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Research Article

Analytic Solution of Multipantograph Equation

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We apply the homotopy analysis method (HAM) for solving the multipantograph equation. The analytical results have been obtained in terms of convergent series with easily computable components. Several examples are given to illustrate the efficiency and implementation of the homotopy analysis method. Comparisons are made to confirm the reliability of the homotopy analysis method.

1. Introduction

The delay differential equation

$$\begin{gathered} y'(t) = \lambda y(t) + \sum_{i=1}^{k} \mu_i y\big(f_i(t)\big), \quad t > 0, \\ y(0) = y_0, \end{gathered} \tag{1.1}$$

where $\lambda, \mu_1, \mu_2, \ldots, \mu_k, y_0 \in \mathbb{C}$, has been studied by numerous authors (e.g., [1–8]). Second-order versions of this equation have also been studied (e.g., [9, 10]). The enduring interest in this equation is due partially to the number of applications it has found such as a current collection system for an electric locomotive, cell growth models, biology, economy, control, and electrodynamics (e.g., [10–13]). The focus of most of the studies made in the complex plane (e.g., [12, 14]) was on solutions on the real line for either the retarded case $0 < q < 1$ or the advanced case $q > 1$.

In 1999, Qiu et al. [15] have studied the delay equation

$$\begin{gathered} y'(t) = \lambda y(t) + \sum_{i=1}^{k} \mu_i y\big(q_i t\big), \\ y(0) = y_0, \end{gathered} \tag{1.2}$$

where $0 < q_k < q_{k-1} < \cdots < q_1 < 1$ and $\lambda, \mu_1, \mu_2, \ldots, \mu_k, y_0 \in \mathbb{C}$, by transforming the proportional delay into the constant delay. They got the sufficient condition of asymptotic stability for the analytic solution, that is,

$$\operatorname{Re}\lambda < 0, \qquad \sum_{i=1}^{k}\left|\mu_i\right| < -\operatorname{Re}\lambda. \tag{1.3}$$

Liu and Li in [16, 17] proved the existence and uniqueness of analytic solution of (1.2) for any $\lambda, \mu_1, \mu_2, \ldots, \mu_k,\ y_0 \in \mathbb{C}$, and the analytic solution is asymptotically stable if

$$\operatorname{Re}\lambda < 0, \qquad \sum_{i=1}^{k}\left|\mu_i\right| < |\lambda|. \tag{1.4}$$

In [17–19] the Dirichlet series solution of (1.2) is constructed, and the sufficient condition of the asymptotic stability for the analytic solution is obtained. It is proved that the θ-methods with a variable stepsize are asymptotically stable if $1/2 < \theta \leq 1$.

It is well known that for the multipantograph equation

$$\begin{gathered} y'(t) = \lambda y(t) + \sum_{i=1}^{k}\mu_i y\left(q_i t\right) + f(t), \quad 0 < t < T, \\ y(0) = \alpha, \end{gathered} \tag{1.5}$$

where $0 < q_k < q_{k-1} < \cdots < q_1 < 1$, the collocation solution associated with the mth degree collocation polynomial possesses the optimal superconvergence order $2m+1$ at the first step $t = h$, provided that the collocation m parameters are properly chosen in $(0,1)$ (e.g., [5] for $f(t) = 0$, and [20] for $f(t) \neq 0$).

Ishiwata and Muroya [21] proposed a piecewise $(2m, m)$-rational approximation with "quasiuniform meshes" which corresponds to the mth collocation method, and established the global error analysis of $O(h^{2m})$ on successive mesh points. This method is more useful than the known collocation method when solving (1.5) in case that a long time integration is needed, that is, if T is large, then the number of steps in the method is less than that of the collocation method. Collocation method is useful for computation, but in these mesh divisions, there are problems. For example, if the end point $t = T$ is larger, then the mesh size near the first mesh point becomes too small, compared with the mesh size near the end point. This implies that the total computational cost is higher (see also [22–25].)

