INTEGRAL REPRESENTATIONS FOR PADÉ-TYPE OPERATORS

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The main purpose of this paper is to consider an explicit form of the Padé-type operators. To do so, we consider the representation of Padé-type approximants to the Fourier series of the harmonic functions in the open disk and of the L^p-functions on the circle by means of integral formulas, and, then we define the corresponding Padé-type operators. We are also concerned with the properties of these integral operators and, in this connection, we prove some convergence results.

1. Introduction

Let f be a function analytic in the open unit disk D, with Taylor power series expansion $\sum_{v=0}^{\infty} a_v \cdot z^v$, and let Λ_f be the linear functional on the space of complex polynomials defined by $\Lambda_f(x^v) = a_v$ $(v = 0,1,2,\ldots)$. By Cauchy's integral formula and by a density argument, the functional Λ_f can be extended to the space $A(\bar{D})$ of all functions which are analytic in D and continuous in the open neighborhood of $\bar{D}$ (see [4]). In particular, we have $f(z) = \Lambda_f((1-x\cdot z)^{-1})$ for any $z \in D$.

Now, let $v_{m+1}(x)$ be an arbitrary polynomial of degree $m+1$, with distinct zeros $\pi_1, \pi_2, \ldots, \pi_n$ of respective multiplicities $(m_1+1), (m_2+1), \ldots, (m_n+1)$ and $(m_1+1)+(m_2+1)+\cdots+(m_n+1) = m+1$.

Let $I(v_{m+1})$ be the linear operator mapping each $g(x) \in A(\bar{D})$ into its Hermite interpolation polynomial G_{m+1} of degree at most m defined by

$$g^{(j)}(\pi_i) = G_{m+1}^{(j)}(\pi_i) \quad \text{for } i = 1,\ldots,n,\ j = 0,1,\ldots,m_i. \tag{1.1}$$

If $g(x,z) = (1-x\cdot z)^{-1}$, then $\Lambda_f(G_{m+1}(x,z))$ is the so-called Padé-type

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approximant to $f(z)$ with generating polynomial $v_{m+1}(x)$. It is a rational function with numerator of degree m and denominator of degree $m+1$, denoted by $(m/(m+1))_f(z)$ and such that

$$f(z)-\left(\frac{m}{m+1}\right)_f(z)=O\left(z^{m+1}\right),\quad \text{if } |z|<\min\left\{\frac{1}{|\pi_1|},\dots,\frac{1}{|\pi_n|}\right\}. \tag{1.2}$$

If $v_{m+1}(x)$ is identical to the orthogonal polynomial $q_{m+1}(x)$ with respect to Λ_f, that is, the polynomial satisfying the orthogonality conditions $\Lambda_f(x^v\cdot q_{m+1}(x))=0$, $v=0,1,\dots,m$, then the Padé-type approximant $(m/(m+1))_f(z)$ becomes identical to the classical Padé approximant $[m/(m+1)]_f(z)$ such that

$$f(z)-\left[\frac{m}{m+1}\right]_f(z)=O\left(z^{2m+2}\right),\quad \text{if } |z|<\min\left\{\frac{1}{|\pi_1|},\dots,\frac{1}{|\pi_n|}\right\}. \tag{1.3}$$

By making use of the notation of duality, we can also write

$$\begin{aligned}\left(\frac{m}{m+1}\right)_f(z)&=\Lambda_f\left(G_{m+1}(x,z)\right)\\&=\left\langle\Lambda_f,\left[I\left(v_{m+1}\right)\right](1-x\cdot z)^{-1}\right\rangle\\&=\left\langle\left[I^*\left(v_{m+1}\right)\right]\left(\Lambda_f\right)(1-x\cdot z)^{-1}\right\rangle.\end{aligned} \tag{1.4}$$

In [3], Brezinski showed that the operator which maps f on $(m/(m+1))_f$ can be understood as the mapping of $A^*(\bar{D})$ into itself which maps Λ_f into $[I^*(v_{m+1})](\Lambda_f)$. This mapping, which depends on the generating polynomial $v_{m+1}(x)$, is called the Padé-type operator for the space $O(D)$ of all analytic functions on D and it is exactly the operator $I^*(v_{m+1})$. If $v_{m+1}(x)$ does not depend on Λ_f, then $I^*(v_{m+1})$ is linear. But for Padé approximants, since $v_{m+1}(x)$ is the orthogonal polynomial $q_{m+1}(x)$ of degree $m+1$ with respect to the functional Λ_f, then $v_{m+1}(x)$ depends on Λ_f and the linearity property holds only if the first $2m+2$ moments of both functionals are the same since, then, both orthogonal polynomial of degree $m+1$ will be the same.

The aim of this paper is to consider the explicit form of the Padé-type operator by means of integral representations. Section 2 deals with the definition of integral representations of Padé-type approximants to real-valued L^2 or harmonic functions and, thus, with the expressions of the Padé-type operator for the spaces $L^2_{\mathbb{R}}(C)$ of all real-valued L^2 functions on C, $L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi,\pi]$ of all real-valued 2π-periodic L^2 functions on $[-\pi,\pi]$, and $H_{\mathbb{R}}(D)$ of all real-valued harmonic functions on D. We also give some examples with applications of these integral representations for the Padé-type operator to the convergence problem of a series of Padé-type approximants and to the problem of finding a sufficient condition permitting the interpretation of any 2π-periodic L^p real-valued function on $[-\pi,\pi]$ as a Padé-type approximant. In [7], by introducing the so-called composed Padé-type approximation, we discussed the general situation of complex-valued harmonic or L^p functions and we showed that any Padé-type approximant in the ordinary sense to a function $f \in O(D)$ is a special case of this composed procedure. It is therefore natural to reflect that any $I^*(\upsilon_{m+1})$ can also be viewed as a special case of the operator which maps every $f \in O(D)$ on a composed Padé-type approximant to f. Such a mapping will be called a composed Padé-type operator for $O(D)$. In Section 3, we define and give the explicit form of the composed Padé-type operators for the spaces $L^2_{\mathbb{C}}(C)$ of all complex-valued L^2 functions on C, $L^2_{\mathbb{C},(2\pi\text{-per})}[-\pi,\pi]$ of all complex-valued 2π-periodic L^2 functions on $[-\pi,\pi]$, and $H_{\mathbb{C}}(D)$ of all complex-valued harmonic functions on D. Since $O(D) \subset H_{\mathbb{C}}(D)$, we thus obtain the desired explicit form of $I^*(\upsilon_{m+1})$.

2. Integral representations and Padé-type operators

In [5, 6], we have defined and studied Padé-type approximation to $L^P 2\pi$-periodic real-valued functions and to harmonic functions in D. In all cases, the development of our theory was analogous to the classical one about analytic functions.

Really, no situation is quite as pleasant as the L^2 case. In this section, we look for another way to introduce Padé-type approximants to L^2 functions and to harmonic functions. Our method is based on integral representation formulas and leads to a number of convergence results.

