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Research Article

Pricing Exotic Options under a High-Order Markovian Regime Switching Model

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We consider the pricing of exotic options when the price dynamics of the underlying risky asset are governed by a discrete-time Markovian regime-switching process driven by an observable, high-order Markov model (HOMM). We assume that the market interest rate, the drift, and the volatility of the underlying risky asset's return switch over time according to the states of the HOMM, which are interpreted as the states of an economy. We will then employ the well-known tool in actuarial science, namely, the Esscher transform to determine an equivalent martingale measure for option valuation. Moreover, we will also investigate the impact of the high-order effect of the states of the economy on the prices of some path-dependent exotic options, such as Asian options, lookback options, and barrier options.

1. Introduction

Regime switching models are important models in econometrics and finance. They have received much attention among academic researchers and practitioners in modeling economic and financial time series. The origin of this important class of models goes back to the seminal work of Hamilton [1] in which a class of Markovian regime-switching autoregressive time series models was first introduced to explain the US business cycle. This class of models has received much attention among (financial) econometricians and various extensions to the model have been introduced in the literature, such as Markovian regime switching ARCH-type models and their variants by Cai [2], Hamilton and Susmel [3], Gray [4], and Klaassen [5]. In finance, regime-switching models are often used to incorporate the switching of model parameters, such as the market interest rates of a bank account and the volatility of equity returns, due to the structural change in macro-economic factors and business cycles. The switching behaviors of market interest rates and volatility of equity returns are well documented in some empirical finance literature. Ang and Bekaert [6] investigate the performance of regime-switching models in fitting interest rate data from United States, Germany, and United Kingdom (see also Ang and Bekaert [7]). They found that regime-switching models have better out-of-sample forecasts than models without switching regimes. They also found that the switching of regimes in interest rates match well with business cycles in the United States. Ang and Bekaert [8] investigate the performance of regime-switching models in fitting equity returns from the United States, Germany, and the United Kingdom and found empirically that the regime-switching effect in the model parameters, such as volatility of equity returns, is significant. Some empirical studies including Schwert [9] and Kim et al. [10] found that the regime-switching effect is present in monthly stock returns and that a Markovian regime-switching specification is appropriate for modeling the monthly stock return volatility.

Recently, the spotlight has turned to the valuation of options under regime-switching models. Some works in this area include Naik [11], Guo [12], Buffington and Elliott [13, 14] and Elliott et al. [15], and others. Most of the literature concerns the pricing of options under a continuous-time Markov-modulated process. However, there is not much work on the valuation of options under a discrete-time Markov-modulated framework. The advantage of a discrete-time framework is its flexibility to incorporate more features in the model, such as the high-order effect in the underlying Markov chain for the model parameters. Incorporating the high-order effect in the underlying Markov chain provides more flexibility in modeling the temporal behavior of the states of an economy and its impact on asset price dynamics. The impact of such a high-order effect on the behavior of option prices is not well explored in the literature. The development of option pricing model with the high-order effect incorporated contributes to the literature by not only advancing the option pricing technology via providing a flexible model, but also helping us to gain a better understanding on the behavior of option prices under the flexible setting.

In this paper, we consider the pricing of exotic options when the price dynamics of the underlying risky asset are governed by a discrete-time Markovian regime-switching process driven by an observable, high-order Markov model (HOMM). The discrete-time framework provides a natural and intuitive way to incorporate the high-order effect in the underlying Markov chain. We assume that the market interest rates of a bank account, the drift, and the volatility of the underlying risky asset's return switch over time according to one of the states of the HOMM. We do not contend that the model we considered is the same as those regime-switching time series models that are ready to fit real interest rates data and volatility of stock returns. However, our model does extract the main feature of those models, namely, the regime-switching effect, and provides a generalization to incorporate the high-order effect. Our goal is to investigate the impact of such a high-order regime-switching effect on the behavior of prices of exotic options, which, we believe, has not been well explored in the literature. Here, we interpret the states of the HOMM as the states of an economy. We will employ the well-known tool in actuarial science, namely, the Esscher transform to determine an equivalent martingale measure for option valuation in the incomplete market setting. We will investigate the impact of the high-order effect of the economic states on the prices of some path-dependent exotic options, such as Asian options, lookback options, and barrier options.

The rest of the paper is organized as follows. In Section 2, we present the Markov-modulated process with the HOMM for modeling the price dynamics of the underlying risky asset. We will illustrate the use of the Esscher transform to determine an equivalent martingale measure for option valuation in Section 3. Section 4 conducts some simulation experiments and investigates the impact of the high-order effect of the economic states on the option prices. Finally, concluding remarks are given in Section 5.

2. Asset price dynamics by the HOMM

In this section, we present a Markovian regime-switching process driven by an observable, high-order Markov chain (HOMM) for modeling asset price dynamics. First, we consider a discrete-time economy with two primary-traded assets, namely, a bank account and a share. Let $\mathcal{T}$ be the time index set $\{0,1,\dots\}$ of the economy. We fix a complete probability space $(\Omega,\mathcal{F},\mathcal{P})$, where $\mathcal{P}$ is a real-world probability. We suppose that the uncertainties due to the fluctuations of market prices and the economic states are described by the probability space $(\Omega,\mathcal{F},\mathcal{P})$. In the sequel, we will define a HOMM for describing the states of an economy.

