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Research Article

Majorization for A Subclass of β-Spiral Functions of Order α Involving a Generalized Linear Operator

Afaf A. Ali Abubaker,[1] Maslina Darus,[1] and Daniel Breaz[2]

[1] *School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, 43600 Bangi, Malaysia*
[2] *Department of Mathematics-Informatics, Faculty of Sciences, 1 Decembrie 1918 University of Alba Iulia, 510009 Alba Iulia, Romania*

Correspondence should be addressed to Maslina Darus, maslina@ukm.my

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Motivated by Carlson-Shaffer linear operator, we define here a new generalized linear operator. Using this operator, we define a class of analytic functions in the unit disk U. For this class, a majorization problem of analytic functions is discussed.

1. Introduction

Let A denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=1}^{\infty} a_{n+1} z^{n+1} \tag{1.1}$$

which are analytic in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

Let f and g be analytic in U. Then, we say that function f is subordinate to g if there exists a Schwarz function $\omega(z)$, analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1$, such that $f(z) = g(\omega(z))$, $z \in U$ (see [1]). We denote this subordination by

$$f \prec g \quad (z \in U). \tag{1.2}$$

Further, f is said to be quasi subordinate to g if there exists an analytic function $\varphi(z)$ such that $f(z)/\varphi(z)$ is analytic in U,

$$\frac{f(z)}{\varphi(z)} \prec g(z), \quad (z \in U) \tag{1.3}$$

and $|\varphi(z)| \leq 1$. Note that the quasi subordination (1.3) is equivalent to

$$f(z) = \varphi(z) g(\omega(z)), \tag{1.4}$$

where $|\varphi(z)| \leq 1$ and $|\omega(z)| \leq |z| < 1$ (see [2]). If $\varphi(z) = 1$, then (1.3) becomes (1.2).

Let functions f and g be analytic functions in U. If $|f(z)| \leq |g(z)|$, then there exists a function φ analytic in U such that $|\varphi(z)| \leq 1$ in U, for which

$$f(z) = \varphi(z) g(z) \quad (z \in U). \tag{1.5}$$

In this case, we say that f is majorized by g in U (see [3]), and we write

$$f(z) \ll g(z) \quad (z \in U). \tag{1.6}$$

If we take $\omega(z) = z$ in (1.4), then the quasi subordination (1.3) becomes the majorization (1.6).

Also, let S denote the subclass of A consisting of all functions which are univalent in U.

In [4], Robertson introduced star-like functions of order α on U.

Definition 1.1. Let $0 \leq \alpha < 1$ and $f \in A$; then, f is a star-like function of order α on U if and only if

$$\Re\left\{\frac{z f'(z)}{f(z)}\right\} > \alpha \quad (z \in U). \tag{1.7}$$

Let $S^*(\alpha)$ denote the whole star-like functions of order α in U.

Spaček [5] extended the class of S^* and obtained the class of β-spiral-like functions. In the same article, the author gave an analytical characterization of spirallikeness of type β on U.

Definition 1.2. Let $-\pi/2 < \beta < \pi/2$ and $f \in A$; then, f is β-spiral-like function on U if and only if

$$\Re\left\{e^{i\beta}\frac{f'(z)}{f(z)}\right\} > 0 \quad (z \in U). \tag{1.8}$$

We denote the whole β-spiral-like functions in U by S^*_β.

Finally, Libera [6] introduced and studied the class of β-spiral-like functions of order α.

Definition 1.3. Let $0 \le \alpha < 1$, $-\pi/2 < \beta < \pi/2$ and $f \in A$; then, f is β-spiral function of order α if and only if

$$\Re\left\{ e^{i\beta} \frac{zf'(z)}{f(z)} \right\} > \alpha \cos\beta \quad (z \in U). \tag{1.9}$$

We denote the whole β-spiral-like functions of order α in U by $S^*_\beta(\alpha)$.

In particular, we consider the convolution with function $\phi(a,c)$ defined by

$$L(a,b)f(z) = z + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1}, \tag{1.10}$$

where $a \in \mathbb{C}$, $b \ne 0, -1, -2, \ldots$, and $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & n = 0, \\ a(a+1)\cdots(a+n-1), & n = \{1,2,3,\ldots\}. \end{cases} \tag{1.11}$$

Function $\phi(a,c)$ is an incomplete beta-function related to the Gauss hypergeometric function by

$$\phi(a,c;z) = z_2F_1(1,a;c;z). \tag{1.12}$$

It has an analytic continuation to the z-plane cut along the positive real line from 1 to ∞. We note that $\phi(a.1;z) = z/(1-z)^a$ and $\phi(2,1;z)$ are the Koebe functions.