In this paper, and in order to overcome such problems, we propose an analytic solution of (1.5) by the HAM addressed in [26–36]. The HAM is based on the homotopy, a basic concept in topology. The auxiliary parameter h is introduced to construct the so-called zero-order deformation equation. Thus, unlike all previous analytic techniques, the HAM provides us with a family of solution expressions in auxiliary parameter h. As a result, the convergence region and rate of solution series are dependent upon the auxiliary parameter h and thus can be greatly enlarged by means of choosing a proper value of h. This provides us with a convenient way to adjust and control convergence region and rate of solution series given by the HAM.

2. Description of the method

In order to obtain an analytic solution of the delay differential equation (1.5), the HAM is employed. Consider the operator N,

$$N\left[y(t)\right] = \frac{\partial y(t)}{\partial t} - \lambda y(t) - \sum_{i=1}^{k} \mu_i y(q_i t) - f(t) = 0, \tag{2.1}$$

where $y(t)$ is unknown function and t the independent variable. Let $y_0(t)$ denote an initial guess of the exact solution $y(t)$ that satisfies $y_0(0) = \alpha$, $h \neq 0$ an auxiliary parameter, $H(t) \neq 0$ an auxiliary function, and L an auxiliary linear operator with the property $L[y(t)] = 0$ when $y(t) = 0$. Then using $q \in [0,1]$ as an embedding parameter, we construct such a homotopy:

$$(1-q)L\left[\phi(t;q) - y_0(t)\right] - qhH(t)N\left[\phi(t;q)\right] = \widehat{H}\left[\phi(t;q); y_0(t), H(t), h, q\right]. \tag{2.2}$$

It should be emphasized that we have great freedom to choose the initial guess $y_0(t)$, the auxiliary linear operator L, the nonzero auxiliary parameter h, and the auxiliary function $H(t)$.

Enforcing the homotopy (2.2) to be zero, that is,

$$\widehat{H}\left[\phi(t;q); y_0(t), H(t), h, q\right] = 0, \tag{2.3}$$

we have the so-called zero-order deformation equation

$$(1-q)L\left[\phi(t;q) - y_0(t)\right] = qhH(t)N\left[\phi(t;q)\right]. \tag{2.4}$$

When $q = 0$, the zero-order deformation equation (2.4) becomes

$$\phi(t;0) = y_0(t) \tag{2.5}$$

and when $q = 1$, since $h \neq 0$ and $H(t) \neq 0$, the zero-order deformation equation (2.4) is equivalent to

$$\phi(t;1) = y(t). \tag{2.6}$$

Thus, according to (2.5) and (2.6), as the embedding parameter q increases from 0 to 1, $\phi(t;q)$ varies continuously from the initial approximation $y_0(t)$ to the exact solution $y(t)$. Such a kind of continuous variation is called deformation in homotopy.

By Taylor's theorem, $\phi(t;q)$ can be expanded in a power series of q as follows:

$$\phi(t;q) = y_0(t) + \sum_{m=1}^{\infty} y_m(t) q^m, \tag{2.7}$$

where

$$y_m(t) = \frac{1}{m!}\frac{\partial^m \phi(t;q)}{\partial q^m}\bigg|_{q=0}. \tag{2.8}$$

If the initial guess $y_0(t)$, the auxiliary linear parameter L, the nonzero auxiliary parameter h, and the auxiliary function $H(t)$ are properly chosen, so that the power series (2.7) of $\phi(t;q)$ converges at $q = 1$. Then, we have under these assumptions the solution series

$$y(t) = \phi(t;1) = \sum_{m=0}^{\infty} y_m(t). \tag{2.9}$$

For brevity, define the vector

$$\vec{y}_n(t) = \{y_0(t), y_1(t), y_2(t), \ldots, y_n(t)\}. \tag{2.10}$$

According to the definition (2.8), the governing equation of $y_m(t)$ can be derived from the zero-order deformation equation (2.4). Differentiating the zero-order deformation equation (2.4) m times with respect to q and then dividing by $m!$ and finally setting $q = 0$, we have the so-called mth-order deformation equation

$$\begin{gathered} L\left[y_m(t) - \chi_m y_{m-1}(t)\right] = hH(t)\Re_m(\vec{y}_{m-1}(t)), \\ y_m(0) = 0, \end{gathered} \tag{2.11}$$

where

$$\begin{aligned} \Re_m(\vec{y}_{m-1}(t)) &= \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\phi(t;q)]}{\partial q^{m-1}}\bigg|_{q=0} \\ &= y'_{m-1}(t) - \lambda y_{m-1}(t) - \sum_{i=1}^{k}\mu_i y_{m-1}(q_i t) - (1-\chi_m) f(t), \\ \chi_m &= \begin{cases} 0, & m \le 1 \\ 1, & m > 1. \end{cases} \end{aligned} \tag{2.12}$$

3. Convergence

Theorem 3.1. *As long as the series* (2.9) *converges, it must be the exact solution of the multipantograph equation* (1.5).