Let $X := \{X_t\}_{t\in\mathcal{T}}$ be an lth-order discrete-time homogeneous HOMM taking values in the state-space:

$$\mathcal{X} := \{x_1,x_2,\dots,x_M\}. \tag{2.1}$$

Write

$$\mathbf{i}(t,l) := (i_t,i_{t-1},\dots,i_{t-l}), \tag{2.2}$$

where $t \geq l$, $l = 1,2,\dots$, and $i_t,i_{t-1},\dots,i_{t-l} \in \{1,2,\dots,M\}$.

The state transition probabilities of X are then specified as follows:

$$P\big(i_{t+1} \mid \mathbf{i}(t,l)\big) := P\big[X_{t+1} = x_{i_{t+1}} \mid X_t = x_{i_t},\dots,X_{t-l} = x_{i_{t-l}}\big], \quad i_{t+1} = 1,2,\dots,M. \tag{2.3}$$

To determine the HOMM completely, we need to define the following initial distributions:

$$P\big(i_{l+1} \mid \mathbf{i}(l,l)\big) := \pi_{i_{l+1}|\mathbf{i}(l,l)}, \quad i_{t+1} = 1,2,\dots,M. \tag{2.4}$$

Now, we will describe the Markov-modulated process for the price dynamics of the underlying risky asset. We assume that the market interest rate of the bank account, the drift, and the volatility of the risky asset switch over time according to the states of the economy modeled by X.

Let $r_{t,j}$ be the market interest rate of the bank account in the tth period. For each $j = 0,1,\dots,l$, we write $\mathbf{X}_{t,j}$ for $(X_t,X_{t-1},\dots,X_{t-j})$, for each $t \geq l$, $j = 0,1,\dots,l$. We suppose that r_t depends on the current value and the past values of the HOMM up to lag j, that is,

$$r_{t,j} := r(\mathbf{X}_{t,j}). \tag{2.5}$$

Then, the price dynamic $B := \{B_t\}_{t\in\mathcal{T}}$ of the bank account is given by

$$B_t = B_{t-1}e^{r_{t,j}}, \quad B_0 = 1, \quad \mathcal{P}\text{-a.s.} \tag{2.6}$$

Let $S := \{S_t\}_{t\in\mathcal{T}}$ be the price process of the risky stock. For each $t \in \mathcal{T}$, let $Y_t := \ln(S_t/S_{t-1})$ be the logarithmic return in the tth-period. We denote by

$$\begin{aligned} \mu_{t,j} &:= \mu(\mathbf{X}_{t,j}), \\ \sigma_{t,j} &:= \sigma(\mathbf{X}_{t,j}) \end{aligned} \tag{2.7}$$

the drift and the volatility, respectively, of the risky stock in the tth-period. In other words, the drift and the volatility depend on the current value and the past values of the HOMM up to lag j. In particular,

$$\begin{aligned} \mu(x_{i_t}, x_{i_{t-1}}, \dots, x_{i_{t-j}}) &= \mu_{\mathbf{i}(t,j)}, \\ \sigma(x_{i_t}, x_{i_{t-1}}, \dots, x_{i_{t-j}}) &= \sigma_{\mathbf{i}(t,j)}, \end{aligned} \tag{2.8}$$

where $\mu_{\mathbf{i}(t,j)} > 0$ and $\sigma_{\mathbf{i}(t,j)} > 0$, for all $\mathbf{i}(t,j)$.

Let $\{\xi_t\}_{t=1,2,\dots}$ be a sequence of i.i.d. random variables with common distribution $N(0,1)$, a standard normal distribution with zero mean and unit variance. We assume that ξ and X are independent. Then, we suppose that the dynamic of Y is governed by the following Markov-modulated model:

$$Y_t = \mu(\mathbf{X}_{t,j}) - \frac{1}{2}\sigma^2(\mathbf{X}_{t,j}) + \sigma(\mathbf{X}_{t,j})\xi_t, \quad t = 1,2,\dots. \tag{2.9}$$

By convention, $Y_0 = 0$, $\mathcal{P}$-a.s.

When $j = 0$, the Markov-modulated model for Y becomes

$$Y_t = \mu(X_t) - \frac{1}{2}\sigma^2(X_t) + \sigma(X_t)\xi_t, \quad t = 1,2,\dots, \tag{2.10}$$

where the drift and the volatility are governed by the current state of the Markov chain X only.

If we further assume that $l = 1$, the Markov-modulated model for Y is similar to the first-order HOMM for logarithmic returns in Elliott et al. [16].

3. Regime-switching Esscher transform

The Esscher transform is a well-known tool in actuarial science. The seminal work of Gerber and Shiu [17] pioneers the use of the Esscher transform for option valuation. Their approach provides a convenient and flexible way for the valuation of options under a general asset price model. The use of the Esscher transform for option valuation can be justified by the maximization of the expected power utility. It also highlights the interplay between actuarial and financial pricing, which is an important topic for contemporary actuarial research as pointed out by Bühlmann et al. [18]. Elliott et al. [15] adopt the regime-switching version of the Esscher transform to determine an equivalent martingale measure for the valuation of options in an incomplete market described by a Markov-modulated geometric Brownian motion. Here, we consider a discrete-time version of the regime-switching Esscher transform and apply it to determine an equivalent martingale measure for pricing options in an incomplete market described by our model.