To begin our discussion, consider any real-valued L^2 function $u(z)$ defined on the unit circle C. Suppose that the Fourier series expansion of $u(e^{it})$ is $\sum_{v=-\infty}^{\infty} \sigma_v \cdot e^{ivt}$. Since u is square integrable, the sequence of partial sums $\{\sum_{v=-n}^{n} \sigma_v \cdot e^{ivt} : n = 0,1,2,\ldots\}$ converges to $u(e^{it})$ in the L^2-norm. Let $P(\mathbb{C})$ be the vector space of all complex-valued analytic polynomials with coefficients in $\mathbb{C}$. For every $p(x) = \sum_{v=0}^{m} \beta_v \cdot x^v \in P(\mathbb{C})$, we denote by $\bar{p}(x)$ the polynomial $\bar{p}(x) = \sum_{v=0}^{m} \bar{\beta}_v \cdot x^v \in P(\mathbb{C})$. Define the linear functionals $T_u : P(\mathbb{C}) \to \mathbb{C}$ and $S_u : P(\mathbb{C}) \to \mathbb{C}$ associated with u by

$$T_u(x^v) := \sigma_v, \quad S_u(x^v) := \sigma_{-v} \quad (v = 0,1,2,\ldots). \tag{2.1}$$

As it is well known, the Poisson integral of $u(z) = u(e^{it})$ ($|z| = 1$) extends to a harmonic real-valued function $u(z) = u(r \cdot e^{it})$ in the unit disk D (where $|z| < 1$, $0 \le r < 1$). This harmonic function being the real part of some analytic function in D, we immediately see that $\overline{T_u(x^v)} = \bar{\sigma}_v = \sigma_{-v} = S_u(x^v)$ for any $v \geq 0$. More generally, we have the following proposition.

PROPOSITION 2.1. *For every $p(x) \in P(\mathbb{C})$ there holds*

$$\overline{S_u(p(x))} = T_u(\bar{p}(x)), \qquad \overline{S_u(\bar{p}(x))} = T_u(p(x)). \tag{2.2}$$

Proof. Let $p(x) = \sum_{v=0}^{m} \beta_v x^v \in P(\mathbb{C})$. By linearity, we obtain

$$\begin{aligned}
S_u(p(x)) &= S_u\left(\sum_{v=0}^{m} \beta_v x^v\right) = \sum_{v=0}^{m} \beta_v S_u(x^v) = \sum_{v=0}^{m} \beta_v \overline{T_u(x^v)} \\
&= \overline{\sum_{v=0}^{m} \bar{\beta}_v \cdot T_u(x^v)} = \overline{T_u\left(\sum_{v=0}^{m} \bar{\beta}_v \cdot x^v\right)} = \overline{T_u(\bar{p}(x))}, \\
\overline{S_u(\bar{p}(x))} &= \overline{S_u\left(\sum_{v=0}^{m} \bar{\beta}_v x^v\right)} = \overline{\sum_{v=0}^{m} \bar{\beta}_v S_u(x^v)} = \overline{\sum_{v=0}^{m} \bar{\beta}_v \overline{T_u(x^v)}} \\
&= \sum_{v=0}^{m} \beta_v \cdot T_u(x^v) = T_u\left(\sum_{v=0}^{m} \beta_v \cdot x^v\right) = T_u(p(x)).
\end{aligned} \tag{2.3}$$

□

COROLLARY 2.2. *For every $p(x) \in P(\mathbb{C})$ there holds*

$$\operatorname{Re} T_u(\bar{p}(x)) = \operatorname{Re} S_u(p(x)), \qquad \operatorname{Re} T_u(p(x)) = \operatorname{Re} S_u(\bar{p}(x)). \tag{2.4}$$

Now, observe that the linear functional S_u can be extended continuously on the space $L^2_{\mathbb{R}}(C)$ of all real-valued square integrable functions on the unit circle C. In fact, if $p(x) = \sum_{v=0}^{m} \beta_v x^v \in P(\mathbb{C})$ then, by Hölder's inequality, we get

$$\begin{aligned}
\left|S_u(p(x))\right|^2 &= \left|\sum_{v=0}^{m} \beta_v \sigma_{-v}\right|^2 \\
&= \left|\sum_{v=0}^{m} \bar{\beta}_v \sigma_v\right|^2 \\
&= \left|\frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} u(e^{it}) \cdot \left(\sum_{v=0}^{m} \bar{\beta}_v \cdot e^{-ivt}\right) dt\right|^2 \\
&= \left|\frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} u(e^{it}) \cdot \overline{p(e^{it})}\, dt\right|^2 \\
&\leq c_u \cdot \|p(x)\|_2^2,
\end{aligned} \tag{2.5}$$

for some positive constant c_u depending only on u, and hence, by the Hahn-Banach theorem, there is a continuous linear extension of S_u on $L^2_{\mathbb{R}}(C)$. It follows, from the Riesz representation theorem, that there exists a unique $F_u \in L^2_{\mathbb{R}}(C)$ such that

$$S_u(g) = \int_C g(\zeta) \cdot \overline{F_u(\zeta)}\, d\zeta = i \cdot \int_{-\pi}^{\pi} g(e^{i\theta}) \cdot \overline{F_u(e^{i\theta})} \cdot e^{i\theta}\, d\theta \tag{2.6}$$

for all $g \in L^2_{\mathbb{R}}(C)$. In particular, if $g(\zeta) = \zeta^v$ then

$$S_u(\zeta^v) = \int_C \zeta^v \overline{F_u(\zeta)}\, d\zeta = i \cdot \int_{-\pi}^{\pi} e^{iv\theta} \cdot \overline{F_u(e^{i\theta})} \cdot e^{i\theta}\, d\theta. \tag{2.7}$$

But

$$S_u(\zeta^v) = \sigma_{-v} = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} u(e^{i\theta}) \cdot e^{iv\theta}\, d\theta, \tag{2.8}$$

and therefore

$$\overline{F_u(e^{i\theta})} = -i \cdot u(e^{i\theta}) \cdot e^{-i\theta}, \tag{2.9}$$

which implies that

$$S_u(g) = \int_{-\pi}^{\pi} g(e^{i\theta}) \cdot u(e^{i\theta})\, d\theta \tag{2.10}$$

for all $g \in L^2_{\mathbb{R}}(C)$. In view of Corollary 2.2, we have thus obtained the following theorem.

THEOREM 2.3. *Let $M = (\pi_{m,k})_{m\geq 0, 0\leq k\leq m}$ be an infinite triangular interpolation matrix with complex entries and, for any $m \geq 0$, let $G_m(x,z)$ be the unique polynomial of degree at most m which interpolates the function $(1 - x \cdot z)^{-1}$ at $x = \pi_{m,0}, \pi_{m,1}, \pi_{m,2}, \ldots, \pi_{m,m}$ (where z is fixed and $|\pi_{m,k}| < 1$).*

(a) *For any real-valued function $u \in L^2_{\mathbb{R}}(C)$, the Padé-type approximant $\mathrm{Re}(m/(m+1))_u(z)$ to $u(z)$ has the following integral representation:*

$$\mathrm{Re}\left(\frac{m}{m+1}\right)_u (z) = \frac{1}{2\pi} \cdot \int_C u(\zeta)\, \mathrm{Re}\left\{\frac{4\pi \bar{G}_m(\zeta, z) - 1}{i\zeta}\right\} d\zeta \quad (|z| = 1), \tag{2.11}$$

or equivalently

$$\begin{aligned}\operatorname{Re}\left(\frac{m}{m+1}\right)_u(e^{it}) &= \int_{-\pi}^{\pi} u(e^{i\theta})\cdot 2\operatorname{Re}\left\{\bar{G}_m(e^{i\theta},e^{it}) - \frac{1}{4\pi}\right\} d\theta \\ &= \frac{1}{2\pi}\cdot\int_{-\pi}^{\pi} u(e^{i\theta})\cdot\operatorname{Re}\left\{4\pi\cdot\bar{G}_m(e^{i\theta},e^{it}) - 1\right\} d\theta,\end{aligned} \tag{2.12}$$

where $-\pi \le t \le \pi$.

