 1. \tag{11}$$

Making use of Theorem 6, we can derive the following G-necessary conditions theorem for problem (P), see Theorem 7, here we require the scalars λ_i^* ($i = 1, \ldots, q^*$) to be positive.

Theorem 7 (G-necessary optimality conditions). *Let problem (P) satisfy Conditions 4 and 5; let x^* be an optimal solution of problem (P). Assume that G_ϕ is both continuously differentiable and strictly increasing on $I_\phi(X, Y)$. If G_{g_j} is both continuously differentiable and strictly increasing on $I_{g_j}(X)$ with $G_{g_j}'(g_j(x^*)) > 0$ for each $j \in M$, then there exist positive integer q^* ($1 \leq q^* \leq n + 1$) and vectors $y_i \in Y(x^*)$ together with scalars λ_i^* ($i \in Q^*$) and μ_j^* ($j \in M$) such that*

$$0 \in \sum_{i=1}^{q^*} \lambda_i^* G_\phi'(\phi(x^*, y_i)) \partial_x \phi(x^*, y_i)$$
$$+ \sum_{j=1}^{m} \mu_j^* G_{g_j}'(g_j(x^*)) \partial g_j(x^*), \tag{12}$$

$$\mu_j^* \left(G_{g_j} \circ g_j(x^*) - G_{g_j}(0) \right) = 0, \quad \mu_j^* \geq 0, \quad j \in M, \tag{13}$$

$$\sum_{i=1}^{q^*} \lambda_i^* = 1, \quad \lambda_i^* \geq 0, \quad i \in Q^*. \tag{14}$$

Proof. Since x^* is an optimal solution to problem (P), it is easy to see that x^* is an optimal solution to problem (G-P). Consider problem (G-P), it is easy to check that problem (G-P) satisfies the assumptions of Theorem 6. Therefore, we choose $y_i \in Y(x^*)$ and $i \in Q^*$ with $q^* \leq n + 1$, such that they satisfy Theorem 6.

Now, for each y_i, we consider the scalar programming $(G\text{-}P_{y_i})$ as follows:

$$\min \quad G_\phi \circ \phi(x, y_i)$$
$$\text{subject to} \quad G_{g_j} \circ g_j(x) \leq 0, \quad j \in M. \qquad (G\text{-}P_{y_i})$$

It is easy to see that x^* is an optimal solution to problem $(G\text{-}P_{y_i})$. Thus, there exist $\lambda_i > 0$ and $\mu_{ji} \geq 0$ for $j \in M$ such that

$$0 \in \lambda_i \partial_x (G_\phi \circ \phi)(x^*, y_i) + \sum_{j=1}^{m} \mu_{ji}^* \partial (G_{g_j} \circ g_j)(x^*), \tag{15}$$

$$\mu_{ji} \left(G_{g_j} \circ g_j(x^*) - G_{g_j}(0) \right) = 0, \quad \mu_{ji} \geq 0, \quad j \in M. \tag{16}$$

So, we obtain from (15) that

$$0 \in \sum_{i=1}^{q^*} \lambda_i \partial_x (G_\phi \circ \phi)(x^*, y_i)$$
$$+ \sum_{j=1}^{m} \left(\sum_{i=1}^{q^*} \mu_{ji} \right) \partial (G_{g_j} \circ g_j)(x^*), \tag{17}$$

or

$$0 \in \sum_{i=1}^{q^*} \lambda_i^* (G_\phi \circ \phi)(x^*, y_i) + \sum_{j=1}^{m} \mu_j^* \partial (G_{g_j} \circ g_j)(x^*), \tag{18}$$

where $\lambda_i^* = \lambda_i / \sum_{j=1}^{q^*} \lambda_j$, $\mu_j^* = \sum_{i=1}^{q^*} \mu_{ji} / \sum_{i=1}^{q^*} \lambda_i$. By Lemma 2, we have

$$\partial_x \left(G_\phi \circ \phi \right)(x^*, y_i) = G_\phi' (\phi(x^*, y_i)) \, \partial_x \phi(x^*, y_i), \quad i \in Q^*,$$

$$\partial \left(G_{g_j} \circ g_j \right)(x^*) = G_{g_j}' \left(g_j(x^*) \right) \partial g_j(x^*), \quad j \in M. \tag{19}$$

Now, from (18), we can deduce the required results. $\square$

Next, we derive G-sufficient optimality conditions for problem (P) under the assumption of (G, β)-invexity proposed in [18].

