ON THE RATIO OF PEARSON TYPE VII AND BESSEL RANDOM VARIABLES

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The exact distribution of the ratio $|X/Y|$ is derived when X and Y are, respectively, Pearson type VII and Bessel function random variables distributed independently of each other. The work is motivated by previously published approximate relationships between these two distributions. An application of the result is provided by computing "correction factors" for some of these approximations.

1. Introduction

For given random variables X and Y, the distribution of the ratio X/Y is of interest in biological and physical sciences, econometrics, and ranking and selection. Examples include Mendelian inheritance ratios in genetics, mass to energy ratios in nuclear physics, target to control precipitation in meteorology, and inventory ratios in economics. Another important example is the stress-strength model in the context of reliability. It describes the life of a component which has a random strength Y and is subjected to random stress X. The component fails at the instant that the stress applied to it exceeds the strength and the component will function satisfactorily whenever $Y > X$. Thus, $\Pr(X < Y)$ is a measure of a component reliability. It has many applications especially in engineering concepts such as structures, deterioration of rocket motors, static fatigue of ceramic components, fatigue failure of aircraft structures and aging of concrete pressure vessels.

The distribution of X/Y has been studied by several authors especially when X and Y are independent random variables and come from the same family. For instance, see Marsaglia [14] and Korhonen and Narula [9] for normal family, Press [18] for Student's t family, Basu and Lochner [1] for Weibull family, Shcolnick [23] for stable family, Hawkins and Han [5] for non-central chi-squared family, Provost [19] for gamma family, and Pham-Gia [17] for beta family. However, there is relatively little work of this kind when X and Y belong to different families. In the applications mentioned above, it is indeed quite possible that X and Y could arise from different but similar distributions.

Pearson type VII distributions (which contain Student's t distributions as particular cases) are becoming of increasing importance in classical as well as in Bayesian statistical

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modeling. These distributions have been perhaps unjustly overshadowed—for at least seventy years—by the normal distribution. Pearson type VII distributions are of central importance in statistical inference. Their applications are a very promising path to take. Classical analysis is soundly bend on the normal distribution while Pearson type VII distributions (in particular, Student's t distributions) offer a more viable alternative with respect to real-world data particularly because its tails are more realistic. We have seen unexpected applications in novel areas such as cluster analysis, discriminant analysis, multiple regression, robust projection indices and missing data imputation.

Pearson type VII distributions (in particular, Student's t distributions) for the past fifty years have also played a crucial role in Bayesian analysis. They serve as the most popular prior distribution (because elicitation of prior information in various physical, engineering and financial phenomena is closely associated with Student's t distributions) and generate meaningful posterior distributions. For further discussion of applications, the reader is referred to Kotz and Nadarajah [11].

On the other hand, Bessel function distributions (which contain logistic distributions as particular cases) have found applications in a variety of areas that range from image and speech recognition and ocean engineering to finance. They are rapidly becoming distributions of first choice whenever "something" with heavier than Gaussian tails is observed in the data. For further discussion of applications, the reader is referred to Kotz et al. [10].

There has been considerable work on the relationship between the Student's t and logistic distributions (which are particular cases of Pearson type VII and Bessel function distributions, resp.). The similarities in the shapes of the logistic and the normal distributions have been noted by several authors. An excellent summary of these properties is found in Johnson and Kotz [7]. However, Mudholkar and George [15] showed that the Student's t distribution function with 9 degrees of freedom, when standardized to have variance one provides a better fit of a standardized logistic distribution than the standard normal. George and Ojo [3] and George et al. [2] extended this result by showing that

$$\Pr(T \leq t) \approx \frac{1}{1+\exp(-at)}, \tag{1.1}$$

where T is a Student's t random variable with ν degrees of freedom and $a = \pi\sqrt{(\nu-2)/(3\nu)}$. This approximation (1.1) was obtained by equating the cumulant of the Student's t distribution with that of the logistic distribution. The approximation was found to be accurate to two decimal places for middle values of t and to three decimal places at the tails. A better approximation between the Student's t and logistic distributions has been recently proposed by Li and De Moor [13].