(b) *Let* $f \in L^2[-\pi,\pi]$ *be a* 2π*-periodic real-valued function, with Fourier coefficients* $\{c_v : v = \pm 0, \pm 1, \pm 2, \ldots\}$. *Since* $f(t) = \sum_{v=-\infty}^{\infty} c_v \cdot e^{ivt}$ *in the* L^2*-norm, the function* $f(t)$ *can be viewed as a function of the unit circle, and therefore the Padé-type approximant* $\operatorname{Re}(m/(m+1))_f(t)$ *to* $f(t)$ *has the following integral representation:*

$$\begin{aligned}\operatorname{Re}\left(\frac{m}{m+1}\right)_f(t) &= \int_{-\pi}^{\pi} f(\theta)\cdot 2\operatorname{Re}\left\{\bar{G}_m(e^{i\theta},e^{it}) - \frac{1}{4\pi}\right\} d\theta \\ &= \frac{1}{2\pi}\cdot\int_{-\pi}^{\pi} f(\theta)\cdot\operatorname{Re}\left\{4\pi\cdot\bar{G}_m(e^{i\theta},e^{it}) - 1\right\} d\theta \quad (-\pi \le t \le \pi).\end{aligned} \tag{2.13}$$

Proof. We have

$$\begin{aligned}\operatorname{Re}\left(\frac{m}{m+1}\right)_u(e^{it}) &= 2\operatorname{Re} T_u\big(G_m(x,e^{it})\big) - u(0) \\ &= 2\operatorname{Re} S_u\big(\bar{G}_m(x,e^{it})\big) - u(0) \\ &= 2\operatorname{Re}\int_{-\pi}^{\pi} \bar{G}_m(e^{i\theta},e^{it})u(e^{i\theta})\,d\theta - \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{i\theta})\,d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{i\theta})\operatorname{Re}\left[4\pi\bar{G}_m(e^{i\theta},e^{it})\right] d\theta \\ &\quad - \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{i\theta})\,d\theta \\ &= \frac{1}{2\pi}\cdot\int_{-\pi}^{\pi} u(e^{i\theta})\cdot\operatorname{Re}\left\{4\pi\bar{G}_m(e^{i\theta},e^{it}) - 1\right\} d\theta \\ &= \int_{-\pi}^{\pi} u(e^{i\theta})\cdot 2\operatorname{Re}\left\{\bar{G}_m(e^{i\theta},e^{it}) - \frac{1}{4\pi}\right\} d\theta \quad (-\pi \le t \le \pi).\end{aligned} \tag{2.14}$$

Setting $z = e^{it}$ and $\zeta = e^{i\theta}$, we also have

$$\begin{aligned}
\operatorname{Re}\left(\frac{m}{m+1}\right)_u(z) &= \operatorname{Re}\left(\frac{m}{m+1}\right)_u(e^{it}) \\
&= \operatorname{Re}\left[\frac{1}{2\pi i}\cdot\int_{-\pi}^{\pi} u(e^{i\theta})\left\{\frac{4\pi\bar{G}_m(e^{i\theta},e^{it})-1}{e^{i\theta}}\right\} ie^{i\theta}\,d\theta\right] \\
&= \operatorname{Re}\left[\frac{1}{2\pi i}\cdot\int_C u(\zeta)\left\{\frac{4\pi\bar{G}_m(\zeta,z)-1}{\zeta}\right\}d\zeta\right] \\
&= \frac{1}{2\pi}\cdot\int_C u(\zeta)\operatorname{Re}\left\{\frac{4\pi\bar{G}_m(\zeta,z)-1}{i\zeta}\right\}d\zeta \quad (|z|=1).
\end{aligned} \tag{2.15}$$

This completes the proof of (a). The proof of (b) is exactly similar and is based on the identification between $L^2_{\mathbb{R}}(C)$ and the space of all 2π-periodic L^2 real-valued functions on $[-\pi,\pi]$ (every $u(z) = u(e^{it}) \in L^2_{\mathbb{R}}(C)$ can be interpreted as a 2π-periodic real-valued function $f(t) \in L^2[-\pi,\pi]$ and conversely). $\square$

In order to simplify the formalism, we make use of the notation

$$\begin{aligned}
\operatorname{Re}\left\{\frac{B_m(\zeta,z)}{i\zeta}\right\} &:= \operatorname{Re}\left\{\frac{4\pi\cdot\bar{G}_m(\zeta,z)-1}{i\zeta}\right\}, \\
\operatorname{Re} B_m(e^{i\theta},e^{it}) &:= \operatorname{Re}\left\{4\pi\cdot\bar{G}_m(e^{i\theta},e^{it})-1\right\}.
\end{aligned} \tag{2.16}$$

As it is well known, the function $\operatorname{Re}(m/(m+1))_u(z)$ $(|z|=1)$ is continuous (see [6]). Hence, the integral operator $\operatorname{Re}(m/(m+1))$ maps $L^2_{\mathbb{R}}(C)$ into $L^2_{\mathbb{R}}(C)$ and therefore, by the closed graph theorem, it is continuous (of course, under the assumption that $|\pi_{m,k}| < 1$ for all $k \le m$). The integral operator

$$\begin{aligned}
&\operatorname{Re}\left(\frac{m}{m+1}\right) : L^2_{\mathbb{R}}(C) \longrightarrow L^2_{\mathbb{R}}(C); \\
&u(z) \longrightarrow \operatorname{Re}\left(\frac{m}{m+1}\right)_u(z) = \frac{1}{2\pi}\cdot\int_C u(\zeta)\cdot\operatorname{Re}\frac{B_m(\zeta,z)}{i\zeta}\,d\zeta
\end{aligned} \tag{2.17}$$

is called the Padé-type operator for $L^2_{\mathbb{R}}(C)$. Its adjoint operator is given by

$$\begin{aligned}
&\operatorname{Re}\left(\frac{m}{m+1}\right)^* : L^2_{\mathbb{R}}(C) \longrightarrow L^2_{\mathbb{R}}(C); \\
&u(z) \longrightarrow \operatorname{Re}\left(\frac{m}{m+1}\right)^*_u(z) = \frac{1}{2\pi}\cdot\int_C u(\zeta)\cdot\operatorname{Re}\frac{B_m(z,\zeta)}{iz}\,d\zeta.
\end{aligned} \tag{2.18}$$