First, for each $t \in \mathcal{T}$, let $\mathcal{F}_t^X$ and $\mathcal{F}_t^Y$ denote the σ-algebras generated by the values of the Markov chain X and the logarithmic returns Y up to and including time t, respectively. We suppose that both $\mathcal{F}_t^X$ and $\mathcal{F}_t^Y$ are observable information sets. We write $\mathcal{G}_t$ for $\mathcal{F}_t^Y \vee \mathcal{F}_T^X$, for each $t \in \mathcal{T}$.

Let $\ominus_t$ be a $\mathcal{F}_T^X$-measurable random variable, for each $t = 1,2,\dots$. That is, the value of $\ominus_t$ is known given the information set $\mathcal{F}_T^X$. We interpret $\ominus_t$ as the regime-switching Esscher parameter at time t conditional on $\mathcal{F}_T^X$. Let $M_Y(t,\ominus_t)$ denote the moment generating function of Y_t given $\mathcal{F}_T^X$ evaluated at $\ominus_t$ under $\mathcal{P}$, that is,

$$M_Y(t,\ominus_t) := E\left(e^{\ominus_t Y_t} \mid \mathcal{F}_T^X\right), \tag{3.1}$$

where $E(\cdot)$ is the expectation under $\mathcal{P}$.

Here we assume that there exists a $\ominus_t$ such that $M_Y(t,\ominus_t) < \infty$. Then, we define a process

$$\Lambda := \{\Lambda_t\}_{t \in \mathcal{T}} \tag{3.2}$$

with $\Lambda_0 = 1$, $\mathcal{P}$-a.s., as follows:

$$\Lambda_t := \prod_{k=1}^{t} \frac{e^{\ominus_k Y_k}}{M_Y(k,\ominus_k)}. \tag{3.3}$$

Lemma 3.1. *Assume that Y_{t+1} is conditionally independent of $\mathcal{F}_t^Y$ given $\mathcal{F}_T^X$. Then, Λ is a $(\mathcal{G},\mathcal{P})$-martingale.*

Proof. We note that Λ_t is $\mathcal{G}_t$-measurable, for each $t \in \mathcal{T}$. Given that Y_{t+1} is conditionally independent of $\mathcal{F}_t^Y$ given $\mathcal{F}_T^X$,

$$\begin{aligned} E\left(\frac{\Lambda_{t+1}}{\Lambda_t} \mid \mathcal{G}_t\right) &= E\left[\frac{e^{\ominus_{t+1} Y_{t+1}}}{M_Y(t+1,\ominus_{t+1})} \mid \mathcal{F}_T^X\right] \\ &= 1, \quad \mathcal{P}\text{-a.s.} \end{aligned} \tag{3.4}$$

Hence, the result follows.

Now, we define a discrete-time version of the regime-switching Esscher transform in Elliott et al. [15] $\mathcal{P}^{\ominus} \sim \mathcal{P}$ on $\mathcal{G}_T$ associated with

$$(\ominus_1, \ominus_2, \dots, \ominus_T) \tag{3.5}$$

as follows:

$$\mathcal{P}^{\ominus}(A) = E(\Lambda_T \cdot I_A), \quad A \in \mathcal{G}_T. \tag{3.6}$$

Let $M_Y(t,z \mid \ominus)$ be the moment generating function of Y_t given $\mathcal{F}_T^X$ under $\mathcal{P}^{\ominus}$ evaluated at z, that is,

$$M_Y(t,z \mid \ominus) = E^{\ominus}(e^{zYt} \mid \mathcal{F}_T^X), \tag{3.7}$$

where $E^{\ominus}(\cdot)$ is an expectation under $\mathcal{P}^{\ominus}$. □

Lemma 3.2. *We have*

$$M_Y(t,z \mid \ominus) = \frac{M_Y(t,\ominus_t + z)}{M_Y(t,\ominus_t)}. \tag{3.8}$$

Proof. By the Bayes' rule, Lemma 3.1, and the fact that Y_t is independent of $\mathcal{F}_{t-1}^Y$ given $\mathcal{F}_T^X$,

$$\begin{aligned} M_Y(t,z \mid \ominus) &= E^{\ominus}(e^{zYt} \mid \mathcal{F}_{t-1}^Y \vee \mathcal{F}_T^X) \\ &= E\left(\frac{\Lambda_t}{\Lambda_{t-1}} e^{zYt} \mid \mathcal{G}_{t-1}\right) \\ &= \frac{E(e^{(z+\ominus_t)Y_t} \mid \mathcal{F}_{t-1}^Y \vee \mathcal{F}_T^X)}{M_Y(t,\ominus_t)} \\ &= \frac{M_Y(t,\ominus_t + z)}{M_Y(t,\ominus_t)}. \end{aligned} \tag{3.9}$$

The seminal works of Harrison and Pliska [19, 20] establish an important link between the absence of arbitrage and the existence of an equivalent martingale measure under which discounted price processes are martingales. This is known as the fundamental theorem of asset pricing and has been extended by several authors, including Dybvig and Ross [21], Back and Pliska [22], and Delbaen and Schachermayer [23], among others. In our case, we specify an equivalent martingale measure by the risk-neutral regime-switching Esscher transform and provide a necessary and sufficient condition on the regime-switching Esscher parameters $(\ominus_1, \ominus_2, \ldots, \ominus_T)$ for $\mathcal{P}^{\ominus}$ to be a risk-neutral regime-switching Esscher transform. □