The above discussion naturally raises the important question: what is the exact distribution of the ratio of Student's t and logistic random variables? This question does not appear to have been addressed in the literature. In this paper, we discuss the more general problem: the exact distribution of the ratio $|X/Y|$ when X and Y are independent random variables having the Pearson type VII and Bessel function distributions with the pdfs

$$f(x) = \frac{\Gamma(M - 1/2)}{\sqrt{N\pi}\Gamma(M - 1)} \left(1 + \frac{x^2}{N}\right)^{1/2-M}, \tag{1.2}$$

$$f(y) = \frac{|y|^m}{\sqrt{\pi}2^m b^{m+1}\Gamma(m + 1/2)} K_m\left(\left|\frac{y}{b}\right|\right), \tag{1.3}$$

respectively, for $-\infty < x < \infty$, $-\infty < y < \infty$, $b > 0$, $m > 1$, $M > 1$ and $N > 0$, where

$$K_m(x) = \frac{\sqrt{\pi}x^m}{2^m\Gamma(m + 1/2)} \int_1^\infty (t^2 - 1)^{m-1/2} \exp(-xt)dt \tag{1.4}$$

is the modified Bessel function of the second kind. We also provide an application section and compute “correction factors” for the approximation provided by (1.1).

The Pearson type VII distribution is related to the Student’s t distribution as follows: if $M = 1 + a/2$ and

$$U = \sqrt{\frac{a}{N}}X \tag{1.5}$$

then U is a Student’s t random variable with degrees of freedom a. Note that the pdf of a Student’s t random variable with degrees of freedom ν is given by

$$f(x) = \frac{1}{\sqrt{\nu}B(\nu/2, 1/2)} \left(1 + \frac{x^2}{\nu}\right)^{-(1+\nu)/2} \tag{1.6}$$

for $-\infty < x < \infty$. Nadarajah and Kotz [16] have shown that the cdf corresponding to (1.6) can be expressed as

$$F(x) = \begin{cases} \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{x}{\sqrt{\nu}}\right) + \frac{1}{2\pi}\sum_{l=1}^{(\nu-1)/2} B\left(l, \frac{1}{2}\right)\frac{\nu^{l-1/2}x}{(\nu + x^2)^l}, & \text{if } \nu \text{ is odd,} \\ \frac{1}{2} + \frac{1}{2\pi}\sum_{l=1}^{\nu/2} B\left(l - \frac{1}{2}, \frac{1}{2}\right)\frac{\nu^{l-1}x}{(\nu + x^2)^{l-1/2}}, & \text{if } \nu \text{ is even.} \end{cases} \tag{1.7}$$

The representations in (1.5) and (1.7) will be crucial for the calculations of this note. The calculations involve the Euler psi function defined by

$$\Psi(x) = \frac{d\log\Gamma(x)}{dx}, \tag{1.8}$$

the Struve function defined by

$$H_\nu(x) = \frac{2x^{\nu+1}}{\sqrt{\pi}2^{\nu+1}\Gamma(\nu+3/2)} \sum_{k=0}^{\infty} \frac{1}{(3/2)_k(\nu+3/2)_k} \left(-\frac{x^2}{4}\right)^k, \tag{1.9}$$

the Bessel function of the first kind defined by

$$J_\nu(x) = \frac{x^\nu}{2^\nu\Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{1}{(\nu+1)_k k!} \left(-\frac{x^2}{4}\right)^k, \tag{1.10}$$

the hypergeometric functions defined by

$$E(a;x) = \sum_{k=0}^{\infty} \frac{1}{(a)_k} \frac{x^k}{k!}, \qquad H(a;b,c;x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k(c)_k} \frac{x^k}{k!} \tag{1.11}$$

and, the Lommel functions defined by

$$\begin{aligned} s_{\mu,\nu}(x) &= \frac{x^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)} H\left(1; \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}; -\frac{x^2}{4}\right), \\ S_{\mu,\nu}(x) &= s_{\mu,\nu}(x) + \frac{2^{\mu+\nu-1}\Gamma(\nu)\Gamma((\mu+\nu+1)/2)}{x^\nu\Gamma((1+\nu-\mu)/2)} E\left(1-\nu; -\frac{x^2}{4}\right) \\ &\quad + \frac{2^{\mu-\nu-1}x^\nu\Gamma(-\nu)\Gamma((\mu-\nu+1)/2)}{\Gamma((1-\nu-\mu)/2)} E\left(1+\nu; -\frac{x^2}{4}\right), \end{aligned} \tag{1.12}$$

where $(e)_k = e(e+1)\cdots(e+k-1)$ denotes the ascending factorial. We also need the following important lemma.