In fact, to $\mathrm{Re}(m/(m+1))$ there corresponds a unique operator $\mathrm{Re}(m/(m+1))^* : L^2_{\mathbb{R}}(C) \to L^2_{\mathbb{R}}(C)$ satisfying $\langle \mathrm{Re}(m/(m+1))_u, w\rangle = \langle u, \mathrm{Re}(m/(m+1))^*_w\rangle$, that is,

$$\int_C \mathrm{Re}\left(\frac{m}{m+1}\right)_u(\zeta)\cdot w(\zeta)\,d\zeta = \int_C u(z)\cdot \mathrm{Re}\left(\frac{m}{m+1}\right)^*_w(z)\,dz \tag{2.19}$$

for all $u,w \in L^2_{\mathbb{R}}(C)$; since, by Fubini's theorem,

$$\begin{aligned}\int_C \mathrm{Re}\left(\frac{m}{m+1}\right)_u(\zeta)\cdot w(\zeta)\,d\zeta\\ &= \int_C \frac{1}{2\pi}\int_C u(z)\cdot \mathrm{Re}\,\frac{B_m(z,\zeta)}{iz}\,dz w(\zeta)\,d\zeta\\ &= \int_C u(z)\cdot\left(\frac{1}{2\pi}\int_C w(\zeta)\cdot \mathrm{Re}\,\frac{B_m(z,\zeta)}{iz}\,d\zeta\right)dz,\end{aligned} \tag{2.20}$$

we conclude that

$$\mathrm{Re}\left(\frac{m}{m+1}\right)^*_w(z) = \frac{1}{2\pi}\int_C w(\zeta)\cdot \mathrm{Re}\,\frac{B_m(z,\zeta)}{iz}\,d\zeta \quad \left(w \in L^2_{\mathbb{R}}(C)\right). \tag{2.21}$$

Similarly, as it is pointed out in [6], for any real-valued 2π-periodic function $f \in L^2[-\pi,\pi]$, the Padé-type approximant $\mathrm{Re}(m/(m+1))_f(t)$ is continuous, and, by construction, 2π-periodic. It follows that the integral operator $\mathrm{Re}(m/(m+1))$ maps the space $L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi,\pi]$ of real-valued 2π-periodic functions of $L^2[-\pi,\pi]$ into itself. Hence, by the closed graph theorem, the operator

$$\begin{gathered}\mathrm{Re}\left(\frac{m}{m+1}\right) : L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi,\pi] \longrightarrow L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi,\pi];\\ f(t) \longrightarrow \mathrm{Re}\left(\frac{m}{m+1}\right)_f(t) = \frac{1}{2\pi}\cdot\int_{-\pi}^{\pi} f(\theta)\cdot \mathrm{Re}\,B_m\left(e^{i\theta},e^{it}\right)d\theta\end{gathered} \tag{2.22}$$

is continuous and is called the *Padé-type operator* for $L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi,\pi]$. Its adjoint operator is then given by

$$\begin{gathered}\mathrm{Re}\left(\frac{m}{m+1}\right)^* : L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi,\pi] \longrightarrow L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi,\pi];\\ f(t) \longrightarrow \mathrm{Re}\left(\frac{m}{m+1}\right)^*_f(t) = \frac{1}{2\pi}\cdot\int_{-\pi}^{\pi} f(\theta)\cdot \mathrm{Re}\,B_m\left(e^{it},e^{i\theta}\right)d\theta.\end{gathered} \tag{2.23}$$

In fact, to $\operatorname{Re}(m/(m+1))$ we associate the unique operator $\operatorname{Re}(m/(m+1))^* : L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi,\pi] \to L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi,\pi]$ satisfying

$$\left\langle \operatorname{Re}\left(\frac{m}{m+1}\right)_f, g\right\rangle = \left\langle f, \operatorname{Re}\left(\frac{m}{m+1}\right)^*_g\right\rangle, \tag{2.24}$$

that is,

$$\int_{-\pi}^{\pi} \operatorname{Re}\left(\frac{m}{m+1}\right)_f(t)\cdot g(t)\,dt = \int_{-\pi}^{\pi} f(\theta)\cdot \operatorname{Re}\left(\frac{m}{m+1}\right)^*_g(\theta)\,d\theta \tag{2.25}$$

for all $f,g \in L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi,\pi]$; it follows, from Fubini's theorem, that

$$\begin{aligned}\int_{-\pi}^{\pi} &\operatorname{Re}\left(\frac{m}{m+1}\right)_f(t)\cdot g(t)\,dt \\ &= \int_{-\pi}^{\pi} \frac{1}{2\pi}\cdot\int_{-\pi}^{\pi} f(\theta)\cdot \operatorname{Re} B_m(e^{i\theta},e^{it})\,d\theta g(t)\,dt \\ &= \int_{-\pi}^{\pi} f(\theta)\cdot\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\cdot \operatorname{Re} B_m(e^{i\theta},e^{it})\,dt\right)d\theta,\end{aligned} \tag{2.26}$$

and consequently

$$\begin{aligned}\operatorname{Re}&\left(\frac{m}{m+1}\right)^*_g(\theta) \\ &= \frac{1}{2\pi}\cdot\int_{-\pi}^{\pi} g(t)\cdot \operatorname{Re} B_m(e^{i\theta},e^{it})\,dt \quad \left(g \in L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi,\pi]\right).\end{aligned} \tag{2.27}$$

Summarizing, we have the following theorem.

Theorem 2.4. *If $m \geq 0$, then for any $u(z) \in L^2_{\mathbb{R}}(C)$ and any $f(t) \in L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi,\pi]$, there holds*

$$\begin{aligned}\operatorname{Re}\left(\frac{m}{m+1}\right)^*_u(z) &= \frac{1}{2\pi}\cdot\int_C u(\zeta)\cdot \operatorname{Re}\frac{B_m(z,\zeta)}{iz}\,d\zeta, \\ \operatorname{Re}\left(\frac{m}{m+1}\right)^*_f(t) &= \frac{1}{2\pi}\cdot\int_{-\pi}^{\pi} f(\theta)\cdot \operatorname{Re} B_m(e^{it},e^{i\theta})\,d\theta.\end{aligned} \tag{2.28}$$

The continuity of the Padé-type operators $\operatorname{Re}(m/(m+1))$ leads immediately to the following convergence results which can be considered as a first example of their application.

Theorem 2.5. (a) *If the sequence* $\{u_n \in L^2_{\mathbb{R}}(C) : n = 0,1,2,\ldots\}$ *converges to* $u \in L^2_{\mathbb{R}}(C)$ *in the* L^2*-norm, then there holds* $\lim_{n\to\infty} \operatorname{Re}(m/(m+1))_{u_n}(z) = \operatorname{Re}(m/(m+1))_u(z)$ *in the* L^2*-norm.*

(b) *If the sequence* $\{f_n \in L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi,\pi] : n = 0,1,2,\ldots\}$ *converges to* $f \in L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi,\pi]$ *in the* L^2*-norm, then there holds* $\lim_{n\to\infty}\operatorname{Re}(m/(m+1))_{f_n}(t) = \operatorname{Re}(m/(m+1))_f(t)$ *in the* L^2*-norm.*

Corollary 2.6. (a) *If the series of functions* $u(z) = \sum_{n=0}^{\infty} a_n \cdot u_n(z)$ *(where* $a_n \in \mathbb{R}$, $u_n \in L^2_{\mathbb{R}}(C)$*) converges in the* L^2*-norm, then* $\operatorname{Re}(m/(m+1))_u(z) = \sum_{n=0}^{\infty} a_n \operatorname{Re}(m/(m+1))u_n(z)$ *in the* L^2*-norm.*

(b) *If the series of functions* $f(t) = \sum_{n=0}^{\infty} a_n \cdot f_n(t)$ *(where* $a_n \in \mathbb{R}$, $f_n \in L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi,\pi]$*) converges in the* L^2*-norm then* $\operatorname{Re}(m/(m+1))_f(t) = \sum_{n=0}^{\infty} a_n \operatorname{Re}(m/(m+1))_{f_n}(t)$ *in the* L^2*-norm.*

Now we determine the conditions under which the integral operator $\operatorname{Re}(m/(m+1))$ is compact onto $L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi,\pi]$. Since, for each fixed $t \in [-\pi,\pi]$, the kernel function $\operatorname{Re} B_m(e^{i\theta}, e^{it})$ is bounded in θ, it follows, from Tonelli's theorem, that the following theorem holds true.

