Research Article

New Multiple Solution to the Boussinesq Equation and the Burgers-Like Equation

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By considering an improved tanh function method, we found some exact solutions of Boussinesq and Burgers-like equations. The main idea of this method is to take full advantage of the Riccati equation which has more new solutions. We found some exact solutions of the Boussinesq equation and the Burgers-like equation.

1. Introduction

In recent years, nonlinear phenomena play a crucial role in applied mathematics and physics. Directly searching for exact solutions of nonlinear partial differential equations (PDEs) has become more and more attractive partly due to the availability of computer symbolic systems like Mathematica or Maple that allow us to perform some complicated and tedious algebraic calculation on a computer as well as help us to find exact solutions of PDEs [1–5] now.

Many explicit exact methods have been introduced in the literature [6–14]. Some of them are Painlevé method, homogeneous balance method, similarity reduction method, sine-cosine method, Darboux transformation, Cole-Hopf transformation, generalized Miura transformation, tanh method, Backlund transformation, and others methods [15, 16].

One of the most effectively straightforward methods constructing exact solution of PDEs is the extended tanh function method [17]. Let us simply describe the tanh function. For doing this, one can consider in two variables general form of nonlinear PDE as follows:

$$H\left(u, u_{tt}, u_x, u_{xx}, \dots\right) = 0 \tag{1}$$

and transform (1) with

$$u\left(x, t\right) = u\left(\xi\right), \quad \xi = k\left(x - \lambda t\right), \tag{2}$$

where k and λ are the wave number and wave speed, respectively. After the transformation, we get a nonlinear ODE for $u(\xi)$ as follows:

$$H'\left(u', u'', u''', \dots\right) = 0. \tag{3}$$

The fact that the solutions of many nonlinear equations can be expressed as a finite series of tanh functions that motivates us to seek for the solutions of (3) in the form

$$u\left(x, t\right) = u\left(\xi\right) = \sum_{i=0}^{m} a_i \tanh^i\left(\xi\right) = \sum_{i=0}^{m} a_i F^i, \tag{4}$$

where $F^i = \tanh^i(\xi)$, an equation for $F(\xi)$, is obtained. m is a positive integer that can be determined by balancing the linear term of highest order with the nonlinear term in (1), and k, λ, a_1, $a_2, \dots, a_m$ are parameters to be determined.

Substituting solution (4) into (3) yields a set of algebraic equations for F^i, then all coefficients of F^i have to vanish. From these relations k, λ, a_1, $a_2, \dots, a_m$ can be determined.

In this work, we will consider to solve general Boussinesq equation and the Burgers-like equation by using the improved tanh function method which is introduced by Chen and Zhang [18].

2. Method and Its Applications

The main idea of this method is to take full advantage of the Riccati equation that tanh function satisfies and uses its

solutions F to replace $\tanh \xi$. The required Riccati equation is given as

$$F' = A + BF + CF^2, \qquad (5)$$

where $F = (dF/d\xi)$ and $A, B,$ and C are constant. In the following, Chen and Zhang [18] have given several cases to get the solution of (5) in the form of finite series of tanh functions (4).

Case 1. If $C = 0, B \neq 0$, then (5) has the solutions

$$\frac{\exp (B\xi) - A}{B}. \qquad (6)$$

Case 2. If $A = 1/2$, $B = 0$, and $C = -1/2$, then (5) has the solutions $\cot h\xi \pm \csc h\xi$, $\tan h\xi \pm i\sec h\xi$ $(i^2 = -1)$.

Case 3. If $A = C = \pm 1/2$, $B = 0$, then (5) has the solutions $\sec \xi \pm \tan \xi$, $\csc \xi \pm \cot \xi$.

Case 4. If $A = 1$, $B = 0$, and $C = -1$, then (5) has the solutions $\tan h\xi$, $\cot h\xi$.

Case 5. If $A = C = 1, B = 0$, then (5) has the solutions $\tan \xi$.

Case 6. If $A = C = -1, B = 0$, then (5) has the solutions $\cot \xi$.

Case 7. If $A = B = 0, C \neq 0$, then (5) has the solutions $-1/(c\xi + c_0)$.

