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Research Article

Discrete Analysis of Portfolio Selection with Optimal Stopping Time

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Most of the investments in practice are carried out without certain horizons. There are many factors to drive investment to a stop. In this paper, we consider a portfolio selection policy with market-related stopping time. Particularly, we assume that the investor exits the market once his wealth reaches a given investment target or falls below a bankruptcy threshold. Our objective is to minimize the expected time when the investment target is obtained, at the same time, we guarantee the probability that bankruptcy happens is no larger than a given level. We formulate the problem as a mix integer linear programming model and make analysis of the model by using a numerical example.

1. Introduction

Portfolio theory deals with the question of how to find an optimal policy to invest among various assets. The mean-variance analysis of Markowitz [1, 2] plays a key role in the theory of portfolio selection, which quantifies the return and the risk in computable terms. The mean-variance model is later extended to the multistage dynamic case. For this and other expected utility-maximization models in dynamic portfolio selection, one is referred to Dumas and Luciano [3], Elton and Gruber [4], Li and Ng [5], Merton [6], and Mossion [7].

An important assumption of the previous portfolio selection model is that the investment horizon is definite. That means an investor knows with certainty the exit time at the beginning of the investment. However, most of the investments in practice are carried out without certain horizons. There are many factors, related to the market or not, which can drive the investment stop. For example, sudden huge consumption, serious illness, and retirement are market-unrelated reasons. Also, those market-related reasons may more strongly affect the investment horizon. A natural example is that the investor may exit the market once his wealth reaches an investment target, which is closely related to the market and also the investment policy itself. Because of the disparity between theory and practice, it seems sympathetic to relax the restrictive assumption that the investment horizon is preknown with certainty.

Research on this subject has been investigated in continuous setting. Yaari [8] first deals with the problem of optimal consumption for an individual with uncertain date of death, under a pure deterministic investment environment. In 2000, Karatzas and Wang [9] address the optimal dynamic investment problem in a complete market with assumption that the uncertain investment horizon is a stopping time of asset price filtration. Multiperiod mean-variance portfolio optimization problem with uncertain exit time is studied by Guo and Hu [10], where the uncertain exit time is market unrelated. A continuous time problem with minimizing the expected time to beat a benchmark is addressed in Browne [11, 12], where the exit time is a random variable related to the portfolio. Literatures of portfolio selection focus on the case that the stopping time is market-state independent. While, the state-dependent exogenous stopping time is considered by Blanchet-Scalliet et al. [13] in dynamic asset pricing theory.

In this paper, we consider a portfolio selection problem with endogenous stopping time in discrete framework, which has not been well discussed in literatures. Specially, we assume that the investor exits the market once his wealth hits an investment target or he is bankrupt. This assumption actually reflects most investors' investment behavior in real life. Our objective is to minimize the expected time that the investment target is obtained, at the same time we guarantee that the probability of which bankruptcy happens is no larger than a given threshold. The investment process is represented by a multistage scenarios tree, in which the discrete stages and notes denote the decision time points and the market states, respectively.

The rest part of the paper is organized as follows. In Section 2, we introduce the statement of the problem, including notations and the general form of the problem. Following, Section 3 is devoted to derive the deterministic formulation of the problem, in which we define a list of integer variables to indicate different states during the investment process. Finally, we make analysis of the model by using a numerical example in Section 4.

2. The Problem Statement

Consider the following investment problem. We distribute the investment budget among a set of assets, and the portfolio can be adjusted at several discrete decision time points during the investment process. At the beginning of the investment, we assign a target wealth and also a bankruptcy threshold. Our objective is to obtain this target wealth and stop the investment as soon as possible. At the same time, we also need to avoid that the bankruptcy occurs before the target wealth obtained.

The problem is based on a finite multistage scenarios tree structure. Our portfolio includes a set of m assets. The underlying dynamic scenarios tree is constructed as follows. There are T stages denoted from time 0 to T. The portfolios can be constructed at the beginning of each stage in the scenarios tree. We denote N_t to be the index set of the scenarios at time t, and S_{nt} as the nth scenario at time t, for $n \in N_t,\ t = 0, 1, \ldots, T$. For those data at this scenario, the price vector of the risky assets is denoted by $u_{nt} \in \Re^m$, and the payoff vector of the assets is denoted by $v_{nt} \in \Re^m$. The decision variables at this scenario are the number of shares of holdings of the assets $x_{nt} \in \Re^m$. We denote the wealth at this scenario to be W_{nt}, and the initial wealth to be B. We denote $a(n)$ and $c(n)$ as the parent node and the children nodes of node n, respectively. Moreover, let $S_{a(n),t-1}$ be the parent scenario of S_{nt}, and $S_{c(n),t+1}$ be the set of immediate children of S_{nt}. The probability of scenario S_{nt} happens is p_{nt}.

