

ANALYZING LINEAR RECURSIVE PROJECTS AS AN ABSORBING CHAIN

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Hardie (2001) used general Markov chains to study the linear “recursive projects” in which some activities may be revisited after a later activity is completed. In this paper, we propose that it is better to treat the project as an absorbing chain. This allows us to calculate the expected value and the variance of the project duration.

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1. Introduction

Project management is usually calculated as a set of static activities, whereas in reality these activities tend to be dynamic involving multiple feedbacks (Park and Peña-Mora [5]). This results in many late project completions. Recently research has been done to formulate and analyze various methods to improve project prediction and control. These include concurrent engineering (Carter and Baker [1]) and stage-gate systems (Cooper [2]) in which projects must meet specifications before they pass from one stage to another. The activities within a project are governed by constraints as well as precedence relationships in traditional network scheduling like CPM (critical path method, Dupont Inc. and Remington Rand, 1958), PDM (procedure diagramming method, IBM Co., 1964) and PERT (program evaluation and review technology, US Navy, Booz-Allen Hamilton and Lockheed Co. 1958).

Normally, in a CPM/PERT project, we assume that the activities are carried out only once. Hardie [3] observed that this is not true in many projects, in which some activities may have to be revisited after a later activity is completed. He called such models the “recursive networks.” For example, consider a project having the following five activities: (1) specification, (2) design, (3) test and integration, (4) manufacture and (5) marketing. We assume that they are listed in the predecessor order; that is, an activity

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cannot start until all activities listed before it have been completed. It is more realistic to assume that the project is recursive, in that there is a non-zero probability that the project is sent back to design (2) after test and integration (3). In this paper, we restrict ourselves only to linear projects; that is, those having no branches. We will present a method to obtain the expected value and the variance of the duration of a linear recursive project.

A linear recursive project can be a “super-activity” inside a larger PERT project. In this case, our method allows us to calculate accurately the mean and the variance of this super-activity, rather than approximating them by the Beta assumption or any other approximating formulas.

2. Methodology

Consider a project with N activities, labeled $1, 2, \dots, N$. Let the time be divided into discrete steps, starting at step 0. Hardie [3] assumes that, if activity i is being carried out at step n , there is a probability p_{ij} ($j < i$) that an early activity j is revisited at step $n + 1$, a probability p_{ii} that activity i is performed again, and a probability $p_{i,i+1}$ that the project advances to the next activity $i + 1$. ($\sum_{j=1}^{i+1} p_{ij} = 1$).

The total duration of activity i thus has a geometric distribution with mean $1/(1 - p_{ii})$.

Similar to Hardie [3], we analyze the system as a Markov chain. The transition matrix representing the chain is of the form

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & \dots & N & E \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ \dots \\ N \\ E \end{matrix} & \left(\begin{array}{cccccccc} p_{11} & p_{12} & 0 & 0 & \dots & 0 & 0 \\ p_{21} & p_{22} & p_{23} & 0 & \dots & 0 & 0 \\ p_{31} & p_{32} & p_{33} & p_{34} & \dots & 0 & 0 \\ p_{41} & p_{42} & p_{43} & p_{44} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ p_{N1} & p_{N2} & p_{N3} & p_{N4} & \dots & p_{NN} & p_{NE} \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{array} \right), \end{matrix} \quad (2.1)$$

where E denotes the end of the project.

Hardie [3] notes that raising the transition matrix \mathbf{P} to the power of n will give the probability that activity i is performed at time n . Especially, the probability that the project is finished by the time n is the $(1, E)$ element of the matrix \mathbf{P}^n . In this paper we recognize this project as an “absorbing chain.” This allows many important results related to this kind of chain be adapted to the current model (see Minh [4]).

First we write the transition matrix (2.1) in the following form:

$$\mathbf{P} = \begin{pmatrix} \mathbf{T} & \mathbf{S} \\ 0 & 1 \end{pmatrix}, \quad (2.2)$$

where

$$\mathbf{T} = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & \cdots & N \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ \cdots \\ N \end{matrix} & \begin{pmatrix} p_{11} & p_{12} & 0 & 0 & \cdots & 0 \\ p_{21} & p_{22} & p_{23} & 0 & \cdots & 0 \\ p_{31} & p_{32} & p_{33} & p_{34} & \cdots & 0 \\ p_{41} & p_{42} & p_{42} & p_{44} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{N1} & p_{N2} & p_{N3} & p_{N4} & \cdots & p_{NN} \end{pmatrix} \end{matrix}, \quad \mathbf{S} = \begin{matrix} & E \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ \cdots \\ N \end{matrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \cdots \\ p_{NE} \end{pmatrix} \end{matrix}. \quad (2.3)$$

Standard results in the study of absorbing chains would immediately yield the following results.

- (1) The expected total duration the project spends in activity i is the $(1, i)$ element of the following “fundamental matrix”:

$$\mathbf{U} = (\mathbf{I} - \mathbf{T})^{-1}. \quad (2.4)$$

- (2) Let D be the project duration. Its expected value $E[D]$ is the sum of the expected durations the project spends in each activity; that is, the sum of the first row of the fundamental matrix \mathbf{U} .
- (3) The second moment of the project duration $E[D^2]$ is the first element of column $(2\mathbf{U} - \mathbf{I})\mathbf{U}\mathbf{e}$, where \mathbf{I} is an identity matrix, and \mathbf{e} a column in which all elements are 1.