Proposition 3.3. *The discounted price process $\{S_t/B_t\}_{t\in\mathcal{T}}$ is a $(\mathcal{G},\mathcal{P}_{\ominus})$-martingale if and only if*

$$\ominus_{t+1} := \ominus(\mathbf{X}_{t+1,j}) = \frac{r_{t+1,j} - \mu_{t+1,j}}{\sigma_{t+1,j}^2}, \quad t = 0,1,\ldots,T-1. \tag{3.10}$$

Proof. By Lemma 3.2,

$$\begin{aligned} E^{\ominus}\left(\frac{S_{t+1}}{B_{t+1}} \mid \mathcal{G}_t\right) &= \frac{S_t}{B_t} e^{-r_{t+1}} E^{\ominus}\left(e^{Y_{t+1}} \mid \mathcal{G}_t\right) \\ &= \frac{S_t}{B_t} e^{-r_{t+1}} M_Y(t+1, 1 \mid \ominus) \\ &= \frac{S_t}{B_t} e^{-r_{t+1}} \frac{M_Y(t+1, \ominus_{t+1}+1)}{M_Y(t+1, \ominus_{t+1})} \\ &= \frac{S_t}{B_t}, \quad \mathcal{P}\text{-a.s.}, \end{aligned} \tag{3.11}$$

if and only if

$$\frac{M_Y(t+1, \ominus_{t+1}+1)}{M_Y(t+1, \ominus_{t+1})} = e^{r_{t+1}}. \tag{3.12}$$

Since $Y_{t+1} \mid \mathcal{F}_T^X \sim N(\mu_{t+1,j} - (1/2)\sigma_{t+1,j}^2, \sigma_{t+1,j}^2)$,

$$M_Y(t+1, \ominus_{t+1}) = \exp\left[\ominus_{t+1}\left(\mu_{t+1,j} - \frac{1}{2}\sigma_{t+1,j}^2\right) + \frac{1}{2}\ominus_{t+1}^2 \sigma_{t+1,j}^2\right]. \tag{3.13}$$

Then,

$$\frac{M_Y(t+1, \ominus_{t+1}+1)}{M_Y(t+1, \ominus_{t+1})} = \exp\left(\mu_{t+1,j} + \ominus_{t+1}\sigma_{t+1,j}^2\right). \tag{3.14}$$

Hence, we have the result that

$$E^{\ominus}\left(\frac{S_{t+1}}{B_{t+1}} \mid \mathcal{G}_t\right) = \frac{S_t}{B_t}, \quad \mathcal{P}\text{-a.s.}, \tag{3.15}$$

if and only if

$$\ominus_{t+1} = \frac{r_{t+1,j} - \mu_{t+1,j}}{\sigma_{t+1,j}^2}. \tag{3.16}$$

The risk-neutral dynamics of Y under $\mathcal{P}^{\ominus}$ are presented in the following corollary. □

Corollary 3.4. *Suppose $v := \{v_t\}_{t=1,2,\dots,T}$ is a sequence of i.i.d. random variables such that $v_t \sim N(0,1)$ under $\mathcal{P}^{\ominus}$. Then, under $\mathcal{P}^{\ominus}$,*

$$Y_{t+1} = r(\mathbf{X}_{t+1,j}) - \frac{1}{2}\sigma^2(\mathbf{X}_{t+1,j}) + \sigma(\mathbf{X}_{t+1,j})v_{t+1}, \quad t = 0, 1, \dots, T-1, \tag{3.17}$$

and the dynamics of X remain unchanged under the change of measures.

Proof. By Lemma 3.2,

$$M_Y(t+1, z \mid \ominus) = \exp\left[z\left(\mu_{t+1,j} - \frac{1}{2}\sigma_{t+1,j}^2\right) + \frac{1}{2}z(2\ominus_{t+1} + z)\sigma_{t+1,j}^2\right]. \tag{3.18}$$

By Proposition 3.3,

$$\ominus_{t+1} = \frac{r_{t+1,j} - \mu_{t+1,j}}{\sigma^2_{t+1,j}}. \tag{3.19}$$

This implies that

$$M_Y(t+1, z \mid \ominus) = \exp\left[z\left(r_{t+1,j} - \frac{1}{2}\sigma^2_{t+1,j}\right) + \frac{1}{2}z^2\sigma^2_{t+1,j}\right]. \tag{3.20}$$

Hence,

$$Y_{t+1} = r(\mathbf{X}_{t+1,j}) - \frac{1}{2}\sigma^2(\mathbf{X}_{t+1,j}) + \sigma(\mathbf{X}_{t+1,j})v_{t+1}, \quad t = 0, 1, \ldots, T-1. \tag{3.21}$$

Since the processes X and ξ are independent, the dynamics of X remain unchanged when we change the measures from $\mathcal{P}$ to $\mathcal{P}^{\ominus}$.