The solutions of (1) can be expressed in the form

$$u(x, t) = u(\xi) = \sum_{i=0}^{m} a_i F^i, \qquad (7)$$

where $\xi = kx - k\omega t$, k and ω are the wave number and the wave speed, respectively, n is a positive integer that can be determined by balancing the linear term of highest order with the nonlinear term in (1), and $a_0, a_1, \ldots, a_n$ are parameters to be determined.

Introducing the similarity variable $\xi = kx - k\omega t$, the travelling wave solutions $u(\xi)$ satisfy

$$H'\left(u', u'', u''', \ldots\right) = 0. \qquad (8)$$

Balancing the highest order of linear term with nonlinear term in (3), we can determine n in (4).

We illustrate the method by considering the Boussinesq equation and Burgers-like equtions.

Example 1 (The Boussinesq equation). Let's consider

$$u_{tt} = \left(c_1 u + c_2 u^n\right)_{xx} + u_{xxtt} = 0. \qquad (9)$$

If we accept that $c_1 \geq 0, c_2 \leq 0$ and $n = 2, c = -1$, we conclude (10) by (9) as follows:

$$u_{tt} - u_{xx} + 2u \cdot u_{xx} - u_{xxtt} = 0 \qquad (10)$$

for doing this example. We could use transformation with (1) for the Boussinesq equation. Let us consider the Boussinesq equation as follows:

$$u(x, t) = u(\xi), \quad \xi = kx - k\omega t. \qquad (11)$$

Substituting (11) into (10), we get

$$w^2 u'' - u'' + 2\left(u'\right)^2 + 2u \cdot u'' - k^2 w^2 u^4 = 0, \qquad (12)$$

and integrating (12) we deduce the following equation:

$$w^2 u' - u' + 2u \cdot u' - k^2 w^2 u''' = 0. \qquad (13)$$

Balancing $(u \cdot u')$ with (u''') gives $m = 2$. Therefore, we may choose

$$u = a_0 + a_1 F + a_2 F^2. \qquad (14)$$

Substituting (14) into (13) along with (5) and using Mathematica yield a system of equations $w, t,$ and F^M. Setting the coefficients of F^M in the obtained system of equations to zero, we can deduce the following set of algebraic polynomials with respect to unknowns $a_0, a_1,$ and a_2

$$u' = a_1 A + a_1 BF + a_1 CF^2$$
$$+ 2a_2 AF + 2a_2 BF^2 + 2a_2 CF^3 = 0,$$

$$u \cdot u' = a_0 a_1 A + a_0 a_1 BF + a_0 a_1 CF^2$$
$$+ 2a_0 a_2 AF + 2a_0 a_2 BF^2 + 2a_0 a_2 CF^3$$
$$+ a_1^2 AF + a_1^2 BF^2 + a_1^2 CF^3$$
$$+ 2a_1 a_2 AF^2 + 2a_1 a_2 BF^3$$
$$+ 2a_1 a_2 CF^4 + a_1 a_2 AF^2$$
$$+ a_1 a_2 BF^3 + a_1 a_2 CF^4$$
$$+ 2a_2^2 AF^3 + 2a_2^2 BF^4 + 2a_2^2 CF^5,$$

$$u''' = a_1 B^2 A + a_1 B^3 F + 7a_1 B^2 CF^2$$
$$+ 8a_1 ABCF + 12a_1 BC^2 F^3 + 2a_1 A^2 C$$
$$+ 8a_1 AC^2 F^2 + 6a_1 C^3 F^4 + 6a_2 A^2 B$$
$$+ 14a_2 AB^2 F + 52a^2 ABCF^2$$
$$+ 16a_2 A^2 CF + 40a_2 AC^2 F^3$$
$$+ 8a_2 B^3 F^2 + 38a^2 B^2 CF^3$$
$$+ 54a_2 BC^2 F^4 + 24a_2 C^3 F^5,$$

$$F^0 : w^2 a_1 A - a_1 A + 2a_0 a_1 A - k^2 w^2 a_1 B^2 A$$
$$- 2k^2 w^2 a_1 A^2 C - 6k^2 w^2 a_2 A^2 B = 0,$$

$$F^1 : w^2 a_1 B + 2w^2 a_2 A - a_1 B - 2a_2 B$$
$$+ 2a_0 a_1 B + 4a_0 a_2 A + 2a_1^2 A - k^2 w^2 a_1 B^3$$
$$- 8k^2 w^2 a_1 ABC - 14k^2 w^2 a_2 AB^2$$
$$- 16k^2 w^2 a_2 A^2 C = 0,$$