Carlson and Shaffer [7] defined a convolution operator on A involving an incomplete beta-function as

$$L(a,b)f(z) = \phi(a,c;z) * f(z) = z + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} a_{n+1} z^{n+1}. \tag{1.13}$$

Definition 1.4. Let function F be given by

$$F(m,\ell,\lambda) = \sum_{n=0}^{\infty} \left(\frac{1+\ell+\lambda n}{1+\ell} \right)^m z^{n+1}, \tag{1.14}$$

where $\ell, \lambda \ge 0$ and $m \in \mathbb{Z}$. The generalized linear operator $L(m,\ell,\lambda,a,c) : A \to A$ is given as

$$L(m,\ell,\lambda,a,b)f(z) = z + \sum_{n=1}^{\infty} \left(\frac{1+\ell+\lambda n}{1+\ell} \right)^m \frac{(a)_n}{(c)_n} a_{n+1} z^{n+1}. \tag{1.15}$$

We note here some special cases.

(1) $L(0,\ell,\lambda,a,b)f(z) = L(a,b)f(z)$ is the Carlson and Shaffer operator [7].

(2) $L(0,\ell,\lambda,\delta+1,1)f(z)$, $\delta \in \mathbb{N}_0$, is the Ruscheweyh derivative [8].

(3) $L(m,0,\lambda,1,1)f(z)$, $m \in \mathbb{N}_0$, is the Al-Oboudi operator [9].

(4) $L(m,0,\lambda,a,b)f(z)$ is the linear operator introduced by Al-Refai and Darus [10].

(5) $F(m,\ell,\lambda)$, $m \in \mathbb{N}_0$, is the generalized multiplier transformation which was introduced and studied by Cátáş [11].

(6) $F(m,\ell,1), m \in \mathbb{N}_0$, is the multiplier transformation which was introduced and studied by Cho and Srivastava [12] and Cho and Kim [13].

Remark 1.5. It follows from the above definition that

$$z\big(L(m,\ell,\lambda,a,c)f(z)\big)' = aL(m,\ell,\lambda,a+1,c)f(z) - (a-1)L(m,\ell,\lambda,a,c)f(z) \quad (z \in U). \tag{1.16}$$

We introduce the class $S^*_\beta(m,\ell,\lambda,a,c,\alpha)$ as follows.

Definition 1.6. Let $a \in \mathbb{C}$, $c \neq 0,-1,-2,\ldots,\ell$, $\lambda \geq 0$, $m \in \mathbb{Z}$, $0 \leq \alpha < 1$, $-\pi/2 < \beta < \pi/2$, and $f \in A$; then, one has $S^*_\beta(m,\ell,\lambda,a,c,\ell,\lambda,\alpha)$ if and only if

$$\Re\left\{e^{i\beta}\frac{z\big(L(a,c)f(z)\big)'}{L(a,c)f(z)}\right\} > \alpha\cos\beta. \tag{1.17}$$

Obviously, when $a = c = 1$ and $m = 0$ we obtain $f \in S^*_\beta(\alpha)$, when $a = c = 1$ and $m = \beta = 0$, we obtain that $f(z)$ is a starl-like function of order α on U, and also when $a = c = 1$ and $m = \alpha = 0$, we obtain that $f(z)$ is spiral-like function of type β on U.

Biernacki [14] in 1936 obtained the first results of majorization-subordination theory. He showed that, if $g(z) \in S$ and $f(z) \prec g(z)$ in U, then $f(z) \ll g(z)$ in $|z| \leq (1/4)$. Goluzin [15] improved the result and Shah [16] obtained the complete solution for S by showing that $f(z) \ll g(z)$ in $|z| \leq (3-\sqrt{5})/2$ and that the result is the best possible. A majorization problem for star-like functions has been given by MacGregor [3]. Also, majorization problem for star-like functions of complex order has recently been investigated by Altintaş et al. [17].

The main object of this paper is to investigate the problem of majorization of the class $S^*_\beta(\ell,\lambda,a,c,\alpha)$ defined by a generalized linear operator.