Lemma 1.1 (Prudnikov et al. [20, 21, equation (2.16.3.14), volume 2]). *For $c > 0$, $z > 0$ and $\nu > -1$,*

$$\int_0^\infty \frac{x^{\nu+1}}{(x^2+z^2)^\rho} K_\nu(cx)dx = (2z)^\nu (z/c)^{1-\rho}\Gamma(\nu+1)S_{-\nu-\rho,1+\nu-\rho}(cz). \tag{1.13}$$

Further properties of the above special functions can be found in Prudnikov et al. [20, 21] and Gradshteyn and Ryzhik [4].

2. Cdf

Theorem 2.1 derives an explicit expression for the cdf of $|X/Y|$ in terms of the Lommel function.

Theorem 2.1. *Suppose X and Y are distributed according to (1.2) and (1.3), respectively. If $a = 2(M-1)$ is an odd integer then the cdf of $Z = |X/Y|$ can be expressed as*

$$F(z) = I(a) + \frac{2a^{m/2}\Gamma(m+1)}{\pi^{3/2}b^{m+1}\Gamma(m+1/2)r^m} \sum_{k=1}^{(a-1)/2} \frac{a^{k/2}B(k,1/2)}{b^{1-k}r^k} S_{-(m+k),1+m-k}\left(\frac{\sqrt{a}}{br}\right), \tag{2.1}$$

where $I(\cdot)$ denotes the integral

$$I(a) = \frac{1}{\pi^{3/2}2^{m-2}b^{m+1}\Gamma(m+1/2)} \int_0^\infty \arctan\left(\frac{ry}{\sqrt{a}}\right) y^m K_m\left(\frac{y}{b}\right) dy, \tag{2.2}$$

and $r = \sqrt{a/N}z$.

Proof. Using the relationship (1.5), one can write the cdf as $\Pr(|X/Y| \le z) = \Pr(|U/Y| \le r)$, which can be expressed as

$$\begin{aligned} F(r) &= \frac{1}{\sqrt{\pi}2^m b^{m+1}\Gamma(m+1/2)} \int_{-\infty}^\infty \{F(r|y|) - F(-r|y|)\} |y|^m K_m\left(\left|\frac{y}{b}\right|\right) dy \\ &= \frac{1}{\sqrt{\pi}2^{m-1}b^{m+1}\Gamma(m+1/2)} \int_0^\infty \{F(ry) - F(-ry)\} y^m K_m\left(\frac{y}{b}\right) dy, \end{aligned} \tag{2.3}$$

where $F(\cdot)$ inside the integrals denotes the cdf of a Student's t random variable with degrees of freedom a. Substituting the form for F given by (1.7) for odd degrees of freedom, (2.3) can be reduced to

$$F(r) = I(a) + \frac{2r}{\pi^{3/2}\sqrt{a}2^m b^{m+1}\Gamma(m+1/2)} \sum_{k=1}^{(a-1)/2} a^k r^{-2k} B\left(k, \frac{1}{2}\right) J(k), \tag{2.4}$$

where $J(k)$ denotes the integral

$$J(k) = \int_0^\infty \frac{y^{m+1}K_m(y/b)}{(y^2 + a/r^2)^k} dy. \tag{2.5}$$

By direct application of Lemma 1.1, one can calculate (2.5) as

$$J(k) = 2^m m! a^{(m-k+1)/2} b^{k-1} r^{k-m-1} S_{-(m+k),1+m-k}\left(\frac{\sqrt{a}}{br}\right). \tag{2.6}$$

The result of the theorem follows by substituting (2.6) into (2.4). □

Theorem 2.2 is the analogue of Theorem 2.1 for the case when the degrees of freedom $2(M-1)$ is an even integer.