$$F^2 : w^2 a_1 C + 2w^2 a_2 B - a_1 C - 2a_2 B$$
$$+ 2a_0 a_1 C + 4a_0 a_2 B + 2a_1^2 B$$
$$+ 6a_1 a_2 A - 7k^2 w^2 a_1 B^2 C - 8k^2 w^2 a_1 AC^2$$
$$- 52k^2 w^2 ABC - 8k^2 w^2 a_2 B^3 = 0,$$

$$F^3 : 2w^2 a_2 C - 2a_2 C + 4a_0 a_2 C + 2a_1^2 C$$
$$+ 4a_1 a_2 B + 2a_1 a_2 B + 4a_2^2 A - 12k^2 w^2 a_1 BC^2$$
$$- 40k^2 w^2 a_2 AC^2 - 38k^2 w^2 B^2 C = 0,$$

$$F^4 : -6a_1 a_2 C + 4a_2^2 B - 6k^2 w^2 a_1 C^3 - 54k^2 w^2 a_2 BC^2 = 0,$$

$$F^5 : 4a_2^2 C - 24k^2 w^2 a_2 C^3 = 0.$$

$$(15)$$

From the solutions of the system, we can find

$$B = 0, \qquad a_0 = 4k^2 w^2 AC - \frac{1}{2} w^2 + \frac{1}{2},$$
$$a_1 = 0, \qquad a_2 = 6k^2 w^2 C^2, \qquad (16)$$
$$w = \frac{1}{4k^2 AC + 1},$$

and with the aid of Mathematica, we find the following.

(i) When we choose $A = 1$, $B = 0$, and $C = 1$ in (16), then we can deduce

$$a_0 = 4k^2 w^2 - \frac{1}{2} w^2 + \frac{1}{2}, \qquad a_2 = 6k^2 w^2. \qquad (17)$$

Therefore, the solution can be found as

$$u(x,t) = \left(4k^2 w^2 - \frac{1}{2} w^2 + \frac{1}{2}\right) + 6k^2 w^2 \left[\tan^2(kx - kwt)\right].$$
$$(18)$$

(ii) In this case, if we take $A = -1$, $B = 0$, and $C = -1$ in (16), then we have

$$a_0 = 4k^2 w^2 - \frac{1}{2} w^2 + \frac{1}{2}, \qquad a_2 = 6k^2 w^2,$$
$$u(x,t) = \left(4k^2 w^2 - \frac{1}{2} w^2 + \frac{1}{2}\right) + 6k^2 w^2 \cot[kx - kwt].$$
$$(19)$$

(iii) Again, when we choose $A = 1$, $B = 0$, $C = -1$ then from (16) is

$$a_0 = -4k^2 w^2 - \frac{1}{2} w^2 + \frac{1}{2}, \qquad a_2 = 6k^2 w^2,$$
$$u(x,t) = \left(-4k^2 w^2 - \frac{1}{2} w^2 + \frac{1}{2}\right) + 6k^2 w^2 \tan h^2(kx - kwt),$$
$$u(x,t) = \left(-4k^2 w^2 - \frac{1}{2} w^2 + \frac{1}{2}\right) + 6k^2 w^2 \cot h^2(kx - kwt).$$
$$(20)$$

(iv) When we choose $A = (1/2)$, $B = 0$, and $C = (1/2)$, then we can find the coefficients of (16) as

$$a_0 = k^2 w^2 - \frac{1}{2} w^2 + \frac{1}{2}, \qquad a_2 = \frac{3}{2} k^2 w^2, \qquad (21)$$

and using the coefficients, the solutions can be found as

$$u(x,t) = \left(k^2 w^2 - \frac{1}{2} w^2 + \frac{1}{2}\right) + \frac{3}{2} k^2 w^2$$
$$\times \left[\sec(kx - kwt) + \tan(kx - kwt)\right]^2,$$
$$u(x,t) = \left(k^2 w^2 - \frac{1}{2} w^2 + \frac{1}{2}\right) + \frac{3}{2} k^2 w^2$$
$$\times \left[\csc(kx - kwt) + \cot(kx - kwt)\right]^2.$$
$$(22)$$