In order to prove our main theorem we need the following lemma.

Lemma 1.7 (see [18]). *Let $\varphi(z)$ be analytic in U satisfying $|\varphi(z)| \leq 1$ for $z \in U$. Then,*

$$\left|\varphi'(z)\right| \leq \frac{1-\left|\varphi(z)\right|^2}{1-|z|^2}. \tag{1.18}$$

2. Main Results

Theorem 2.1. *Let function* $f \in A$ *and suppose that* $g \in S^*_\beta(m, \ell, \lambda, a, c, \alpha)$. *If* $L(m, \ell, \lambda, a, c)f$ *is majorized by* $L(m, \ell, \lambda, a, c)g$ *in* U, *then*

$$\left|L(m, \ell, \lambda, a+1, c)f(z)\right| \le \left|L(m, \ell, \lambda, a+1, c)g(z)\right| \quad (|z| \le r_1), \tag{2.1}$$

where

$$r_1 = r(m, \ell, \lambda, a, c, \alpha, \beta) = \frac{2 + |a| + \left|2(1-\alpha)\cos\beta - ae^{i\beta}\right|}{2\left|2(1-\alpha)\cos\beta - ae^{i\beta}\right|} - \frac{\sqrt{\Theta(a, \alpha, \beta)}}{2\left|2(1-\alpha)\cos\beta - a\ e^{i\beta}\right|}, \tag{2.2}$$

$$\begin{aligned}\Theta(a, \alpha, \beta) = 4 + |a|^2 + \left|2(1-\alpha)\cos\beta - ae^{i\beta}\right|^2 + 4|a| + 4\left|2(1-\alpha)\cos\beta - ae^{i\beta}\right| \\ - 2|a|\left|2(1-\alpha\cos\beta) - ae^{i\beta}\right|,\end{aligned} \tag{2.3}$$

for $a \in \mathbb{C}$, $c \neq 0, -1, -2, \ldots, , \ell$, $\lambda \ge 0$, $m \in \mathbb{Z}$, $0 \le \alpha < 1$, $-\pi/2 < \beta < \pi/2$, *and* $|a| \ge |2(1-\alpha)\cos\beta - ae^{i\beta}|$.

Proof. Since $g \in S^*_\beta(m, \ell, \lambda, a, c, \alpha)$, we have

$$e^{i\beta}\frac{z(L(m, \ell, \lambda, a, c)g(z))'}{L(m, \ell, \lambda, a, c)g(z)} = \frac{1 + (1-2\alpha)\omega}{1-\omega}\cos\beta + i\sin\beta, \tag{2.4}$$

where ω is analytic in U, with $\omega(0) = 0$ and

$$|\omega| \le |z| < 1 \quad (z \in U). \tag{2.5}$$

By using (1.16) in (2.4), we get

$$e^{i\beta}\frac{\left[aL(m, \ell, \lambda, a+1, c)g(z) - (a-1)L(m, \ell, \lambda, a, c)g(z)\right]}{L(m, \ell, \lambda, a, c)g(z)} = \frac{1 + (1-2\alpha)\omega}{1-\omega}\cos\beta + i\sin\beta. \tag{2.6}$$

Hence,

$$\frac{L(m, \ell, \lambda, a+1, c)g(z)}{L(m, \ell, \lambda, a, c)g(z)} = \frac{ae^{i\beta} + \left(2(1-\alpha)\cos\beta - ae^{i\beta}\right)\omega}{ae^{i\beta}(1-\omega)}, \tag{2.7}$$

which, in view of (2.5), immediately yields the inequality

$$\left|L(m, \ell, \lambda, a, c)g(z)\right| \le \frac{\left|e^{i\beta}\right||a|(1+|z|)}{|a| - \left|2(1-\alpha)\cos\beta - ae^{i\beta}\right||z|}\left|L(m, \ell, \lambda, a+1, c)g(z)\right|. \tag{2.8}$$

Next, since $L(m,\ell,\lambda,a,c)f$ is majorized by $L(m,\ell,\lambda,a,c)g$ in U, from (1.5) we have

$$z\left(L(m,\ell,\lambda,a,c)f(z)\right)' = z\varphi'(z)L(m,\ell,\lambda,a,c)g(z) + z\varphi(z)\left(L(m,\ell,\lambda,a,c)g(z)\right)'. \tag{2.9}$$