Theorem 8 (*G-sufficient optimality conditions*). *Let $(x^*, \mu^*, v^*, q^*, \lambda^*, \overline{y})$ satisfy conditions (12)–(14), where $v^* = \phi(x^*, y_1) = \cdots = \phi(x^*, y_{q^*})$; let G_ϕ be both continuously differentiable and strictly increasing on $I_\phi(X, Y)$; let G_{g_j} be both continuously differentiable and strictly increasing on $I_{g_j}(X)$ for each $j \in M$. Assume that $\phi(\cdot, y_i)$ is (G_ϕ, β_i^ϕ)-invex at x^* on E for each $i \in Q^*$ and g_j is (G_j^g, β_j^g)-invex at x^* on E for each $j \in M$. Then, x^* is an optimal solution to (P).*

Proof. Suppose, contrary to the result, that x^* is not an optimal solution for problem (P). Hence, there exists $x_0 \in E$ such that

$$\sup_{y \in Y} \phi(x_0, y) < \phi(x^*, y_i), \quad i \in Q^*. \tag{20}$$

By the monotonicity of G_ϕ, we have

$$G_\phi \circ \phi(x_0, y_i) < G_\phi \circ \phi(x^*, y_i), \quad i \in Q^*. \tag{21}$$

Employing (13), (14), and the fact that

$$G_{g_j} \circ g_j(x_0) \leq G_{g_j} \circ (0) = G_{g_j} \circ g_j(x^*), \quad j \in J(x^*), \tag{22}$$

we can write the following statement

$$\frac{\sum_{i=1}^{q^*} \lambda_i^* \left(G_\phi \circ \phi(x_0, y_i) - G_\phi \circ \phi(x^*, y_i) \right)}{\beta_i^\phi (x_0, x^*)}$$

$$+ \frac{\sum_{j=1}^{m} \mu_j^* \left(G_{g_j} \circ g_j(x_0) - G_{g_j} \circ g_j(x^*) \right)}{\beta_j^g (x_0, x^*)} < 0. \tag{23}$$

By the generalized invexity assumptions of $\phi(\cdot, y_i)$ and g_j, we have

$$G_\phi \circ \phi(x_0, y_i) - G_\phi \circ \phi(x^*, y_i)$$

$$\geq (>) \, \beta_i^\phi (x_0, x^*) \, G_\phi' (\phi(x^*, y_i)) \left\langle \xi_i^\phi, \eta(x_0, x^*) \right\rangle,$$

$$\forall \xi_i^\phi \in \partial_x \phi(x^*, y_i), \quad i \in Q^*,$$

$$G_{g_j} \circ g_j(x_0) - G_{g_j} \circ g_j(x^*)$$

$$\geq (>) \, \beta_j^g (x_0, x^*) \, G_{g_j}' \left(g_j(x^*) \right) \left\langle \xi_j^g, \eta(x_0, x^*) \right\rangle,$$

$$\forall \xi_j^g \in \partial g_j(x^*), \quad j \in M. \tag{24}$$

Employing (24) to (23), we have

$$\sum_{i=1}^{q^*} \lambda_i^* G_\phi' (\phi(x^*, y_i)) \left\langle \xi_i^\phi, \eta(x_0, x^*) \right\rangle$$

$$+ \sum_{j=1}^{m} \mu_j^* G_{g_j}' \left(g_j(x^*) \right) \left\langle \xi_j^g, \eta(x_0, x^*) \right\rangle < 0 \tag{25}$$

or

$$\left\langle \sum_{i=1}^{q^*} \lambda_i^* G_\phi' (\phi(x^*, y_i)) \, \xi_i^\phi \right.$$

$$\left. + \sum_{j=1}^{m} \mu_j^* G_{g_j}' \left(g_j(x^*) \right) \xi_j^g, \eta(x_0, x^*) \right\rangle < 0, \tag{26}$$

which implies that

$$0 \notin \sum_{i=1}^{q^*} \lambda_i^* G_\phi' (\phi(x^*, y_i)) \, \partial_x \phi(x^*, y_i)$$

$$+ \sum_{j=1}^{m} \mu_j^* G_{g_j}' \left(g_j(x^*) \right) \partial g_j(x^*). \tag{27}$$