Proof. If the series (2.9) converges, we can write

$$S(t) = \sum_{m=0}^{\infty} y_m(t) \tag{3.1}$$

and it holds that

$$\lim_{m\to\infty} y_m(t) = 0. \tag{3.2}$$

We can verify that

$$\sum_{m=1}^{n}\left[y_m(t) - \chi_m y_{m-1}(t)\right] = y_1 + \left(y_2 - y_1\right) + \cdots + \left(y_n - y_{n-1}\right) = y_n(t), \tag{3.3}$$

which gives us, according to (3.2),

$$\sum_{m=1}^{\infty}\left[y_m(t) - \chi_m y_{m-1}(t)\right] = \lim_{n\to\infty} y_n(t) = 0. \tag{3.4}$$

Furthermore, using (3.3) and the definition of the linear operator L, we have

$$\sum_{m=1}^{\infty} L\left[y_m(t) - \chi_m y_{m-1}(t)\right] = L\left[\sum_{m=1}^{\infty}\left[y_m(t) - \chi_m y_{m-1}(t)\right]\right] = 0. \tag{3.5}$$

According to (2.11), we can obtain that

$$\sum_{m=1}^{\infty} L\left[y_m(t) - \chi_m y_{m-1}(t)\right] = hH(t)\sum_{m=1}^{\infty}\Re_m\left(\vec{y}_{m-1}(t)\right) = 0, \tag{3.6}$$

which gives, since $h \neq 0$ and $H(t) \neq 0$,

$$\sum_{m=1}^{\infty}\Re_m\left(\vec{y}_{m-1}(t)\right) = 0. \tag{3.7}$$

By the definition (2.12) of $\Re_m(\vec{y}_{m-1}(t))$, it holds that

$$\begin{aligned}\sum_{m=1}^{\infty}\Re_m\left(\vec{y}_{m-1}(t)\right) &= \sum_{m=1}^{\infty}\left[y'_{m-1}(t) - \lambda y_{m-1}(t) - \sum_{i=1}^{k}\mu_i y_{m-1}\left(q_i t\right) - \left(1-\chi_m\right) f(t)\right] \\ &= \sum_{m=0}^{\infty} y'_m(t) - \lambda\sum_{m=0}^{\infty} y_m(t) - \sum_{m=0}^{\infty}\sum_{i=1}^{k}\mu_i y_n\left(q_i t\right) - f(t) \\ &= S^{'}(t) - \lambda S(t) - \sum_{i=1}^{k}\mu_i S\left(q_i t\right) - f(t).\end{aligned} \tag{3.8}$$

From (3.7) and (3.8), we have

$$S'(t) = \lambda S(t) + \sum_{i=1}^{k}\mu_i S\left(q_i t\right) + f(t) \tag{3.9}$$

and, moreover, with the help of (2.11), it holds that

$$S(0) = \sum_{m=0}^{\infty} y_m(0) = y_0(0) + \sum_{m=1}^{\infty} y_m(0) = y_0(0) = \alpha. \tag{3.10}$$

In view of (3.9) and (3.10), $S(t)$ must be the exact solution of (1.5). □

4. Examples

The HAM provides an analytical solution in terms of an infinite power series. However, there is a practical need to evaluate this solution, and to obtain numerical values from the infinite power series. The consequent series truncation, as well as the practical procedure conducted to accomplish this task, transforms the otherwise analytical results into an exact solution, which is evaluated to a finite degree of accuracy. In order to investigate the accuracy of the HAM solution with a finite number of terms, three examples were solved. The HAM results were compared with the exact solutions. The impact of the term numbers in the series solution and truncation process was assessed by evaluating the HAM results for different terms in the series. By increasing the number of the HAM terms, the percentage of error decreases. It is also observed that the HAM results with 10 terms have acceptable accuracy compared to the exact solutions. Therefore, it may be concluded that the use of 10 terms in the series yields accurate results with HAM solution sufficiently. MATLAB 7 is used to carry out the computations.