We will consider the pricing of three different types of exotic options, namely, Asian options, lookback options, and barrier options. First, we deal with an arithmetic average floating-strike Asian call option with maturity T. The payoff of the Asian option at the maturity T is given by

$$P_{\mathrm{AA}}(T) = \max(S_T - J_T, 0), \tag{3.22}$$

where the arithmetic average J_T of the underlying stock price is

$$J_T = \frac{1}{T}\sum_{t=0}^{T} S_t. \tag{3.23}$$

Then, we consider the pricing of a down-and-out European call option with barrier level L, strike price K, and maturity at time T. The payoff of the barrier option at time T is

$$P_B(T) = \max(S_T - K, 0)I_{\{\min_{0\le t\le T} S_t > L\}}, \tag{3.24}$$

where I_E is the indicator function of an event E. Finally, we deal with a European-style lookback floating-strike call option with maturity at time T. The payoff of the lookback option is

$$P_{\mathrm{LB}}(T) = \max(S_T - m_{0,T}, 0), \tag{3.25}$$

where $m_{0,T} := \min_{0\le t\le T} S_t$. $\square$

4. Simulation experiments

In this section, we give some simulation experiments to investigate the effect of the order of the HOMM on the pricing of the following options: Asian option, barrier option, and lookback option described in the previous section. In particular, we will investigate the behaviors of the option prices implied by the second-order HOMM (Model I), the first-order HOMM (Model II), and the model without switching regimes (Model III). For illustration, we assume that the Markov chain has two states in each of the three models. That is, the economy has two states with State "1" and State "2" representing a "Good" economy and a "Bad" economy, respectively. We employ the Monte Carlo simulation to compute the option prices. 5000 simulation runs are generated for computing each option price. All computations were done in a standard PC with C codes. We remark that the simulation of option prices can be done in EXCEL, see for example, Sundaresan [24, Chapter 14]. Moreover, the simulation of the high-order Markov chain can also be done in EXCEL as in Ching et al. [25]. Hence, the simulation process of our models can also be done in EXCEL.

We specify some specimen values for the model parameters. First, we specify these values for Model I. Let r_{ij} be the daily market interest rate when the economy in the current period is in the jth state and the economy in the last period is in the ith state, for $i, j = 1, 2$. We suppose that

$$\begin{aligned} r_{11} &= \frac{0.06}{252} = 0.0238\%, \qquad r_{12} = \frac{0.02}{252} = 0.00794\%, \\ r_{21} &= \frac{0.04}{252} = 0.0159\%, \qquad r_{22} = \frac{0.01}{252} = 0.00397\%. \end{aligned} \tag{4.1}$$

Here, we assume that one year has 252 trading days. In other words, the corresponding annual market interest rates are 6%, 2%, 4%, and 1%, respectively. Let σ_{ij} denote the daily volatility when the economy in the current period is in the jth state and the economy in the last period is in the ith state. We assume that

$$\begin{aligned} \sigma_{11} &= \frac{0.1}{\sqrt{252}} = 0.63\%, \qquad \sigma_{12} = \frac{0.3}{\sqrt{252}} = 1.89\%, \\ \sigma_{21} &= \frac{0.2}{\sqrt{252}} = 1.26\%, \qquad \sigma_{22} = \frac{0.4}{\sqrt{252}} = 2.52\%. \end{aligned} \tag{4.2}$$

In other words, the corresponding annual volatilities are 10%, 30%, 20%, and 40%, respectively. Let

$$\pi_{ijk} := P(X_t = k \mid X_{t-1} = i,\ X_{t-2} = j) \quad \text{for } i, j, k = 1, 2. \tag{4.3}$$

We suppose that

$$\pi_{111} = 0.7, \qquad \pi_{121} = 0.3, \qquad \pi_{211} = 0.6, \qquad \pi_{221} = 0.2. \tag{4.4}$$

We assume that the two initial states of the second-order HOMM $X_0 = 1$ and $X_1 = 2$. Then, we specify the values of the model parameters for Model II. For each $i = 1, 2$, let r_i and σ_i denote the daily market interest rate and the daily volatility when the current economy is in the ith state, respectively. We suppose that

$$\begin{aligned} r_1 &= r_{11} = 0.0238\%, \qquad r_2 = r_{12} = 0.00794\%, \\ \sigma_1 &= \sigma_{11} = 0.63\%, \qquad \sigma_2 = \sigma_{12} = 1.89\%. \end{aligned} \tag{4.5}$$

Let

$$\pi_{ij} := P(X_t = j \mid X_{t-1} = i,) \quad \text{for } i, j = 1, 2. \tag{4.6}$$

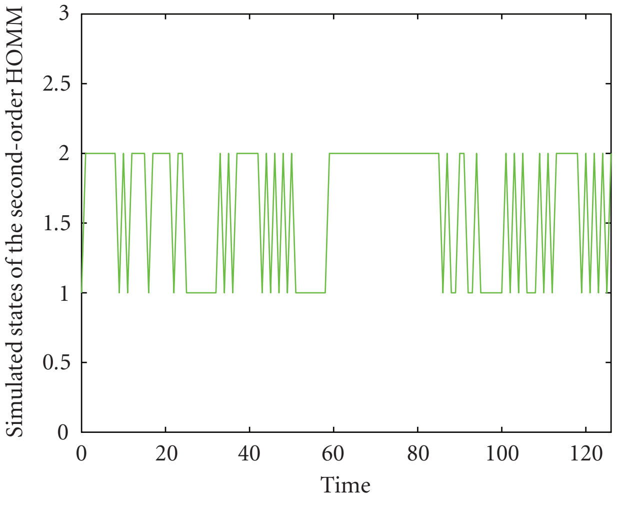

Figure 4.1. Simulated states of the second-order HOMM.