Theorem 2.7. *If there is a constant* $c_* < \infty$ *such that*

$$\int_{-\pi}^{\pi} \left|\operatorname{Re} B_m(e^{i\theta}, e^{it})\right|^2 d\theta \le (2\pi)^2 \cdot c_* \tag{2.29}$$

for almost all $t \in [-\pi,\pi]$*, then the Padé-type operator* $\operatorname{Re}(m/(m+1)) : L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi,\pi] \to L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi,\pi]$ *is compact. Moreover,*

$$\left\| \operatorname{Re}\left(\frac{m}{m+1}\right) \right\| \le (2\pi)^{5/2} \cdot c_* \tag{2.30}$$

and $\operatorname{Re}(m/(m+1))^*$ *is also compact.*

It is readily seen that if the Padé-type operator $\operatorname{Re}(m/(m+1)) : L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi,\pi] \to L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi,\pi]$ is compact, then it is not one-to-one. This follows from the fact that $\dim L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi,\pi] = \infty$ and therefore 0 must be an eigenvalue of $\operatorname{Re}(m/(m+1))$. However, it would be interesting to know a necessary and sufficient condition under which for any $h \in L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi,\pi]$ there is an $f \in L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi,\pi]$ such that $\operatorname{Re}(m/(m+1))_f = h$. Of course, such a general condition is the inequality

$$\left\| \mathrm{Re}\left(\frac{m}{m+1}\right)^{*}_{f} \right\|_2 \geq c \cdot \|f\|_2 \tag{2.31}$$

or alternatively,

$$\int_{-\pi}^{\pi} |f(t)|^2 \, dt \leq c \cdot \int_{-\pi}^{\pi} \left| \int_{-\pi}^{\pi} f(\theta) \cdot \mathrm{Re}\, B_m(e^{it}, e^{i\theta}) \, d\theta \right|^2 dt \tag{2.32}$$

for some constant $c > 0$ and for every $f \in L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi, \pi]$. Obviously, this inequality is true if and only if

$$|f(t)| \leq c \cdot \left| \int_{-\pi}^{\pi} f(\theta) \cdot \mathrm{Re}\, B_m(e^{it}, e^{i\theta}) \, d\theta \right| \tag{2.33}$$

for almost all $t \in [-\pi, \pi]$, and thus we have proved the following theorem describing a sufficient condition under which every function in $L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi, \pi]$ is a Padé-type approximant.

Theorem 2.8. *If there is a constant $c > 0$ such that*

$$|f(t)| \leq c \cdot \left| \int_{-\pi}^{\pi} f(\theta) \cdot \mathrm{Re}\, B_m(e^{it}, e^{i\theta}) \, d\theta \right| \tag{2.34}$$

almost everywhere on $[-\pi, \pi]$, for every $f \in L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi, \pi]$, then the range of $\mathrm{Re}(m/(m+1))$ *equals $L^2_{\mathbb{R},(2\pi\text{-per})}[-\pi, \pi]$.*

Finally, we turn to integral representation formulas in the harmonic case. If u is harmonic and real-valued in the unit disk, then, for any $0 \leq r < 1$, the restriction $u_r(t) = u(r \cdot e^{it})$ $(-\pi \leq t \leq \pi)$ of $u(z)$ to the circle of radius r can be interpreted as a real-valued, 2π-periodic function in $L^2[-\pi, \pi]$. According to Theorem 2.3, the Padé-type approximant $\mathrm{Re}(m/(m+1))_{u_r}(t)$ to $u_r(t)$ is given by the integral representation formula

$$\begin{aligned} \mathrm{Re}\left(\frac{m}{m+1}\right)_{u_r}(t) &= \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} u_r(\theta) \cdot \mathrm{Re}\left\{4\pi \cdot \bar{G}_m(r \cdot e^{i\theta}, r \cdot e^{it}) - 1\right\} d\theta \\ &= \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} u_r(r \cdot e^{i\theta}) \cdot \mathrm{Re}\left\{4\pi \cdot \bar{G}_m(r \cdot e^{i\theta}, r \cdot e^{it}) - 1\right\} d\theta. \end{aligned} \tag{2.35}$$

After a simple change of variables $z = r \cdot e^{it}$ and $\zeta = r \cdot e^{i\theta}$, we obtain

$$\begin{aligned}\operatorname{Re}\left(\frac{m}{m+1}\right)_u(z) &= \frac{1}{2\pi} \cdot \int_{|\zeta|=r} u(\zeta) \cdot \operatorname{Re}\left\{\frac{4\pi \cdot \bar{G}_m(\zeta, z) - 1}{\zeta i}\right\} d\zeta \\ &= \frac{1}{2i} \cdot \int_{|\zeta|=r} u(\zeta) \cdot \operatorname{Re}\left\{\frac{B_m(\zeta, z)}{\zeta i}\right\} d\zeta,\end{aligned} \tag{2.36}$$

and hence we can state the following theorem.

THEOREM 2.9. *Let $M = (\pi_{m,k})_{m \geq 0, 0 \leq k \leq m}$ be an infinite triangular interpolation matrix with complex entries and, for any $m \geq 0$, let $G_m(x,z)$ be the unique polynomial of degree at most m which interpolates the function $(1 - x \cdot z)^{-1}$ at $x = \pi_{m,0}, \pi_{m,1}, \pi_{m,2}, \ldots, \pi_{m,m}$ (where z is fixed and $|\pi_{m,k}| < 1$ for each $k \leq m$).*

The Padé-type approximant $\operatorname{Re}(m/(m+1))_u(z)$ to the harmonic real-valued function $u(z)$ in the disk is given by the following integral representation formula:

$$\operatorname{Re}\left(\frac{m}{m+1}\right)_u(z) = \frac{1}{2\pi} \cdot \int_{|\zeta|=|z|} u(\zeta) \cdot \operatorname{Re}\left\{\frac{B_m(\zeta, z)}{i\zeta}\right\} d\zeta \quad (z \in D). \tag{2.37}$$