This is a contradiction to condition (12). $\square$

Example 9. Let $Y = [0, 1]$. Define

$$\phi(x, y) = \begin{cases} e^{x+2y}, & y \geq x \\ e^{2x+y}, & x < y, \end{cases} \tag{28}$$

$$g(x) = \left| x - \frac{3}{2} \right| - \frac{1}{2}.$$

Then, $\phi(\cdot, y)$ is $(\log, 1)$-invex at $x = 1$ for each $y \in Y$, g is 1-invex at $x = 1$, and

$$f(x) := \sup_{y \in Y} \phi(x, y) = \begin{cases} e^{x+2}, & y \geq x \\ e^{2x+1}, & x < y. \end{cases} \tag{29}$$

Since

$$\log(\phi(x, y)) = \begin{cases} x + 2y, & y \geq x \\ 2x + y, & x < y, \end{cases} \tag{30}$$

then

$$\partial_x \log(\phi(x, y)) = \begin{cases} 1, & y \geq x \\ [1, 2], & x = y \\ 2, & x < y. \end{cases} \tag{31}$$

Consider $x_0 = 1$. Since $Y(x_0) = \{1\}$, then we can assume that $q = 1$. Therefore,

$$0 \in \lambda G_\phi' (\phi(1, 1)) \, \partial_x \phi(1, 1) - \mu g_j'(1) = \lambda [1, 2] - \mu, \tag{32}$$

where $\lambda = \mu = 1$. Now, from Theorem 8, we can say that $x_0 = 1$ is an optimal solution to (P).

4. Duality

Making use of the optimality conditions of the preceding section, we present dual problem (DI) to the primal one (P) and establish G-weak, G-strong, and G-strict converse duality theorems. For convenience, we use the following notations:

$$K(x) = \left\{ (q, \lambda, \overline{y}) \in N \times \dot{\mathbb{R}}_+^q \times \mathbb{R}^{pq} \mid 1 \le q \le n+1, \right.$$

$$\lambda = (\lambda_1, \dots, \lambda_q) \in \dot{\mathbb{R}}_+^q$$

$$\text{with } \sum_{i=1}^q \lambda_i = 1, \overline{y} = (y_1, \dots y_q) \tag{33}$$

$$\left. \text{with } y_i \in Y(x), i = 1, \dots q \right\}.$$

$H_1(q, \lambda, \overline{y})$ denotes the set of all triplets $(z, \mu, \nu) \in \mathbb{R}^n \times \mathbb{R}_+^m \times \mathbb{R}_+$ satisfying

$$0 \in \sum_{i=1}^q \lambda_i G'_\phi (\phi(z, y_i)) \partial_z \phi(z, y_i) + \sum_{j=1}^m \mu_j G'_{g_j} (g_j(z)) \partial g_j(z), \tag{34}$$

$$\phi(z, y_i) \ge \nu, \quad i = 1, 2, \dots, q, \tag{35}$$

$$\mu_j g_j(z) \ge 0, \quad j \in M, \tag{36}$$

$$y_i \in Y(z), \quad (q, \lambda, \overline{y}) \in K(z). \tag{37}$$

Our dual problem (DI) can be stated as follows:

$$\max_{(q, \lambda, \overline{y}) \in K(z)} \sup_{(z, \mu, \nu) \in H_1(q, \lambda, \overline{y})} \nu. \tag{DI}$$

Note that if $H_1(q, \lambda, \overline{y})$ is empty for some triplet $(q, \lambda, \overline{y}) \in K(z)$, then define $\sup_{(z, \mu, \nu) \in H_1(q, \lambda, \overline{y})} \nu = -\infty$.