We consider an objective related to the achievement of performance goal and bankruptcy. The investment stops once the goal is reached or the bankruptcy occurs, the related stopping time is denoted as t_u and t_l, respectively. Specifically, for given wealth levels l and u, with $l < B < u$, we say that the performance goal u is reached if $W_{nt} \geq u$, denoting this time as t_u, that is, $t_u = \inf\{t > 0; W_{nt} \geq u\}$; that the bankruptcy occurs if $W_{nt} < l$, denoting this time as t_l, that is, $t_l = \inf\{t > 0; W_{nt} < l\}$. Our objective is to minimize the expected time that the goal is reached, at the same time we guarantee the probability that the bankruptcy happens before the goal is reached is no more than a given level, say q, $0 < q < 1$. Thus, the investment problem can be represented in the general form

$$\begin{aligned} \min\ & E[t_u] \\ \text{s.t.}\ & P(W < l) \leq q \\ & \text{budget constraints} \\ & t \in \{0, 1, 2, \ldots, T\}, \end{aligned} \tag{M}$$

where the first constraint is a probability constraint of bankruptcy, in which W generally represents the realized wealth by investment. Moreover, the budget constraints are the wealth dynamics during the investment horizon. We will continue the discussion on the determistic formulation of the model in Section 3.

3. The Problem Formulation

In this section, we will derive the deterministic formulation of the problem (M). Most efforts are devoted to present the objective function and the probability constraint. Actually, we do this by introducing a list of indicator variables. Before we start this work, let us first consider the budget constraints first.

3.1. The Budget Constraints

Based on the previously given notations on the scenarios tree, we first have the allocation of the initial investment wealth represented as $B = u_0' x_0$. At scenario $S_{nt},\ n \in N_t,\ t = 0, 1, \ldots, T$, the wealth W_{nt} should be the realized payoff during the previous period, that is,

$$W_{nt} = v_{nt}' x_{a(n),t-1}. \tag{3.1}$$

Also, for a self-financing process that we are considering here, the realized wealth will be reinvested at this decision point, which means

$$W_{nt} = u_{nt}' x_{nt}. \tag{3.2}$$

Therefore, we conclude the budget constraints at scenario S_{nt}, $n \in N_t$, $t = 0, 1, \ldots, T$, by the set of equations as follows:

$$
\begin{gathered}
u_0' x_0 = B, \\
v_{nt}' x_{a(n),t-1} = u_{nt}' x_{nt}, \quad n \in N_t,\ t = 1, 2, \ldots, T.
\end{gathered}
\tag{3.3}
$$

3.2. The Objective and the Probability Constraint

We come to the formulation of the objective function and the probability constraint. Let us consider the investment process. There are basically three different outputs at a given scenario S_{nt}. The first one is that we succeed to obtain the target wealth and stop the investment on this scenario, and this is really the objective. The second one is that we unfortunately fall into bankruptcy on this scenario and have to exit the investment. In either case, we cannot restart it again. In addition to the above two cases, the investment may be continued to next period.

Now, we define two 0-1 variables to describe the investment story. On scenario S_{nt}, $n \in N_t$, $t = 0, 1, \ldots, T$, first define $\varepsilon_{nt} \in \{0, 1\}$ such that

$$
\varepsilon_{nt} = \begin{cases} 1, & W_{nt} \geq u,\ 1 \leq W_{a(n),j} < u,\ \forall j < t, \\ 0, & \text{otherwise.} \end{cases}
\tag{3.4}
$$

Parallel to ε_{nt}, we define $\eta_{nt} \in \{0, 1\}$ such that

$$
\eta_{nt} = \begin{cases} 1, & W_{nt} < l,\ l \leq W_{a(n),j} < u,\ \forall j < t, \\ 0, & \text{otherwise.} \end{cases}
\tag{3.5}
$$