As it is mentioned in [5], the function $\operatorname{Re}(m/(m+1))_u(z)$ is the real part of an analytic function in the unit disk, and therefore, it is a harmonic real-valued function in D (of course, under the assumption that $|\pi_{m,k}| < 1$ for all $k \leq m$). If $H_{\mathbb{R}}(D)$ is the space of all harmonic real-valued functions in D, the integral operator

$$\begin{gathered}\operatorname{Re}\left(\frac{m}{m+1}\right) : H_{\mathbb{R}}(D) \longrightarrow H_{\mathbb{R}}(D); \\ u(z) \longrightarrow \operatorname{Re}\left(\frac{m}{m+1}\right)_u(z) = \frac{1}{2\pi} \cdot \int_{|\zeta|=|z|} u(\zeta) \cdot \operatorname{Re}\left\{\frac{B_m(\zeta, z)}{i\zeta}\right\} d\zeta\end{gathered} \tag{2.38}$$

is said to be a *Padé-type operator* of $H_{\mathbb{R}}(D)$. It is easily seen that a Padé-type operator of $H_{\mathbb{R}}(D)$ is continuous. For, if $\{u_n \in H_{\mathbb{R}}(D) : n = 0, 1, 2, \ldots\}$ and if $\lim_{n\to\infty} u_n = u \in H_{\mathbb{R}}(D)$ compactly in the disk D, then, by the maximum principle for harmonic functions, we have

$$\begin{aligned}
&\sup_{|z|\le r}\left|\operatorname{Re}\left(\frac{m}{m+1}\right)_{u_n}(z)-\operatorname{Re}\left(\frac{m}{m+1}\right)_{u}(z)\right| \\
&\quad=\sup_{|z|=r}\left|\operatorname{Re}\left(\frac{m}{m+1}\right)_{u_n}(z)-\operatorname{Re}\left(\frac{m}{m+1}\right)_{u}(z)\right| \\
&\quad=\frac{1}{2\pi}\cdot\sup_{|z|=r}\left|\int_{|\zeta|=r}\left[u_n(\zeta)-u(\zeta)\right]\cdot\operatorname{Re}\left\{\frac{B_m(\zeta,z)}{\zeta i}\right\}d\zeta\right| \qquad (2.39)\\
&\quad\le\frac{1}{2\pi r}\cdot 2\pi r\cdot\left\{\sup_{|z|=r,|\zeta|=r}\left|\operatorname{Re}B_m(\zeta,z)\right|\right\}\cdot\left\{\sup_{|\zeta|=r}\left|u_n(\zeta)-u(\zeta)\right|\right\} \\
&\quad\le L(r,m)\cdot\left\{\sup_{|\zeta|=r}\left|u_n(\zeta)-u(\zeta)\right|\right\}
\end{aligned}$$

for any $r<1$, and the continuity of $\operatorname{Re}(m/(m+1)) : H_{\mathbb{R}}(D)\to H_{\mathbb{R}}(D)$ follows.

As for the L^2-case, the continuity of the Padé-type operator for $H_{\mathbb{R}}(D)$ leads to the following convergence results.

Theorem 2.10. *If the sequence $\{u_n : n=0,1,2,\ldots\}$ of harmonic real-valued functions in the open unit disk converges compactly to $u\in H_{\mathbb{R}}(D)$, then there holds*

$$\lim_{n\to\infty}\operatorname{Re}\left(\frac{m}{m+1}\right)_{u_n}(z)=\operatorname{Re}\left(\frac{m}{m+1}\right)_{u}(z) \qquad (2.40)$$

compactly in D.

Corollary 2.11. *If the series of harmonic real-valued functions*

$$u(z)=\sum_{n=0}^{\infty}a_n\cdot u_n(z)\quad\left(a_n\in\mathbb{R},\ u_n\in H_{\mathbb{R}}(D)\right) \qquad (2.41)$$

converges compactly in the disk, then

$$\operatorname{Re}\left(\frac{m}{m+1}\right)_{u}(z)=\sum_{n=0}^{\infty}a_n\operatorname{Re}\left(\frac{m}{m+1}\right)_{u_n}(z), \qquad (2.42)$$

the convergence of the series being compact in D.

Remark 2.12. In [2], Brezinski showed that the (Hermite) interpolation polynomial $G_m(x,z)$ of $(1-xz)^{-1}$ at $x=\pi_{m,0},\pi_{m,1},\dots,\pi_{m,m}$ is given by

$$G_m(x,z)=\frac{1}{1-x\cdot z}\cdot\left(1-\frac{\upsilon_{m+1}(x)}{\upsilon_{m+1}(z^{-1})}\right)\quad\left(z\neq\pi_{m,k}^{-1},\ k=0,1,\dots,m\right),\tag{2.43}$$

where $\upsilon_{m+1}(x)$ is any generating polynomial $\upsilon_{m+1}(x)=\gamma\cdot\prod_{k=0}^{m}(x-\pi_{m,k})$ $(\gamma\neq 0)$. We thus obtain the following expressions for the kernels $\operatorname{Re}\{B_m(\zeta,z)/\zeta i\}$ and $\operatorname{Re}B_m(e^{i\theta},e^{it})$:

$$\begin{aligned}\operatorname{Re}\left\{\frac{B_m(\zeta,z)}{\zeta i}\right\}&=\operatorname{Re}\left\{\frac{-4i\zeta^{-1}}{1-\zeta\cdot\bar{z}}\left(1-\bar{z}^{m+1}\cdot\prod_{k=0}^{m}\frac{\zeta-\overline{\pi_{m,k}}}{1-\overline{z\cdot\pi_{m,k}}}\right)-\zeta^{-1}\right\},\\ \operatorname{Re}B_m(e^{i\theta},e^{it})&=\operatorname{Re}\left\{\frac{4\pi}{1-e^{i(\theta-t)}}\left(1-\prod_{k=0}^{m}\frac{e^{i\theta}-\overline{\pi_{m,k}}}{e^{it}-\overline{\pi_{m,k}}}\right)-1\right\}.\end{aligned}\tag{2.44}$$

If, for example, $\pi_{m,0}=\cdots=\pi_{m,m}=0$, then for any $u\in L^2_{\mathbb{R}}(C)$, we have

$$\begin{aligned}&\operatorname{Re}\left(\frac{m}{m+1}\right)_u(z)\\&\quad=\operatorname{Re}\left\{\frac{2}{\pi i}\cdot\sum_{v=0}^{m}\bar{z}^{v}\int_C u(\zeta)\cdot\zeta^{v-1}\,d\zeta-\frac{2}{\pi i}\cdot\int_C u(\zeta)\cdot\zeta^{-1}\,d\zeta\right\}\quad(z\in C)\end{aligned}\tag{2.45}$$

or

$$\begin{aligned}&\operatorname{Re}\left(\frac{m}{m+1}\right)_u(e^{it})\\&\quad=2\cdot\int_{-\pi}^{\pi}u(e^{i\theta})\cos(\theta-t)\,d\theta-2\cdot\int_{-\pi}^{\pi}u(e^{i\theta})\cos\left[m-(\theta-t)\right]d\theta\\&\quad=4\cdot\int_{-\pi}^{\pi}u(e^{i\theta})\cdot\sin\left[\frac{(m+1)\theta-(m+1)t}{2}\right]\\&\qquad\qquad\cdot\sin\left[\frac{(m-1)\theta-(m-1)t}{2}\right]d\theta\quad(-\pi\leq t\leq\pi).\end{aligned}\tag{2.46}$$