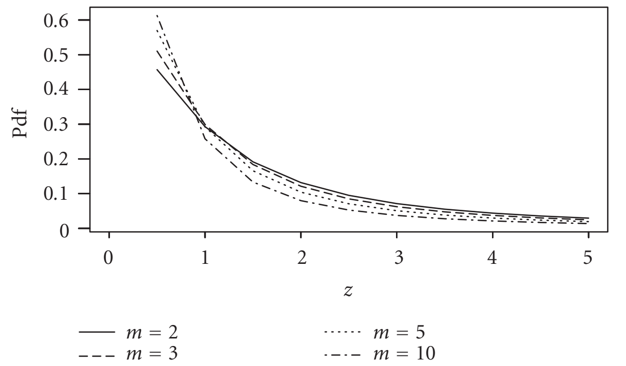

Figure 2.1. Plots of the pdf of (2.1) and (2.7) for $M = 6$, $N = 1$ and $m = 2,3,5,10$.

Theorem 2.2. *Suppose X and Y are distributed according to (1.2) and (1.3), respectively. If $a = 2(M-1)$ is an even integer then the cdf of $Z = |X/Y|$ can be expressed as*

$$F(z) = \frac{2a^{m/2}\Gamma(m+1)}{\pi^{3/2}b^{m+1}\Gamma(m+1/2)r^m}\sum_{k=1}^{a/2}\frac{a^{k/2}B(k-1/2,1/2)}{b^{3/2-k}r^k}S_{1/2-m-k,3/2+m-k}\left(\frac{\sqrt{a}}{br}\right), \tag{2.7}$$

where $r = \sqrt{a/N}z$.

Proof. Substituting the form for F given by (1.7) for odd degrees of freedom, (2.3) can be reduced to

$$F(r) = \frac{2r}{\pi^{3/2}\sqrt{a}2^m b^{m+1}\Gamma(m+1/2)}\sum_{k=1}^{a/2}a^{k-1/2}r^{1-2k}B\left(k-\frac{1}{2},\frac{1}{2}\right)J(k), \tag{2.8}$$

where $J(k)$ denotes the integral

$$J(k) = \int_0^\infty \frac{y^{m+1}K_m(y/b)}{\left(y^2+a/r^2\right)^{k-1/2}}dy. \tag{2.9}$$

By direct application of Lemma 1.1, one can calculate (2.9) as

$$J(k) = 2^m m! a^{(m-k+3/2)/2}b^{k-3/2}r^{k-m-3/2}S_{1/2-m-k,3/2+m-k}\left(\frac{\sqrt{a}}{br}\right). \tag{2.10}$$

The result of the theorem follows by substituting (2.10) into (2.8). □

Figure 2.1 illustrates possible shapes of the pdf of $|X/Y|$ for $M = 6$, $N = 1$ and a range of values of m. Note that the shapes are unimodal and that the value of m largely dictates the behavior of the pdf near $z = 0$.

Table 2.1 provides percentage points of the random variable $Z = |X/Y|$, where X is Student's t random variable and Y is the logistic random variable (see the next section).

Table 2.1. Percentage points of $Z = |X/Y|$, where X is Student's t and Y is logistic.