Defining that $L[\phi(t;q)] = \partial\phi(t;q)/\partial t$, with the property $L[C] = 0$, where C is the integral constant and using $H(t) = 1$, the mth-order deformation equations (2.11) for $m \geq 1$ becomes

$$y_m(t) = \chi_m y_{m-1}(t) + h\int_0^t \left[y'_{m-1}(\tau) - \lambda y_{m-1}(\tau) - \sum_{i=1}^{k} \mu_i y_{m-1}(q_i\tau) - (1-\chi_m) f(\tau) \right] d\tau. \tag{4.1}$$

Example 4.1. We consider the following pantograph differential equation:

$$\begin{gathered} y'(t) = -y(t) + \frac{1}{4} y\left(\frac{1}{2}t\right) - \frac{1}{4}e^{-0.5t}, \\ y(0) = 1. \end{gathered} \tag{4.2}$$

The exact solution is $y(t) = e^{-t}$. Note that we still have freedom to choose the auxiliary parameter h. To investigate the influence of h on the solution series (2.9), we can consider the convergence of some related series such as $y'(0)$, $y''(0)$, $y'''(0)$, and so on. However, $y''(0)$ is dependent of h. Let R_h denote a set of all possible values of h by means of which the corresponding series of $y''(0)$ converges. According to Theorem 3.1, for each $h \in R_h$, the corresponding series of $y''(0)$ converges to the same result. The curve $y''(0)$ versus h contains a horizontal line segment above the the valid region R_h. We call such a kind of curve the h-curve [33], which clearly indicates the the valid region R_h of a solution series. The so-called h-curve of $y''(0)$ is as shown in Figure 1. From Figure 1 it is clear that the series of $y''(0)$ is

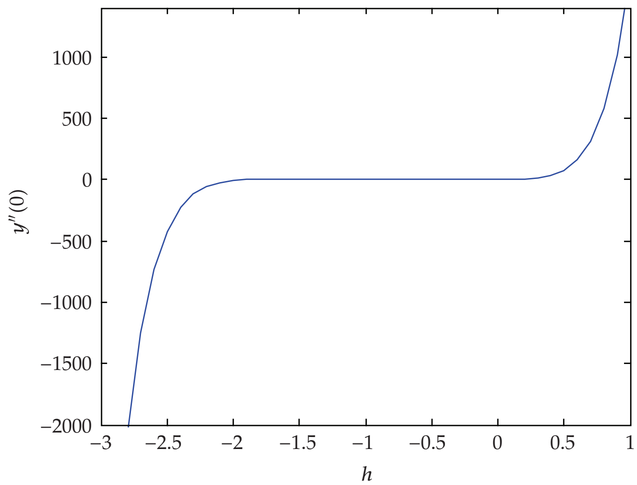

Figure 1: The h-curve of $y''(0)$. Solid line: 10th-order approximation of $y''(0)$.

convergent when $-2 \le h \le 0$. Using $h = -1$, we have from (2.9) and (4.1) that the ten terms approximate solution obtained by HAM are

$$\begin{aligned}\sum_{m=0}^{10} y_m(t) &= 1 - t + \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{1}{24}t^4 - \frac{1}{120}t^5 + \frac{1}{720}t^6 \\ &\quad - \frac{1}{5040}t^7 + \frac{1}{40320}t^8 - \frac{1}{362880}t^9 + 6.3 \times 10^{-8}t^{10} \\ &\simeq \sum_{k=0}^{10} \frac{(-1)^k t^k}{k!}.\end{aligned} \tag{4.3}$$

We see that HAM solution is very close to the exact solution. It may be concluded that the use of 10 terms in the homotopy series yields accurate results.