Theorem 10 (G-weak duality). *Let x and $(z, \mu, \nu, q, \lambda, \overline{y})$ be (P)-feasible and (DI)-feasible, respectively; let G_ϕ be both continuously differentiable and strictly increasing on $I_\phi(X, Y)$; let G_{g_j} be both continuously differentiable and strictly increasing on $I_{g_j}(X)$ for each $j \in M$. If $\phi(\cdot, y_i)$ is (G_ϕ, β_i^ϕ)-invex at z for each $i \in Q$ and g_j is (G_{g_j}, β_j^g)-invex at z for each $j \in M$, then*

$$\sup_{y \in Y} \phi(x, y) \ge \nu. \tag{38}$$

Proof. Suppose to the contrary that $\sup_{y \in Y} \phi(x, y) < \nu$. Therefore, we obtain

$$\phi(x, y) < \nu \le \phi(z, y_i), \quad y_i \in Y(z), \quad \forall y \in Y, \ i \in Q. \tag{39}$$

Thus, we obtain from the monotonicity assumption of G_ϕ that

$$G_\phi \circ \phi(x, y_i) < G_\phi \circ \phi(z, y_i), \quad y_i \in Y(z), \quad i \in Q. \tag{40}$$

Again, we obtain from the monotonicity assumption of G_{g_j} and the fact

$$g_j(x) \le 0, \quad \mu_j g_j(z) \ge 0, \quad \mu_j \ge 0, j \in M \tag{41}$$

that

$$G_{g_j} \circ g_j(x) \le G_{g_j} \circ g_j(z), \quad j \in M. \tag{42}$$

Hence,

$$\sum_{i=1}^q \lambda_i \frac{G_\phi \circ \phi(x, y_i) - G_\phi \circ \phi(z, y_i)}{\beta_i^\phi(x, z)}$$
$$+ \sum_{j=1}^m \mu_j \frac{G_{g_j} \circ g_j(x) - G_{g_j} \circ g_j(z)}{\beta_j^g(x, z)} < 0. \tag{43}$$

Similar to the proof of Theorem 8, by (43) and the generalized invexity assumptions of $\phi(\cdot, y_i)$ and g_j, we have

$$\left\langle \sum_{i=1}^q \lambda_i G'_\phi (\phi(z, y_i)) \xi_i^\phi + \sum_{j=1}^m \mu_j G'_{g_j} (g_j(z)) \xi_j^g, \eta(x, z) \right\rangle < 0, \tag{44}$$

which follows that

$$0 \notin \sum_{i=1}^q \lambda_i G'_\phi (\phi(z, y_i)) \partial_x \phi(z, y_i) + \sum_{j=1}^m \mu_j G'_{g_j} (g_j(z)) \partial g_j(z). \tag{45}$$

Thus, we have a contradiction to (34). So $\sup_{y \in Y} \phi(x, y) \ge \nu$. $\square$

Theorem 11 (G-strong duality). *Let problem (P) satisfy Conditions 4 and 5; let x^* be an optimal solution of problem (P). Suppose that G_ϕ is both continuously differentiable and strictly increasing on $I_\phi(X, Y)$ and G_{g_j} is both continuously differentiable and strictly increasing on $I_{g_j}(X)$ with $G'_{g_j}(g_j(x^*)) > 0$ for each $j \in M$. If the hypothesis of Theorem 10 holds for all (DI)-feasible points $(z, \mu, \nu, q, \lambda, \overline{y})$, then there exist $(q^*, \lambda^*, \overline{y}^*) \in K(z)$ and $(x^*, \mu^*, \nu^*) \in H_1(q^*, \lambda^*, \overline{y}^*)$ such that $(q^*, \lambda^*, \overline{y}^*, x^*, \mu^*, \nu^*)$ is a (DI) optimal solution, and the two problems (P) and (DI) have the same optimal values.*

Proof. By Theorem 7, there exists $\nu^* = \phi(x^*, y_i^*), i = 1, \dots, q^*$, satisfying the requirements specified in the theorem, such that $(q^*, \lambda^*, \overline{y}^*, x^*, \mu^*, \nu^*)$ is a (DI) feasible solution, then the optimality of this feasible solution for (DI) follows from Theorem 10. $\square$