We assume that

$$\pi_{11} = \pi_{111} = 0.7, \qquad \pi_{21} = \pi_{121} = 0.3. \tag{4.7}$$

We further assume that the initial state $X_0 = 1$. For Model III, we assume that the daily market interest rate

$$r = r_{11} = 0.0238\%, \tag{4.8}$$

and the daily volatility

$$\sigma = \sigma_{11} = 0.63\%. \tag{4.9}$$

To understand the impact of the order of the HOMM on the dynamics of the states of the economy and the return process of the underlying share, we provide plots of the realizations of the processes X and Y under Models I, II, and III with the parameter values described as above in the following figures.

Figures 4.1, 4.2, and 4.3 depict simulated paths of the second-order HOMM, the first-order HOMM, and the zero-order HOMM, respectively.

Comparing Figures 4.1, 4.2, and 4.3, it becomes apparent that the level of persistency of the states of the HOMM increases as the order of the HOMM becomes higher.

In the sequel, we assume that the current price of the underlying share $S_0 = 100$. Figure 4.4 depicts the simulated log return processes Y from the second-order HOMM, the first-order HOMM, and the zero-order HOMM.

From Figure 4.4, it is clear that the log return process Y becomes more volatile when the order of the HOMM becomes higher. If the log return process Y of the stock is more volatile, the prices of options written on the stock become higher. We will see in the following that the prices of options will become higher when the order of the HOMM is

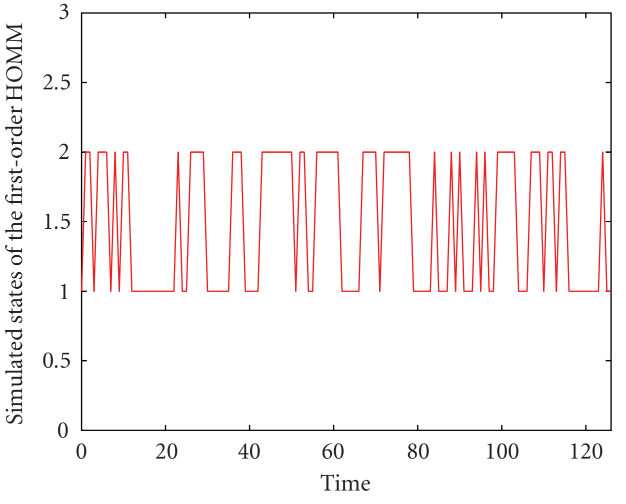

FIGURE 4.2. Simulated states of the first-order HOMM.

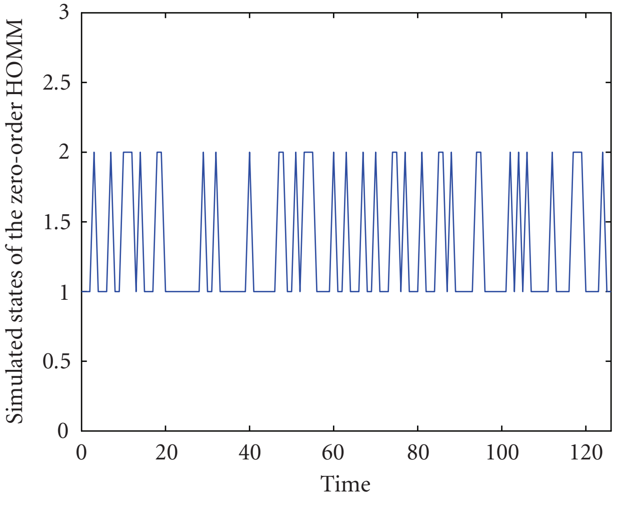

FIGURE 4.3. Simulated states of the zero-order HOMM.

higher. Hence, the simulated state processes X and log return processes Y here explain the simulated option prices for the exotic options and provide us with a better understanding on the impact of the order of the HOMM on the option prices.

In all cases, we assume that the time to maturity ranges from 21 trading days (one month) to 126 trading days (six months) with an increment of 21 trading days. Figure 4.5 depicts the prices of the Asian options implied by Model I, Model II, and Model III for various maturities.

Assume the barrier level $L = 80$ and the strike price $K = 100$. Figure 4.6 depicts the prices of the barrier options implied by the three models for various maturities.

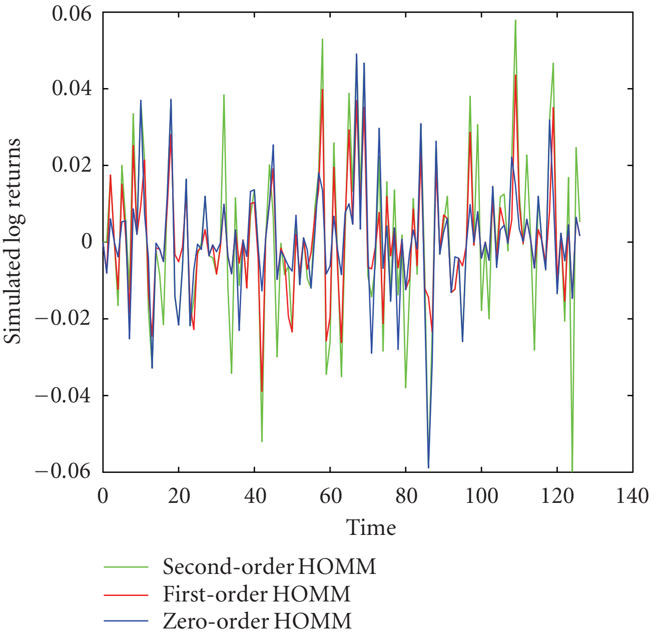

Figure 4.4. Simulated log returns.