Reading the definitions, $\varepsilon_{nt} = 1$ indicates the first case, where the investment reaches the target and stops at scenario S_{nt}, and $\eta_{nt} = 1$ represents the second case that bankruptcy happens at scenario S_{nt}. By using ε_{nt} and η_{nt}, we can write our objective as

$$
E[t_u] = \sum_{t=0}^{T} \left(t \cdot \sum_{n \in N_t} p_{nt} \varepsilon_{nt} \right),
\tag{3.6}
$$

and the deterministic form of the probability as

$$
P(W < l) = \sum_{t=0}^{T} \sum_{n \in N_t} p_{nt} \eta_{nt} \leq q.
\tag{3.7}
$$

We consider again the indicator variables ε_{nt} and η_{nt}. Their values on scenario S_{nt} actually depend on both the current state and also all of the ancestor states. Take ε_{nt} as an example, $\varepsilon_{nt} = 1$ holds if and only if the following two conditions are both satisfied. One is that the investment continues to the current scenario, and the other is that the payoff at the current scenario is no less than the target wealth. If either of the above conditions is not achieved, we should get $\varepsilon_{nt} = 0$. Moreover, the case of η_{nt} is for the same logic but about the bankruptcy part. Thus, we introduce another two sets of variables to track the current state and the historical states separately.

For the current state, we define $\delta_{nt}, \xi_{nt} \in \{0,1\}$ as follows:

$$\delta_{nt} = \begin{cases} 1, & W_{nt} \geq u, \\ 0, & W_{nt} < u, \end{cases} \qquad \xi_{nt} = \begin{cases} 1, & W_{nt} < l, \\ 0, & W_{nt} \geq l, \end{cases} \tag{3.8}$$

and for the ancestor states, we define $\phi_{nt} \in \{0,1\}$ such that

$$\phi_{nt} = \begin{cases} 1, & l \leq W_{a(n),j} < u, \ \forall j < t, \\ 0, & \text{otherwise}, \end{cases} \tag{3.9}$$

where $\phi_{nt} = 1$ means that the investment has kept going on to the current scenario and $\phi_{nt} = 0$ means that it has stopped on the parent scenario or other ancestor scenarios before.

Combine the above definitions and review ε_{nt} and η_{nt}, we realize the relations

$$\begin{aligned} \varepsilon_{nt} &= \delta_{nt} \cdot \phi_{nt}, \\ \eta_{nt} &= \xi_{nt} \cdot \phi_{nt}. \end{aligned} \tag{3.10}$$

If we replace these nonlinear constraints by a set of linear constraints, then the problem can be hopefully formulated as a linear programming problem, which will benefit for the further research on solution methods and applications. Since the indicator variables are all defined as binary 0-1 variables, we derive the transformation

$$\varepsilon_{nt} = \delta_{nt} \cdot \phi_{nt} \Longleftrightarrow \begin{cases} \delta_{nt} + \phi_{nt} - \varepsilon_{nt} \leq 1, \\ \delta_{nt} + \phi_{nt} - 2\varepsilon_{nt} \geq 0. \end{cases} \tag{3.11}$$

It is direct to check that for given values of $\{\delta_{nt}, \phi_{nt}\}$, ε_{nt} must realize the same value either by $\varepsilon_{nt} = \delta_{nt} \cdot \phi_{nt}$ or by the constraints

$$\begin{aligned} \delta_{nt} + \phi_{nt} - \varepsilon_{nt} &\leq 1, \\ \delta_{nt} + \phi_{nt} - 2\varepsilon_{nt} &\geq 0, \end{aligned} \tag{3.12}$$

and similar case for η_{nt},

$$\eta_{nt} = \xi_{nt} \cdot \phi_{nt} \Longleftrightarrow \begin{cases} \xi_{nt} + \phi_{nt} - \eta_{nt} \leq 1, \\ \xi_{nt} + \phi_{nt} - 2\eta_{nt} \geq 0. \end{cases} \tag{3.13}$$

Therefore, we now replace (3.10) by the following set of inequalities:

$$\begin{aligned}
\delta_{nt} + \phi_{nt} - \varepsilon_{nt} &\leq 1, \\
\delta_{nt} + \phi_{nt} - 2\varepsilon_{nt} &\geq 0, \\
\xi_{nt} + \phi_{nt} - \eta_{nt} &\leq 1, \\
\xi_{nt} + \phi_{nt} - 2\eta_{nt} &\geq 0.
\end{aligned} \tag{3.14}$$

Up to now, we have almost derived out the formulation of the model based on a series of indicator variables, including ε, η, δ, ξ, ϕ. The remaining task is to construct the dynamics of ϕ_{nt} and also the constraints of δ_{nt} and ξ_{nt}, so that the definitions here can be implemented in the model.