(v) When we choose $A = -(1/2)$, $B = 0$, and $C = -(1/2)$, then we can find the coefficients of (16) as follows:

$$a_0 = k^2 w^2 - \frac{1}{2} w^2 + \frac{1}{2}, \qquad a_2 = \frac{3}{2} k^2 w^2,$$
$$u(x,t) = \left(k^2 w^2 - \frac{1}{2} w^2 + \frac{1}{2}\right) + \frac{3}{2} k^2 w^2$$
$$\times \left[\sec(kx - kwt) + \tan(kx - kwt)\right]^2,$$
$$u(x,t) = \left(k^2 w^2 - \frac{1}{2} w^2 + \frac{1}{2}\right) + \frac{3}{2} k^2 w^2$$
$$\times \left[\csc(kx - kwt) + \cot(kx - kwt)\right]^2.$$
$$(23)$$

(vi) When we choose $A = (1/2)$, $B = 0$, and $C = -(1/2)$, then we can find the coefficients of (16) as follows:

$$a_0 = -k^2 w^2 - \frac{1}{2} w^2 + \frac{1}{2}, \qquad a_2 = \frac{3}{2} k^2 w^2,$$
$$u(x,t) = \left(-k^2 w^2 - \frac{1}{2} w^2 + \frac{1}{2}\right) + \frac{3}{2} k^2 w^2$$
$$\times \left[\coth(kx - kwt) \pm \csc h(kx - kwt)\right]^2,$$
$$u(x,t) = \left(-k^2 w^2 - \frac{1}{2} w^2 + \frac{1}{2}\right) + \frac{3}{2} k^2 w^2$$
$$\times \left[\tanh(kx - kwt) \pm i \sec h(kx - kwt)\right]^2.$$
$$(24)$$

(vii) When we choose $A = 0$, $B = 0$, and $C \neq 0$, then we can find the coefficients of (16) as follows:

$$a_0 = -\frac{1}{2} w^2 + \frac{1}{2}, \qquad a_2 = 6k^2 w^2 C^2,$$
$$u(x,t) = \left(-\frac{1}{2} w^2 + \frac{1}{2}\right) + \frac{3}{2} k^2 w^2 C^2 \left(-\frac{1}{(kx - kwt) + C_0}\right)^2.$$
$$(25)$$

Figure 1 gives to us 2D and 3D graphics for (25).

Example 2 (Burger-like equations). Let's consider

$$u_t + u_x + u u_x + \lambda u_{xx} = 0, \qquad (26)$$

(a)

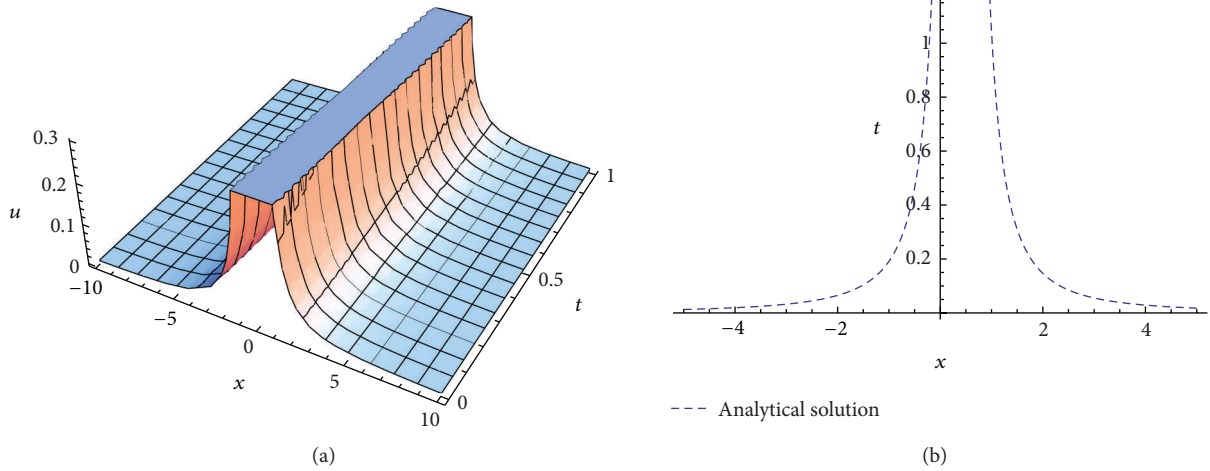--- Analytical solution

(b)