Theorem 12 (G-strict converse duality). *Let $\overline{x}$ and $(z, \mu, \nu, q, \lambda, \overline{y})$ be optimal solutions for (P) and (DI), respectively. Suppose that G_ϕ is both continuously differentiable and strictly increasing on $I_\phi(X, Y)$ and G_{g_j} is both continuously differentiable and strictly increasing on $I_{g_j}(X)$ for each $j \in M$. If $\phi(\cdot, y_i)$ is (G_ϕ, β_i^ϕ)-invex at z for each $i \in Q$ and g_j is (G_{g_j}, β_j^g)-invex at z for each $j \in M$, then $\overline{x} = z$; that is, z is a (P)-optimal solution and $\sup_{y \in Y} \phi(\overline{x}, y) = \nu$.*

Proof. Suppose to the contrary that $\overline{x} \neq z$. Similar to the arguments as in the proof of Theorem 8, there exist $\xi_i^{\phi} \in \phi(z, y_i)$ and $\xi_j^g \in g_j(z)$ such that

$$
\begin{aligned}
0 &= \left\langle \sum_{i=1}^{q} \lambda_i G_{\phi}' \left(\phi(z, y_i) \right) \xi_i^{\phi} + \sum_{j=1}^{m} \mu_j G_{g_j}' \left(g_j(z) \right) \xi_j^g, \eta(\overline{x}, z) \right\rangle \\
&< \sum_{i=1}^{q} \lambda_i \frac{G_{\phi} \circ \phi(\overline{x}, y_i) - G_{\phi} \circ \phi(z, y_i)}{\beta_i^{\phi}(\overline{x}, z)} \\
&\quad + \sum_{j=1}^{m} \mu_j \frac{G_{g_j} \circ g_j(\overline{x}) - G_{g_j} \circ g_j(z)}{\beta_j^g(\overline{x}, z)}, \\
&\sum_{j=1}^{m} \mu_j \frac{G_{g_j} \circ g_j(\overline{x}) - G_{g_j} \circ g_j(z)}{\beta_j^g(\overline{x}, z)} \leq 0.
\end{aligned}
\tag{46}
$$

Therefore,

$$
\sum_{i=1}^{q} \lambda_i \frac{G_{\phi} \circ \phi(\overline{x}, y_i) - G_{\phi} \circ \phi(z, y_i)}{\beta_i^{\phi}(\overline{x}, z)} > 0.
\tag{47}
$$

From the above inequality, we can conclude that there exists $i_0 \in Q$, such that

$$
G_{\phi} \circ \phi\left(\overline{x}, y_{i_0}\right) - G_{\phi} \circ \phi\left(z, y_{i_0}\right) > 0
\tag{48}
$$

or

$$
\phi\left(\overline{x}, y_{i_0}\right) > \phi\left(z, y_{i_0}\right).
\tag{49}
$$

It follows that

$$
\sup_{y \in Y} \phi(\overline{x}, y) \geq \phi\left(\overline{x}, y_{i_0}\right) > \phi\left(z, y_{i_0}\right) > \nu.
\tag{50}
$$

On the other hand, we know from Theorem 10 that

$$
\sup_{y \in Y} \phi(\overline{x}, y) = \nu.
\tag{51}
$$

This contradicts to (50).   □

5. Conclusion

In this paper, we have discussed the applications of (G, β)-invexity for a class of nonsmooth minimax programming problem (P). Firstly, we established G-necessary optimality conditions for problem (P). Under the nondifferential (G, β)-invexity assumptions, we have also derived the sufficiency of the G-necessary optimality conditions for the same problem. Further, we have constructed a dual model (DI) and derived G-duality results between problems (P) and (DI). Note that many researchers are interested in dealing with the minimax programming under generalized invexity assumptions; see [1, 10, 11, 14–17]. However, we have not found results for minimax programming problems under the G-invexity or its extension assumptions. Hence, this work extends the applications of G-invexity to the generalized minimax programming as well as to the nonsmooth case.