3. Examples

We now present two examples—the first can be calculated numerically and the second analytically.

In the first example, consider a project A having 4 activities X, Y, Z, T and the following transition matrix (in days):

$$\mathbf{P}_A = \begin{matrix} & \begin{matrix} X & Y & Z & T & E \end{matrix} \\ \begin{matrix} X \\ Y \\ Z \\ T \\ E \end{matrix} & \begin{pmatrix} .1 & .9 & 0 & 0 & 0 \\ .2 & .1 & .7 & 0 & 0 \\ .1 & 0 & .3 & .6 & 0 \\ 0 & .1 & .1 & .4 & .4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}. \quad (3.1)$$

Hardie [3] shows that the probability that project A reaches the end E after day 5 is .28728, which is the $(1, E)$ element of $(\mathbf{P}_A)^5$.

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Now treating project A as an absorbing chain, we write

$$\mathbf{T}_A = \begin{matrix} & X & Y & Z & T \\ \begin{matrix} X \\ Y \\ Z \\ T \end{matrix} & \begin{pmatrix} .1 & .9 & 0 & 0 \\ .2 & .1 & .7 & 0 \\ .1 & 0 & .3 & .6 \\ 0 & .1 & .1 & .4 \end{pmatrix}, & \mathbf{S}_A = \begin{matrix} & E \\ \begin{matrix} X \\ Y \\ Z \\ T \end{matrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ .4 \end{pmatrix}. \end{matrix} \quad (3.2)$$

The fundamental matrix for this chain is

$$\mathbf{U}_A = \begin{pmatrix} 1 & -.1 & -.9 & 0 & 0 \\ -.2 & 1 & -.1 & -.7 & 0 \\ -.1 & 0 & 1 & -.3 & -.6 \\ 0 & -.1 & -.1 & 1 & -.4 \end{pmatrix}^{-1} \quad (3.3)$$

$$= \begin{matrix} & X & Y & Z & T \\ \begin{matrix} X \\ Y \\ Z \\ T \end{matrix} & \begin{pmatrix} 1.8651 & 2.1429 & 2.5 & 2.5 \\ .75397 & 2.1429 & 2.5 & 2.5 \\ .43651 & .71429 & 2.5 & 2.5 \\ .19841 & .47619 & .83333 & 2.5 \end{pmatrix}. \end{matrix}$$

This means that the expected total duration that project A spends in activity Y is 2.1429 days, which is the (X, Y) element of \mathbf{U}_A . Also, the expected duration of project A is

$$E[D] = 1.8651 + 2.1429 + 2.5 + 2.5 = 9.008 \text{ days.} \quad (3.4)$$

Furthermore, because

$$(2\mathbf{U} - \mathbf{I})\mathbf{U}\mathbf{e} = \begin{pmatrix} 109.23 \\ 90.324 \\ 63.789 \\ 37.379 \end{pmatrix} \quad (3.5)$$

we have $E[D^2] = 109.23$. The variance of the project duration is thus

$$\text{Var}[D] = E[D^2] - E^2[D] = 109.23 - 9.008^2 = 28.086. \quad (3.6)$$

In the second example, we assume that the project can only revert to the beginning of the project (Back to the drawing board). We write the transition matrix in the form

$$\mathbf{P}_B = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & \cdots & N & E \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ \cdots \\ N \\ E \end{matrix} & \left(\begin{array}{cccccccc} q_1 & s_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ q_2 & r_2 & s_2 & 0 & \cdots & 0 & 0 & 0 \\ q_3 & 0 & r_3 & s_3 & \cdots & 0 & 0 & 0 \\ q_4 & 0 & 0 & r_4 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots \\ q_N & 0 & 0 & 0 & \cdots & r_N & s_N & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{array} \right), \end{matrix} \quad (3.7)$$

where $q_i + r_i + s_i = 1$ for all $1 \leq i \leq N$.

It can be proved by induction that the closed-form for $E[D]$ is

$$E[D] = \frac{1}{s_N} \left(\frac{R_2 R_3 \cdots R_N}{s_1 s_2 \cdots s_{N-1}} + \frac{R_3 \cdots R_N}{s_2 \cdots s_{N-1}} + \cdots + \frac{R_{N-1} R_N}{s_{N-2} s_{N-1}} + \frac{R_N}{s_{N-1}} + 1 \right), \quad (3.8)$$

where $R_i = 1 - r_i$ for all $2 \leq i \leq N$. Especially, when $q_i = q$, $r_i = r$ and $s_i = s$ for all $1 \leq i \leq N$, we obtain

$$E[D] = \frac{1}{s} (\gamma^{N-1} + \gamma^{N-2} + \cdots + \gamma + 1) = \frac{1}{s} \left(\frac{1 - \gamma^N}{1 - \gamma} \right), \quad (3.9)$$

where $\gamma = (1 - r)/s$.

This confirms that the expected project duration is longer when the reversion probability q is larger, and/or the probability s of advancing to the next activity is smaller.

4. Conclusion

In this paper, we show that by considering a linear recursive project as an absorbing chain, we can calculate the expected value and the variance of its total duration. These results can give a project manager better insight into planning the overall project management.

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