3.3. *The Dynamics of Indicator Variables*

Consider the constraints of δ_{nt} and ξ_{nt} first. Given a large enough number $M_1 > u$ and a small enough number $M_2 < l$, we have for δ_{nt},

$$\begin{aligned}
\delta_{nt} &= \begin{cases} 1, & W_{nt} \geq u, \\ 0, & W_{nt} < u, \end{cases} \\
&\Longleftrightarrow \begin{cases} W_{nt} - (M_1 - u) \cdot \delta_{nt} < u, \\ W_{nt} + (u - M_2) \cdot (1 - \delta_{nt}) \geq u \end{cases}
\end{aligned} \tag{3.15}$$

and for ξ_{nt}, we have

$$\begin{aligned}
\xi_{nt} &= \begin{cases} 1, & W_{nt} < l, \\ 0, & W_{nt} \geq l, \end{cases} \\
&\Longleftrightarrow \begin{cases} W_{nt} - (M_1 - l) \cdot (1 - \xi_{nt}) < l, \\ W_{nt} + (l - M_2) \cdot \xi_{nt} \geq l. \end{cases}
\end{aligned} \tag{3.16}$$

We combine the constraints of δ_{nt} and ξ_{nt} as the constraint set

$$\begin{aligned}
W_{nt} - (M_1 - u) \cdot \delta_{nt} &< u, \\
W_{nt} + (u - M_2)(1 - \delta_{nt}) &\geq u, \\
W_{nt} - (M_1 - l)(1 - \xi_{nt}) &< l, \\
W_{nt} + (l - M_2) \cdot \xi_{nt} &\geq l.
\end{aligned} \tag{3.17}$$

Next, let us focus on the dynamics of ϕ_{nt}. At the beginning point of the investment, $\phi_0 = 1$ holds. During the investment process, we first write out the dynamics and then explain the underlying reasons:

$$\begin{aligned} &\phi_0 = 1, \\ &\phi_{nt} = \phi_{a(n),t-1} - \left(\varepsilon_{a(n),t-1} + \eta_{a(n),t-1}\right). \end{aligned} \tag{3.18}$$

The dynamic equation holds for the following reasons.

First, suppose the investment has been continued to the scenario $S_{a(n),t-1}$ and does not stop at that scenario, which means we already held $\phi_{a(n),t-1} = 1$, and $\varepsilon_{a(n),t-1} = \eta_{a(n),t-1} = 0$, then, the investment must keep going on to the current scenario S_{nt}. In this case, we should have $\phi_{nt} = 1$ base on the definition of ϕ. The recursive equation in (3.18) succeeds to realize this case and gives $\phi_{nt} = 1 - 0 = 1$.

Second, if the investment has stopped, either on the parent scenario $S_{a(n),t-1}$ or on any of the ancestor scenarios before, we should hold $\phi_{nt} = 0$. This case can also be realized by the dynamic equation (3.18). In case that the investment stopped on the parent scenario $S_{a(n),t-1}$, that is, $\phi_{a(n),t-1} = 1$, and either $\varepsilon_{a(n),t-1} = 1$ or $\eta_{a(n),t-1} = 1$, then (3.18) gives $\phi_{nt} = 0$; in the other case of stopping before the previous stage, we already had $\phi_{a(n),t-1} = 0$, also both $\varepsilon_{a(n),t-1} = 0$ and $\eta_{a(n),t-1} = 0$, the result of (3.18) is still $\phi_{nt} = 0$.

3.4. The Deterministic Formulation

Now, we have derived all the constraints of the indicator variables by (3.3), (3.14), (3.17), (3.18). Together with the objective function and the probability constraint represented by (3.6) and (3.7), respectively, the problem (M) can be finally written as a mix integer linear programming problem:

$$\begin{aligned} \min\ & \sum_{t=1}^{T}\left(t \cdot \sum_{n\in N_t} p_{nt}\varepsilon_{nt}\right) \\ \text{s.t.}\ & \sum_{t=1}^{T}\sum_{n\in N_t} p_{nt}\eta_{nt} \leq q \\ & (3.3),\ (3.14),\ (3.17),\ (3.18) \\ & \varepsilon_{nt}, \eta_{nt}, \delta_{nt}, \xi_{nt}, \phi_{nt} \in \{0,1\}, \\ & n \in N_t, \quad t \in \{1,2,\ldots,T\}. \end{aligned} \tag{P}$$

Next, we construct an example to analyze the model and illustrate the solving process. The problem is input by an MATLAB program, and numerically solved by using Cplex software.