Example 4.2. Next, we consider the nonhomogeneous delay equation

$$\begin{gathered} y'(t) = -y(t) + \frac{1}{2}y\left(\frac{1}{2}t\right) + \cos t + \sin t - \frac{1}{2}\sin\frac{1}{2}t, \quad 0 \le t \le 2\pi, \\ y(0) = 0. \end{gathered} \tag{4.4}$$

By means of the h-curve, it is reasonable to choose $h = -1.5$. We have for $t > 0$ the ten terms approximate solution obtained by HAM as follows:

$$\begin{aligned}\sum_{m=0}^{10} y_m(t) &= t - \frac{1}{6}t^3 + \frac{1}{120}t^5 - \frac{1}{5040}t^7 + \frac{1}{362880}t^9 + 1.6 \times 10^{-7}t^{10} \\ &\simeq \sum_{m=0}^{9} \frac{(-1)^k}{(2k+1)!}t^{2k+1}.\end{aligned} \tag{4.5}$$

Table 1: Comparison of the results of the HAM and the $(2m, m)$-rational approximation.

n	HAM	$(2m, m)$-rational approximation
0	0	$3.8391471 \cdots E-07$
1	$6.93 \cdots E-18$	$2.613675 \cdots E-08$
2	$3.46 \cdots E-18$	$1.70118 \cdots E-09$
3	$1.23 \cdots E-31$	$1.0844 \cdots E-10$
4	$4.04 \cdots E-36$	$6.83 \cdots E-12$

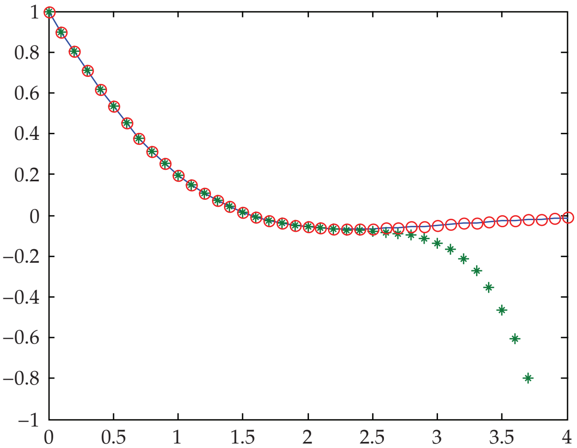

Figure 2: Plots of ten "$**$" and twenty "oo" terms approximations for $y(t)$ "$-$" versus t.

In view of (4.5), we can conclude that the exact solution is $y(t) = \sin t$. Ishiwata and Muroya [21] proposed a piecewise $(2m, m)$-rational approximation $Q_{2m,m}(t)$ with "quasiuniform meshes" which corresponds to the mth collocation method. For $m = 2$, and $h = 2^{(6+n)}$, $n = 0, 1, \ldots, 4$, the errors $e(h) = |Q_{4,2}(h) - y(h)|$ at the first mesh point $t_1 = h$ are shown in the third column of Table 1. In Table 1, The accuracy of the HAM is examined by comparing (4.5) with the available exact and the $(2m, m)$-rational approximation method.

Example 4.3. In the last example, we consider the pantograph equation

$$y'(t) = -y(t) - e^{-0.5t} \sin(0.5t) y(0.5t) - 2e^{-0.75t} \cos(0.5t) \sin(0.25t) y(0.25t), \quad y(0) = 1. \tag{4.6}$$

The exact solution is $y(t) = e^{-t} \cos t$. By means of the h-curve, it is reasonable to choose $h = -1$. We have for $t > 0$,

$$\sum_{m=0}^{10} y_m(t) = 1 - t + \frac{1}{3}t^3 - \frac{1}{6}t^4 + \frac{1}{30}t^5 - \frac{1}{630}t^7 + \frac{1}{2520}t^8 - \frac{1}{22680}t^9 - \frac{1}{3628800}t^{10}. \tag{4.7}$$

The first nine terms of the series (4.7) are coinciding with the first nine terms of the Taylor series of $e^{-t} \cos t$. Figure 2 shows plots of ten and twenty terms approximation of $y(t)$.

5. Discussion and conclusion

In this paper, the HAM was employed to solve the multipantograph differential equation. Unlike the traditional methods, the solutions here are given in series form. The approximate solution to the equation was computed with no need for special transformations, linearization, or discretization. It was shown that the HAM solutions are very close to the exact solutions. It may be concluded that the use of a few terms in the series yields accurate results with HAM solution sufficiently. HAM is a powerful tool for solving analytically nonlinear equations.

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