FIGURE 1: The solution (25) is shown at $k = w = 1$, $C = 0.5$, and $C_0 = 0.1$, and the second graph represents the exact analytical solution of (25) for $t = 0.5$.
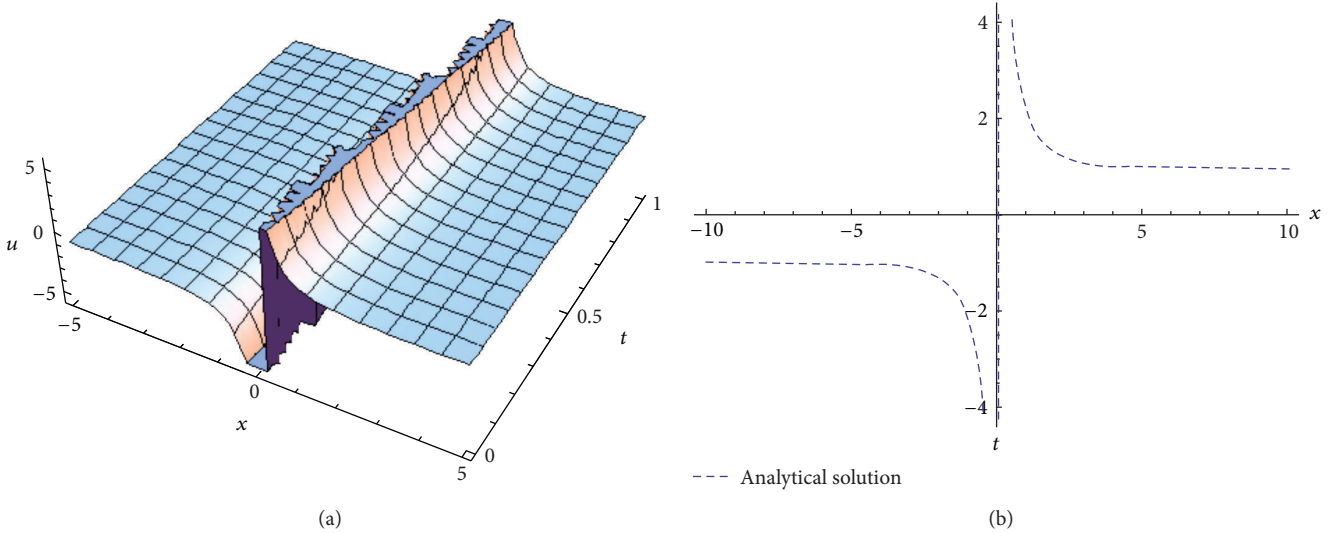

(a)

--- Analytical solution

(b)

FIGURE 2: The solution (39) is shown at $k = w = 1$, and the second graph represents the exact analytical solution of (39) for $t = 0.5$.

where $\lambda = 1$, and in order to obtain Burger-like solution of equation, we get

$$u(x,t) = u(\xi), \quad \xi = k(x - wt). \tag{27}$$

Substituting (27) into (26), we get

$$-wu' + u' + uu' + ku'' = 0. \tag{28}$$

Balancing (uu') with u'' gives $m = 1$. Therefore, we may choose the following ansatz:

$$u = a_0 + a_1 F. \tag{29}$$

Substituting (29) into (28) along with (5) and using Mathematica yield the following set of algebraic polynomials with respect to unknowns a_0, a_1, a_2, and a_3

$$u' = a_1 A + a_1 BF + a_1 CF^2 + 2a_2 AF \tag{30}$$
$$+ 2a_2 BF^2 + 2a_2 CF^3 = 0,$$

$$uu' = (a_0 + a_1 F)(a_1 A + a_1 BF + a_1 CF^2),$$

$$u''' = a_1 BA + a_1 B^2 F + a_1 CAF$$
$$+ 3a_1 BCF^2 + 2a_1 C^2 F^3,$$

$$F^0 : w^2 a_1 A - a_1 A + 2a_0 a_1 A - k^2 w^2 a_1 B^2 A$$
$$- 2k^2 w^2 a_1 A^2 C - 6k^2 w^2 a_2 A^2 B = 0,$$