4. An Example

The investment process is represented by a 3-stage triple tree, noted from time 1 to time 4, as showed in Figure 1. The portfolio can be organized and reorganized at the beginning of each

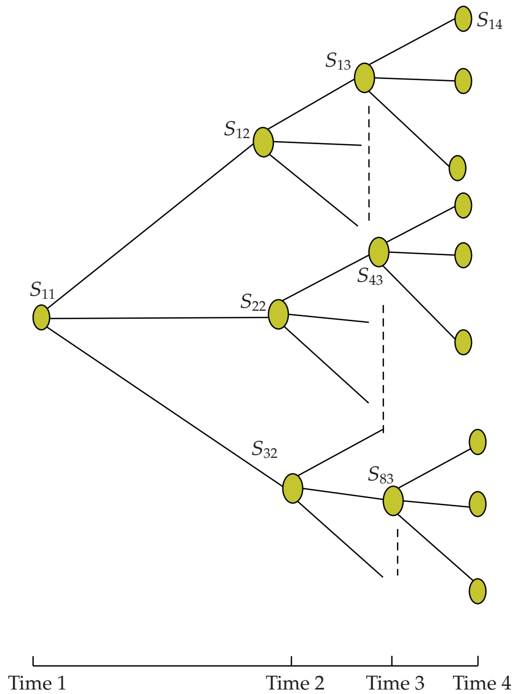

Figure 1: The scenario tree of example.

Table 1: The selected solutions of example.

		Time 1	Time 2	Time 3		Time 4	
Payoff target obtained	Scenario	None	None	$S_{1,3}$	$S_{8,3}$	None	
	Solution			$\varepsilon_{1,3} = 1$	$\varepsilon_{8,3} = 1$		
				$\delta_{1,3} = 1$	$\delta_{8,3} = 1$		
				$\phi_{1,3} = 1$	$\phi_{8,3} = 1$		
Bankruptcy happens	Scenario	None	None	$S_{4,3}$		$S_{19,4}$	$S_{27,4}$
	Solution			$\eta_{4,3} = 1$		$\eta_{19,4} = 1$	$\eta_{27,4} = 1$
				$\xi_{4,3} = 1$		$\xi_{19,4} = 1$	$\xi_{27,4} = 1$
				$\phi_{4,3} = 1$		$\phi_{19,4} = 1$	$\phi_{27,4} = 1$

stage. We simply consider a portfolio of two assets, and the prices on each decision point are given. Also, the conditional probabilities of the three notes in any single-stage subtree are $P = \{.3, .36, .34\}$ in order. For other essential constants, we assume the initial budget $B = \$100$, the target payoff $u = \$104$, and the bancruptcy banchmark $l = \$95$. In addition, we take $M_1 = 10\,000$ and $M_2 = -10\,000$ as those two large enough numbers for formulating the problem. Finally, we assign the largest accetable bancruptcy probability to be $q = 0.2$.

Cplex takes 0.41 second to optimize the problem. Reading the solution file, we find that there are chances to obtain the payoff target before the investment horizon, as clearly as in the third stage on the scenarios of $\{S_{1,3}, S_{8,3}\}$, respectively. Accordingly, the bankruptcy possibly happens on the third and the fourth stages, on the scenarios of $\{S_{4,3}, S_{19,4}, S_{27,4}\}$, which makes the total probability of bankruptcy is 0.178. Details of selected optimal solutions are shown in Table 1. Other solutions are also carefully checked, it turns out that the construction of indicator variables does work. For example, on the children scenarios of the stopping scenarios $\{S_{1,3}, S_{8,3}, S_{4,3}\}$, the values of ϕ are all zero as the investment has been stopped before.

For two-stage problem, there are well-known algorithms such as branch-and-bound, Lagrangian relaxation, or cutting plane methods for solving it. When we extend it into the multistage case, as we are doing now, the problem becomes much more complex. As the size of the problem increases, the existing solution methods become less efficient. We will further investigate on more applicable solution methodologies. In addition to the solution methodology, another relevant research topic is to compare the investment policies under different objectives and risk constraints.

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