3. Integral representations and composed Padé-type approximation

We are now in a position to generalize the definitions and results of Section 2 to the context of composed Padé-type approximation. Recall that a composed Padé-type approximant to a harmonic complex-valued function $u=u_1+iu_2$ in the disk D (resp., to an L^p complex-valued function $u = u_1 + iu_2$ on the circle C or to a 2π-periodic complex-valued function $f = f_1 + if_2 \in L^p[-\pi,\pi]$) is a coordinate approximant given by the formula

$$\left(\frac{m}{m+1}\right)_u(z) = \mathrm{Re}\left(\frac{m_1}{m_1+1}\right)_{u_1}(z) + i\,\mathrm{Re}\left(\frac{m_2}{m_2+1}\right)_{u_2}(z) \quad (z \in D), \tag{3.1}$$

respectively, by the formula

$$\left(\frac{m}{m+1}\right)_u(z) = \mathrm{Re}\left(\frac{m_1}{m_1+1}\right)_{u_1}(z) + i\,\mathrm{Re}\left(\frac{m_2}{m_2+1}\right)_{u_2}(z) \quad (z \in C) \tag{3.2}$$

or

$$\left(\frac{m}{m+1}\right)_f(t) = \mathrm{Re}\left(\frac{m_1}{m_1+1}\right)_{f_1}(z) + i\,\mathrm{Re}\left(\frac{m_2}{m_2+1}\right)_{f_2}(t), \tag{3.3}$$

where $-\pi \le t \le \pi$ (see [7]).

Set

$$\begin{aligned} L^p_{\mathbb{C}}(C) &:= \{u \in L^p(C) : u \text{ is complex-valued}\}, \\ L^p_{\mathbb{C},(2\pi\text{-per})}[-\pi,\pi] &:= \{f \in L^p[-\pi,\pi] : f \text{ is complex-valued and} \\ &\qquad 2\pi\text{-periodic } (f(-\pi) = f(\pi))\}, \\ H_{\mathbb{C}}(D) &:= \{u : D \longrightarrow \mathbb{C} : u \text{ is harmonic and complex-valued}\}. \end{aligned} \tag{3.4}$$

From Theorems 2.3 and 2.9, the following theorem follows immediately.

Theorem 3.1. *For $j = 1,2$, let $M^{(j)} = (\pi^{(j)}_{m_j,k})_{m_j \ge 0, 0 \le k \le m_j}$ be an infinite triangular interpolation matrix with complex entries $\pi^{(j)}_{m_j,k} \in D$, and, for any $m_j \ge 0$, let $G^{(j)}_{m_j}(x,z)$ be the unique polynomial of degree at most m_j which interpolates the function $(1-xz)^{-1}$ at $x = \pi^{(j)}_{m_j,0}, \pi^{(j)}_{m_j,1}, \dots, \pi^{(j)}_{m_j,m_j}$ (where z is regarded as a parameter).*

If $G^{(j)}_{m_j}(x,z) = \sum_{v=0}^{m_j} g_v^{(j,m_j)}(z) \cdot x^v$, denote by $\overline{G^{(j)}_{m_j}}(x,z)$ the polynomial

$$\sum_{v=0}^{m_j} \overline{g_v^{(j,m_j)}(z)} \cdot x^v. \tag{3.5}$$

Put

$$B^{(j)}_{m_j}(x,z) = 4\pi \cdot \overline{G^{(j)}_{m_j}}(x,z) - 1. \tag{3.6}$$

(a) *For any* $u = u_1 + i \cdot u_2 \in L^2_{\mathbb{C}}(C)$, *the corresponding composed Padé-type approximant* $(m/(m+1))_u(z)$ *to* $u(z)$ *has the following integral representation*

$$\left(\frac{m}{m+1}\right)_u(z) = \frac{1}{2\pi} \cdot \int_C \left\{ u_1(\zeta) \cdot \operatorname{Re}\left[\frac{B^{(1)}_{m_1}(\zeta,z)}{\zeta i}\right] + i \cdot u_2(z) \cdot \operatorname{Re}\left[\frac{B^{(2)}_{m_2}(\zeta,z)}{i\zeta}\right]\right\} d\zeta \quad (|z| = 1), \tag{3.7}$$

or equivalently

$$\left(\frac{m}{m+1}\right)_u(e^{it}) = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} \left\{ u_1(e^{i\theta}) \cdot \operatorname{Re} B^{(1)}_{m_1}(e^{i\theta}, e^{it}) + i \cdot u_2(e^{i\theta}) \cdot \operatorname{Re} B^{(2)}_{m_2}(e^{i\theta}, e^{it}) \right\} d\theta \quad (-\pi \le t \le \pi). \tag{3.8}$$

(b) *For any* $f = f_1 + i \cdot f_2 \in L^2_{\mathbb{C},(2\pi\text{-per})}[-\pi,\pi]$, *the corresponding composed Padé-type approximant* $(m/(m+1))_f(t)$ *to* $f(t)$ *has the following integral representation:*

$$\left(\frac{m}{m+1}\right)_f(t) = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} \left\{ f_1(\theta) \cdot \operatorname{Re} B^{(1)}_{m_1}(e^{i\theta}, e^{it}) + i \cdot f_2(\theta) \cdot \operatorname{Re} B^{(2)}_{m_2}(e^{i\theta}, e^{it}) \right\} d\theta \quad (-\pi \le t \le \pi). \tag{3.9}$$

(c) *For any* $u = u_1 + i \cdot u_2 \in H_{\mathbb{C}}(D)$, *the corresponding composed Padé-type approximant* $(m/(m+1))_u(z)$ *to* $u(z)$ *has the following integral representation:*

$$\left(\frac{m}{m+1}\right)_u(z) = \frac{1}{2\pi} \cdot \int_{|\zeta|=|z|} \left\{ u_1(z) \cdot \operatorname{Re}\left[\frac{B^{(1)}_{m_1}(\zeta,z)}{\zeta i}\right] + i \cdot u_2(\zeta) \cdot \operatorname{Re}\left[\frac{B^{(2)}_{m_2}(\zeta,z)}{\zeta i}\right]\right\} d\zeta \quad (|z| < 1). \tag{3.10}$$

In particular, since any Padé-type approximant in the ordinary sense is a composed Padé-type approximant, we can give integral representation for the classical Padé-type approximants to analytic functions.