$$F^1 : w^2 a_1 B + 2w^2 a_2 A - a_1 B - 2a_2 B + 2a_0 a_1 B$$

$$+ 4a_0 a_2 A + 2a_1^2 A - k^2 w^2 a_1 B^3$$

$$- 8k^2 w^2 a_1 ABC - 14k^2 w^2 a_2 AB^2$$

$$- 16k^2 w^2 a_2 A^2 C = 0,$$

$$F^2 : w^2 a_1 C + 2w^2 a_2 B - a_1 C - 2a_2 B + 2a_0 a_1 C$$

$$+ 4a_0 a_2 B + 2a_1^2 B + 6a_1 a_2 A - 7k^2 w^2 a_1 B^2 C$$

$$- 8k^2 w^2 a_1 AC^2 - 52k^2 w^2 a_2 B^3 = 0,$$

$$F^3 : 2w^2 a_2 C - 2a_2 C + 4a_0 a_2 C + 2a_1^2 C + 4a_1 a_2 B$$

$$+ 2a_1 a_2 B + 4a_2^2 A - 12k^2 w^2 a_1 BC^2$$

$$- 40k^2 w^2 a_2 AC^2 - 38k^2 w^2 B^2 C = 0.$$

$$(31)$$

From the solutions of the system, we can find

$$a_0 = w - 1, \qquad a_1 = -2kc, \tag{32}$$

and we obtain the following multiple solution and triangular periodic solutions of (26).

(i) When we choose $A = C = 1$, $B = 0$ in (31), then

$$a_1 = -2k, \qquad a_0 = w - 1. \tag{33}$$

Therefore, the solution can be found as

$$u(x,t) = (w - 1) - 2k \tan[k(x - wt)]. \tag{34}$$

(ii) In the case if we take $A = C = -1$, $B = 0$ in (31), then we have

$$a_1 = 2k, \qquad a_0 = w - 1,$$
$$u(x,t) = (w - 1) + 2k \cot[k(x - wt)]. \tag{35}$$

(iii) Again, when we choose $A = 1$, $B = 0$, and $C = -1$,

$$a_0 = w - 1, \qquad a_1 = -2k,$$

$$u(x,t) = (w - 1) + 2k \tanh[k(x - wt)], \tag{36}$$

$$u(x,t) = (w - 1) + 2k \coth[k(x - wt)].$$

(iv) When we choose $A = (1/2)$, $B = 0$, and $C = -(1/2)$,

$$a_0 = w - 1, \qquad a_1 = -k,$$

$$u(x,t) = (w - 1) - k[\sec[k(x - wt)] + \tan[k(x - wt)]],$$

$$u(x,t) = (w - 1) - k[\cos ec[k(x - wt)] + \cot[k(x - wt)]]. \tag{37}$$

(v) When we choose $A = C = -(1/2)$, $B = 0$, then we can find the coefficients of (31) as follows:

$$a_0 = k, \qquad a_0 = w - 1,$$

$$u(x,t) = (w - 1) + k[\sec[k(x - wt)] - \tan[k(x - wt)]],$$

$$u(x,t) = (w - 1) + k[\cos ec[k(x - wt)] - \cot[k(x - wt)]]. \tag{38}$$

(vi) When we choose $A = (1/2)$, $B = 0$, and $C = -(1/2)$, then we can find the coefficients of (31) as follows:

$$u(x,t) = (w - 1) + k[\coth[k(x - wt)] \pm \cos ech[k(x - wt)]],$$

$$u(x,t) = (w - 1) + k[\tanh[k(x - wt)] \pm i\sec h[k(x - wt)]]. \tag{39}$$

Figure 2 gives to us 2D and 3D graphics for (39).

3. Conclusion

We have presented a generalized tanh function method and used it to solve the Boussinesq equation and the Burgers-like equation. In fact this method is readily applicable to a large variety of nonlinear PDE*s*.

Firstly, all the nonlinear PDE*s* which can be solved by other tanh function method can be solved easily by this method. Secondly we have used only the special solutions of (4). If we use only the special solutions of (4), we can obtain more solutions. We are also aware of the fact that not all fundamental equations can be treated with the method.

We also obtain some new and more general solutions at the same time. Furthermore, this method is also computerizable, which allows us to perform complicated and tedious algebraic calculation on a computer.

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