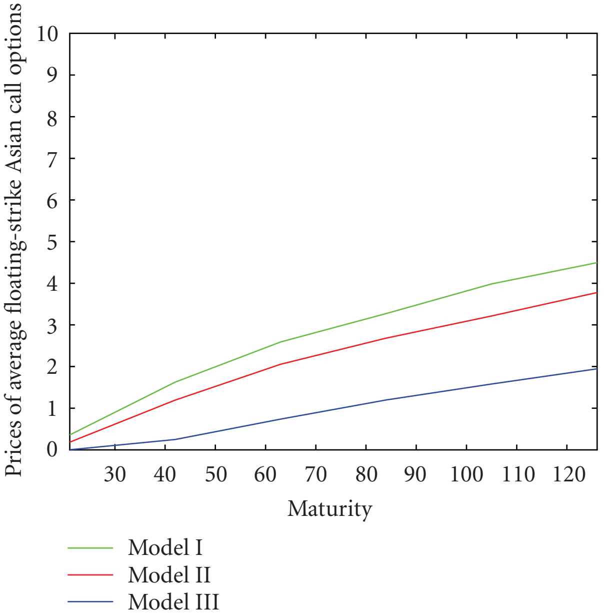

Figure 4.5. Prices of Asian options versus maturities.

Figure 4.7 depicts the prices of the lookback options implied by the three models for various maturities.

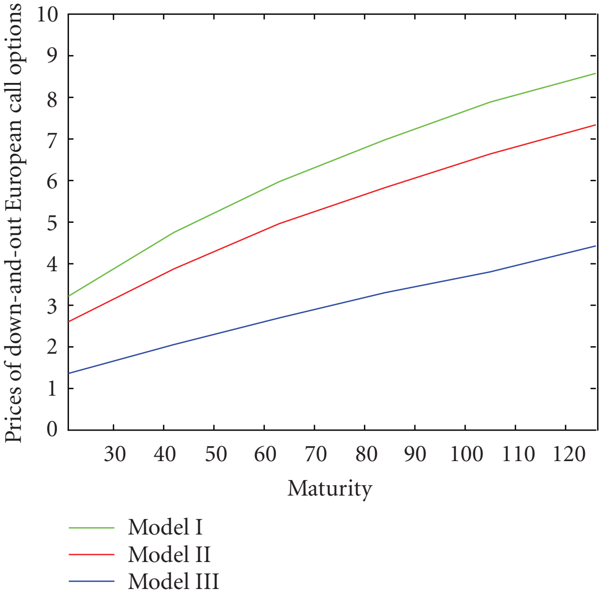

Figure 4.6. Prices of Barrier options versus maturities.

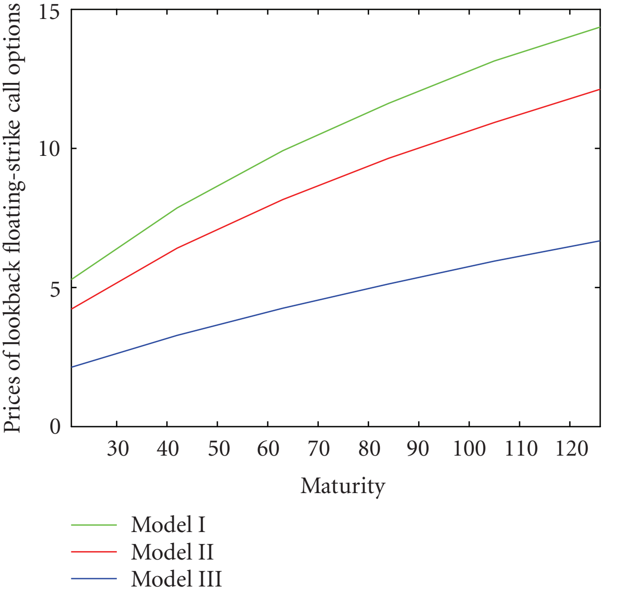

Figure 4.7. Prices of lookback options versus maturities.

We can regard Model III (i.e., the no-regime-switching case) as a zero-order HOMM and Model I as a first-order HOMM. Then, from Figures 4.5, 4.6, and 4.7, we see that the prices of the Asian options, the barrier options, and the lookback options increase substantially as the order of the HOMM does. These prices are sensitive to the order of the HOMM. This is true for the options with various maturities. In other words, the high-order effect in the states of economy has significant impact on the prices of these path-dependent exotic options. The differences between the prices implied by the first-order HOMM and those implied by the zero-order HOMM are more substantial than the difference between the prices obtained from the second-order HOMM and those obtained from the first-order HOMM.