References

[1] T. Antczak, "Generalized fractional minimax programming with B-(p, r)-invexity," *Computers & Mathematics with Applications*, vol. 56, no. 6, pp. 1505–1525, 2008.

[2] M. A. Hanson, "On sufficiency of the Kuhn-Tucker conditions," *Journal of Mathematical Analysis and Applications*, vol. 80, no. 2, pp. 545–550, 1981.

[3] A. Ben-Israel and B. Mond, "What is invexity?" *Australian Mathematical Society Journal B*, vol. 28, no. 1, pp. 1–9, 1986.

[4] T. Antczak, "New optimality conditions and duality results of G type in differentiable mathematical programming," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 66, no. 7, pp. 1617–1632, 2007.

[5] T. Antczak, "On G-invex multiobjective programming. Part I. Optimality," *Journal of Global Optimization*, vol. 43, no. 1, pp. 97–109, 2009.

[6] T. Antczak, "On G-invex multiobjective programming. Part II. Duality," *Journal of Global Optimization*, vol. 43, no. 1, pp. 111–140, 2009.

[7] T. Antczak, "(p, r)-invex sets and functions," *Journal of Mathematical Analysis and Applications*, vol. 263, no. 2, pp. 355–379, 2001.

[8] V. Preda, "On efficiency and duality for multiobjective programs," *Journal of Mathematical Analysis and Applications*, vol. 166, no. 2, pp. 365–377, 1992.

[9] Z. A. Liang, H. X. Huang, and P. M. Pardalos, "Optimality conditions and duality for a class of nonlinear fractional programming problems," *Journal of Optimization Theory and Applications*, vol. 110, no. 3, pp. 611–619, 2001.

[10] D. H. Yuan, X. L. Liu, A. Chinchuluun, and P. M. Pardalos, "Nondifferentiable minimax fractional programming problems with (C, α, ρ, d)-convexity," *Journal of Optimization Theory and Applications*, vol. 129, no. 1, pp. 185–199, 2006.

[11] M. V. Ştefănescu and A. Ştefănescu, "Minimax programming under new invexity assumptions," *Revue Roumaine de Mathématiques Pures et Appliquées*, vol. 52, no. 3, pp. 367–376, 2007.

[12] T. Antczak, "The notion of V-r-invexity in differentiable multiobjective programming," *Journal of Applied Analysis*, vol. 11, no. 1, pp. 63–79, 2005.

[13] H. C. Lai and J. C. Lee, "On duality theorems for a nondifferentiable minimax fractional programming," *Journal of Computational and Applied Mathematics*, vol. 146, no. 1, pp. 115–126, 2002.

[14] T. Antczak, "Minimax programming under (p, r)-invexity," *European Journal of Operational Research*, vol. 158, no. 1, pp. 1–19, 2004.

[15] I. Ahmad, S. Gupta, N. Kailey, and R. P. Agarwal, "Duality in nondifferentiable minimax fractional programming with B-(p, r)-invexity," *Journal of Inequalities and Applications*, vol. 2011, article 75, 2011.

[16] T. Antczak, "Nonsmooth minimax programming under locally Lipschitz (Φ, ρ)-invexity," *Applied Mathematics and Computation*, vol. 217, no. 23, pp. 9606–9624, 2011.

[17] S. K. Mishra and K. Shukla, "Nonsmooth minimax program-
ming problems with V-r-invex functions," *Optimization*, vol. 59,
no. 1, pp. 95–103, 2010.

[18] D. H. Yuan, X. L. Liu, S. Y. Yang, and G. M. Lai, "Nondifferential
mathematical programming involving (G, β)-invexity," *Journal
of Inequalities and Applications*, vol. 2012, article 256, 2012.

[19] F. H. Clarke, *Optimization and Nonsmooth Analysis*, John Wiley
& Sons, New York, NY, USA, 1983.

[20] M. Studniarski and A. W. A. Taha, "A characterization of strict
local minimizers of order one for nonsmooth static minmax
problems," *Journal of Mathematical Analysis and Applications*,
vol. 259, no. 2, pp. 368–376, 2001.