Corollary 3.2. *Let* $M = (\pi_{m,k})_{m\ge 0, 0\le k\le m}$ *be an infinite triangular interpolation matrix with complex entries* $\pi_{m,k} \in D$, *and, for any* $m \ge 0$, *let* $G_m(x,z)$

be the unique polynomial of degree at most m *which interpolates the function* $(1-xz)^{-1}$ *at* $x=\pi_{m,0},\pi_{m,1},\dots,\pi_{m,m}$ *(*z *is regarded as a parameter).*

If $G_m(x,z)=\sum_{v=0}^{m} g_v^{(m)}(z)\cdot x^v$, *denote by* $\bar{G}_m(x,z)$ *the polynomial* $\sum_{v=0}^{m}\overline{g_v^{(m)}(z)}\cdot x^v$, *and put*

$$B_m(x,z)=4\pi\cdot\bar{G}_m(x,z)-1. \tag{3.11}$$

For any $f\in O(D)$, *the corresponding Padé-type approximant* $(m/(m+1))_f(z)$ *to* $f(z)$ *(in the Brezinski's sense of* [1]*) has the following integral representation:*

$$\left(\frac{m}{m+1}\right)_f(z)=\frac{1}{2\pi}\cdot\int_{|\zeta|=|z|} f(\zeta)\cdot\operatorname{Re}\left[\frac{B_m(\zeta,z)}{\zeta i}\right]d\zeta \quad (|z|<1). \tag{3.12}$$

Under the assumptions of Theorem 3.1, the integral operators

$$\begin{aligned}
&\left(\frac{m}{m+1}\right): L^2_{\mathbb{C}}(C)\longrightarrow L^2_{\mathbb{C}}(C);\\
&\qquad u=u_1+iu_2\longmapsto\left(\frac{m}{m+1}\right)_u(z)\\
&\qquad =\frac{1}{2i}\cdot\int_C\left\{u_1(\zeta)\cdot\operatorname{Re}\left[\frac{B^{(1)}_{m_1}(\zeta,z)}{\zeta i}\right]+i\cdot u_2(\zeta)\cdot\operatorname{Re}\left[\frac{B^{(2)}_{m_2}(\zeta,z)}{\zeta i}\right]\right\}d\zeta,\\
&\left(\frac{m}{m+1}\right): L^2_{\mathbb{C},(2\pi\text{-per})}[-\pi,\pi]\longrightarrow L^2_{\mathbb{C},(2\pi\text{-per})}[-\pi,\pi];\\
&\qquad f=f_1+i\cdot f_2\longmapsto\left(\frac{m}{m+1}\right)_f(t)\\
&\qquad =\frac{1}{2\pi}\cdot\int_{-\pi}\left\{f_1(\theta)\cdot\operatorname{Re}B^{(1)}_{m_1}(e^{i\theta},e^{it})+i\cdot f_2(\theta)\cdot\operatorname{Re}B^{(2)}_{m_2}(e^{i\theta},e^{it})\right\}d\theta,\\
&\left(\frac{m}{m+1}\right): H_{\mathbb{C}}(D)\longrightarrow H_{\mathbb{C}}(D);\\
&\qquad u=u_1+iu_2\longmapsto\left(\frac{m}{m+1}\right)_u(z)\\
&\qquad =\frac{1}{2\pi}\cdot\int_{|\zeta|=|z|}\left\{u_1(\zeta)\cdot\operatorname{Re}\left[\frac{B^{(1)}_{m_1}(\zeta,z)}{\zeta i}\right]\right.\\
&\qquad\qquad\left.+i\cdot u_2(\zeta)\cdot\operatorname{Re}\left[\frac{B^{(2)}_{m_2}(\zeta,z)}{\zeta i}\right]\right\}d\zeta
\end{aligned} \tag{3.13}$$

are called *composed Padé-type operators* for $L^2_{\mathbb{C}}$, $L^2_{\mathbb{C},(2\pi\text{-per})}[-\pi,\pi]$, and $H_{\mathbb{C}}(D)$, respectively. Under the assumptions of Corollary 3.2, the integral operator

$$\left(\frac{m}{m+1}\right) : O(D) \longrightarrow O(D);$$
$$f \longmapsto \left(\frac{m}{m+1}\right)_f(z) = \frac{1}{2\pi}\cdot\int_{|\zeta|=|z|} f(\zeta)\cdot \operatorname{Re}\left[\frac{B_m(\zeta,z)}{\zeta i}\right] d\zeta \tag{3.14}$$

is called a *Padé-type operator* for $O(D)$.

The continuity of these integral operators follows directly from the arguments of Section 2 and leads to the following result.

Theorem 3.3. *Under the assumptions and notations of Theorem 3.1 and Corollary 3.2,*

(a) *if the sequence $\{u_n \in L^2_{\mathbb{C}}(C) : n = 0,1,2,\ldots\}$ converges to $u \in L^2_{\mathbb{C}}(C)$ in the L^2-norm, then $\lim_{n\to\infty}(m/(m+1))_{u_n}(z) = (m/(m+1))_u(z)$ in the L^2-norm;*

(b) *if the sequence $\{f_n \in L^2_{\mathbb{C},(2\pi\text{-per})}[-\pi,\pi] : n = 0,1,2,\ldots\}$ converges to $f \in L^2_{\mathbb{C},(2\pi\text{-per})}[-\pi,\pi]$, with respect to the L^2-norm, then $\lim_{n\to\infty}(m/(m+1))_{f_n}(t) = (m/(m+1))_f(t)$ in the L^2-norm;*

(c) *if the sequence $\{u_n \in H_{\mathbb{C}}(D) : n = 0,1,2,\ldots\}$ converges to $u \in H_{\mathbb{C}}(D)$ compactly in D, then $\lim_{n\to\infty}(m/(m+1))_{u_n}(z) = (m/(m+1))_u(z)$ compactly in D;*

(d) *if the sequence $\{f_n \in O(D) : n = 0,1,2,\ldots\}$ converges to $f \in O(D)$ compactly in D, then $\lim_{n\to\infty}(m/(m+1))_{f_n}(z) = (m/(m+1))_f(z)$ compactly in D.*

Especially, for series of functions, there is an obvious consequence of this theorem.

Corollary 3.4. *Under the assumptions of Theorem 3.1 and Corollary 3.2,*

(a) *if the series of functions $u(z) = \sum_{n=0}^{\infty} a_n \cdot u_n(z)$ ($a_n \in \mathbb{C}$, $u_n \in L^2_{\mathbb{C}}(C)$) converges in the L^2-norm, then $(m/(m+1))_u(z) = \sum_{n=0}^{\infty} a_n\cdot(m/(m+1))_{u_n}(z)$ in the L^2-norm;*

(b) *if the series of functions $f(t) = \sum_{n=0}^{\infty} a_n \cdot f_n(t)$ (where $a_n \in \mathbb{C}$, $f_n \in L^2_{\mathbb{C},(2\pi\text{-per})}[-\pi,\pi]$) converges in the L^2-norm, then $(m/(m+1))_f(t)\sum_{n=0}^{\infty} a_n\cdot(m/(m+1))_{f_n}(t)$ in the L^2-norm;*

(c) *if the series of functions $u(z) = \sum_{n=0}^{\infty} a_n \cdot u_n(z)$ ($a_n \in \mathbb{C}$, $u_n \in H_{\mathbb{C}}(D)$) converges compactly in the disk D, then $(m/(m+1))_u(z) = \sum_{n=0}^{\infty} a_n\cdot(m/(m+1))_{u_n}(z)$ compactly in D;*

(d) *if the series of analytic functions* $f(z) = \sum_{n=0}^{\infty} a_n \cdot f_n(z)$ $(a_n \in \mathbb{C}$, $f_n O(D))$ *converges compactly in* D*, then* $(m/(m+1))_f(z) = \sum_{n=0}^{\infty} a_n \cdot (m/(m+1))_{f_n}(z)$ *compactly in* D.

Remark 3.5. Padé and Padé-type approximants to arbitrary series of functions were first considered by Brezinski in [1, 2].

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