5. Conclusion

We investigated the pricing of exotic options under a discrete-time Markovian regime-switching process driven by an observable HOMM, which can incorporate the high-order effect in the states of the economy. We supposed that the market interest rate, the stock appreciation rate, and the stock volatility switch over time according to the states of the economy. The Esscher transform has been employed to select a pricing measure under the incomplete market setting. We investigated the impact of the high-order effect on the prices of some path-dependent exotic options, including Asian options, lookback options, and barrier options, through simulation experiments. We found that the presence of the high-order effect in the states of the economy has significant impact on the prices of the path-dependent exotic options with various maturities.

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References

[1] J. D. Hamilton, "A new approach to the economic analysis of nonstationary time series and the business cycle," *Econometrica*, vol. 57, no. 2, pp. 357–384, 1989.

[2] J. Cai, "A Markov model of switching-regime ARCH," *Journal of Business & Economic Statistics*, vol. 12, no. 3, pp. 309–316, 1994.

[3] J. D. Hamilton and R. Susmel, "Autoregressive conditional heteroskedasticity and changes in regime," *Journal of Econometrics*, vol. 64, no. 1-2, pp. 307–333, 1994.

[4] S. Gray, "Modeling the conditional distribution of interest rates as a regime-switching process," *Journal of Financial Economics*, vol. 42, no. 1, pp. 27–62, 1996.

[5] F. Klaassen, "Improving GARCH volatility forecasts with regime-switching GARCH," *Empirical Economics*, vol. 27, no. 2, pp. 363–394, 2002.

[6] A. Ang and G. Bekaert, "Regime switches in interest rates," *Journal of Business & Economic Statistics*, vol. 20, no. 2, pp. 163–182, 2002.

[7] A. Ang and G. Bekaert, "Short rate nonlinearities and regime switches," *Journal of Economic Dynamics & Control*, vol. 26, no. 7-8, pp. 1243–1274, 2002.

[8] A. Ang and G. Bekaert, "International asset allocation with regime shifts," *Review of Financial Studies*, vol. 15, no. 4, pp. 1137–1187, 2002.

[9] G. W. Schwert, "Business cycles, financial crises, and stock volatility," *Carnegie-Rochester Conference Series on Public Policy*, vol. 31, pp. 83–125, 1989.

[10] C.-J. Kim, C. R. Nelson, and R. Startz, "Testing for mean reversion in heteroskedastic data based on Gibbs-sampling-augmented randomization," *Journal of Empirical Finance*, vol. 5, no. 2, pp. 131–154, 1998.

[11] V. Naik, "Option valuation and hedging strategies with jumps in the volatility of asset returns," *The Journal of Finance*, vol. 48, no. 5, pp. 1969–1984, 1993.

[12] X. Guo, "Information and option pricings," *Quantitative Finance*, vol. 1, no. 1, pp. 38–44, 2001.

[13] J. Buffington and R. J. Elliott, "Regime switching and European options," in *Stochastic Theory and Control (Lawrence, KS, 2001)*, vol. 280 of *Lecture Notes in Control and Information Sciences*, pp. 73–82, Springer, Berlin, Germany, 2002.

[14] J. Buffington and R. J. Elliott, "American options with regime switching," *International Journal of Theoretical and Applied Finance*, vol. 5, no. 5, pp. 497–514, 2002.

[15] R. J. Elliott, L. Chan, and T.-K. Siu, "Option pricing and Esscher transform under regime switching," *Annals of Finance*, vol. 1, no. 4, pp. 423–432, 2005.

[16] R. J. Elliott, W. C. Hunter, and B. M. Jamieson, "Drift and volatility estimation in discrete time," *Journal of Economic Dynamics & Control*, vol. 22, no. 2, pp. 209–218, 1998.

[17] H. Gerber and E. Shiu, "Option pricing by Esscher transforms (with discussions)," *Transactions of the Society of Actuaries*, vol. 46, pp. 99–140, 1994.

[18] H. Bühlmann, F. Delbaen, P. Embrechts, and A. Shiryaev, "On Esscher transforms in discrete finance models," *ASTIN Bulletin*, vol. 28, no. 2, pp. 171–186, 1998.

[19] J. M. Harrison and S. R. Pliska, "Martingales and stochastic integrals in the theory of continuous trading," *Stochastic Processes and Their Applications*, vol. 11, no. 3, pp. 215–260, 1981.

[20] J. M. Harrison and S. R. Pliska, "A stochastic calculus model of continuous trading: complete markets," *Stochastic Processes and Their Applications*, vol. 15, no. 3, pp. 313–316, 1983.

[21] P. H. Dybvig and S. A. Ross, "Arbitrage," in *The New Palgrave: Finance*, pp. 57–71, MacMillan Press, New York, NY, USA, 1987.

[22] K. Back and S. R. Pliska, "On the fundamental theorem of asset pricing with an infinite state space," *Journal of Mathematical Economics*, vol. 20, no. 1, pp. 1–18, 1991.

[23] F. Delbaen and W. Schachermayer, "A general version of the fundamental theorem of asset pricing," *Mathematische Annalen*, vol. 300, no. 3, pp. 463–520, 1994.

[24] S. Sundaresan, *Fixed Income Markets and Their Derivatives*, South-Western, Cincinnati, Ohio, USA, 2nd edition, 2002.

[25] W.-K. Ching, E. S. Fung, and M. K. Ng, "Building higher-order Markov chain models with EXCEL," *International Journal of Mathematical Education in Science and Technology*, vol. 35, no. 6, pp. 921–932, 2004.

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