ν	$p = 0.9$	$p = 0.95$	$p = 0.975$	$p = 0.99$	$p = 0.995$	$p = 0.999$
3	5.562765	11.38354	22.97968	58.01899	116.9898	590.5066
4	6.262781	12.71831	25.40326	64.11429	126.6723	631.2136
5	6.571752	13.28467	26.69873	66.59717	134.2463	683.6856
6	6.713695	13.57901	27.17702	68.55331	137.8122	661.8825
7	6.785755	13.71471	27.67064	69.98242	139.2899	706.187
8	6.851048	13.84042	27.70334	69.84954	137.9891	718.6892
9	6.890786	13.90781	27.73966	69.55903	137.6793	699.746
10	6.896549	14.00395	28.24329	70.84583	142.5509	725.8325
11	6.95357	14.02722	28.11031	69.5793	137.904	696.0801
12	6.972723	14.09180	28.12042	69.88332	137.7253	672.2092
13	6.98896	14.14476	28.17688	70.4751	141.4761	693.2014
14	7.009541	14.11228	28.33336	71.51355	140.8694	705.5214
15	7.037168	14.21723	28.54814	70.52729	140.4317	672.6095
16	7.016891	14.15984	28.35549	71.11684	143.1923	749.5653
17	7.036384	14.22758	28.58657	71.57068	141.4651	688.2397
18	6.997855	14.16037	28.31962	70.2939	141.4549	705.8531
19	7.013366	14.24295	28.55525	71.99197	144.5075	760.897
20	7.05649	14.23309	28.37539	71.53259	142.1804	711.128
21	7.079385	14.23683	28.82255	71.48645	140.3166	698.9362
22	7.068098	14.29088	28.87514	71.84303	144.8472	720.8385
23	7.065508	14.24519	28.45713	71.72072	142.1739	707.4103
24	7.08719	14.26229	28.42829	71.54304	144.2094	744.5381
25	7.088949	14.22109	28.56216	71.27478	141.8147	700.2782
26	7.075956	14.22041	28.37632	69.83216	138.6226	714.1018
27	7.061162	14.19190	28.37419	71.1976	140.5687	766.0238
28	7.103002	14.28380	28.55252	72.33449	144.1237	715.3463
29	7.087898	14.35654	28.80674	72.79021	144.8118	713.6059
30	7.07424	14.2707	28.63380	71.4855	141.3878	698.7528
31	7.089486	14.25152	28.65639	72.30395	145.2743	741.7811
32	7.06058	14.19111	28.56882	72.30109	146.1088	691.8094
33	7.137295	14.41838	28.77831	71.79008	143.2821	708.5934
34	7.09389	14.28261	28.41822	70.6438	139.7121	716.6218
35	7.085203	14.25356	28.53685	71.72688	143.7028	682.4963
36	7.109764	14.32743	28.82298	71.98	143.0642	729.9266
37	7.103148	14.32635	28.89781	72.73917	144.9194	720.3259
38	7.119519	14.37951	29.07110	72.8409	147.2216	729.187
39	7.099915	14.36532	28.65924	71.41699	144.1128	748.3013
40	7.097136	14.24074	28.39764	70.28753	139.6711	709.3462
41	7.121472	14.36992	28.91728	71.5884	145.9546	687.5026
42	7.122682	14.34184	28.83802	72.30898	143.8715	708.7313

Table 2.1. Continued.

ν	$p = 0.9$	$p = 0.95$	$p = 0.975$	$p = 0.99$	$p = 0.995$	$p = 0.999$
43	7.136523	14.31962	28.82072	72.56791	145.7568	726.197
44	7.073623	14.31739	28.59732	70.96285	143.9753	684.8288
45	7.098625	14.29619	28.63336	70.971	141.1438	771.8231
46	7.076823	14.29069	28.68125	71.09681	141.0267	691.0396
47	7.133042	14.37075	28.72546	72.23762	144.6101	741.7482
48	7.111953	14.33463	28.78489	71.15746	141.5647	746.4949
49	7.109182	14.25147	28.80318	72.11692	142.8293	700.7086
50	7.119438	14.31630	28.71464	72.46818	144.9791	702.7882

3. Application

In this section, we provide "correction factors" for the approximation (1.1). These factors are computed as the percentage points z_p associated with the cdfs (2.1) and (2.7) when X is a Student's t random variable with degrees of freedom ν and Y is a logistic random variable with the scale parameter $\pi\sqrt{(\nu-2)/(3\nu)}$. Evidently, this involves computation of the Lommel and Bessel functions. Fortunately routines for these calculations are widely available. We used the functions LommelS1$(\cdot)$, LommelS2$(\cdot)$, and BesselK$(\cdot)$ in the algebraic manipulation package, MAPLE. Table 2.1 provides the numerical values of z_p for $\nu = 3, 4, \ldots, 50$ and $p = 0.9, 0.95, 0.975, 0.99, 0.995, 0.999$. We hope these numbers will be of use to the practitioners of the approximation (1.1). Similar tabulations could be easily derived for other values of ν by using the LommelS1$(\cdot)$, LommelS2$(\cdot)$, and BesselK$(\cdot)$ functions in MAPLE.

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