Also, by using (1.16) in (2.11), we get

$$\begin{aligned} &aL(m,\ell,\lambda,a+1,c)f(z) - (a-1)L(m,\ell,\lambda,a,c)f(z) \\ &\quad = z\varphi'(z)L(m,\ell,\lambda,a,c)g(z) + \varphi(z)\left[aL(m,\ell,\lambda,a+1,c)g(z) - (a-1)L(m,\ell,\lambda,a,c)g(z)\right]; \end{aligned} \tag{2.10}$$

then, we have

$$L(m,\ell,\lambda,a+1,c)f(z) = \frac{1}{a}z\varphi'(z)L(m,\ell,\lambda,a,c)g(z) + \varphi(z)L(m,\ell,\lambda,a+1,c)g(z). \tag{2.11}$$

Thus, by Lemma 1.7, since the Schwarz function ϕ satisfies the inequality in (1.18) and using (2.8) in (2.11), we get

$$\begin{aligned} \left|L(m,\ell,\lambda,a+1,c)f(z)\right| \leq{}& \frac{\left(1-\left|\varphi(z)\right|^2\right)|z|}{(1-|z|)\left(|a|-\left|2(1-\alpha)\cos\beta - ae^{i\beta}\right||z|\right)} \\ &\times \left|L(m,\ell,\lambda,a+1,c)g(z)\right| + \left|\varphi(z)\right| \left|L(m,\ell,\lambda,a+1,c)g(z)\right|. \end{aligned} \tag{2.12}$$

Hence,

$$\begin{aligned} \left|L(m,\ell,\lambda,a+1,c)f(z)\right| \leq{}& \frac{\left(1-\left|\varphi(z)\right|^2\right)|z| + (1-|z|)\left(|a|-\left|2(1-\alpha)\cos\beta - ae^{i\beta}\right||z|\right)\left|\varphi(z)\right|}{(1-|z|)\left(|a|-\left|2(1-\alpha)\cos\beta - ae^{i\beta}\right||z|\right)} \\ &\times \left|L(m,\ell,\lambda,a+1,c)g(z)\right|, \end{aligned} \tag{2.13}$$

which, upon setting

$$|z| = r, \qquad \left|\varphi(z)\right| = \rho \quad (0 \leq \rho \leq 1) \tag{2.14}$$

yields

$$\left|L(m,\ell,\lambda,a+1,c)f(z)\right| \leq \frac{\theta(\rho)}{(1-r)\left(|a|-\left|2(1-\alpha)\cos\beta - ae^{i\beta}\right|r\right)} \left|L(m,\ell,\lambda,a+1,c)g(z)\right|, \tag{2.15}$$

where function $\theta(\rho)$ defined by

$$\theta(\rho) = \left(1-\rho^2\right)r + (1-r)\left(|a| - \left|2(1-\alpha)\cos\beta - ae^{i\beta}\right|r\right)\rho \tag{2.16}$$

takes on its maximum value at $\rho = 1$ with

$$r_1 = r(m,\ell,\lambda,a,c,\alpha,\beta) = \max\{r \in [0,1] : \varphi(r,\rho) \le 1, \quad \forall \rho \in [0,1]\}, \tag{2.17}$$

where

$$\varphi(r,\rho) = \frac{\theta(\rho)}{(1-r)\left(|a| - \left|2(1-\alpha)\cos\beta - ae^{i\beta}\right|r\right)}; \tag{2.18}$$

then, we have

$$\frac{\theta(\rho)}{(1-r)\left(|a| - \left|2(1-\alpha)\cos\beta - ae^{i\beta}\right|r\right)} \le 1. \tag{2.19}$$

A simple calculus in (2.19) is equivalent to

$$-(1+\rho)r + (1-r)\left(|a| - \left|2(1-\alpha)\cos\beta - ae^{i\beta}\right|r\right) \ge 0, \tag{2.20}$$

while the inequality in (2.19) takes its minimum value at $\rho = 1$, that is,

$$\left|2(1-\alpha)\cos\beta - ae^{i\beta}\right|r^2 - \left(2|a| + \left|2(1-\alpha)\cos\beta - ae^{i\beta}\right|\right)r + |a| \ge 0, \tag{2.21}$$

for all $r \in [0, r_1]$, where $r_1 = r(m,\ell,\lambda,a,c,\alpha,\beta)$ given in (2.2) holds true for $|z| \le r(m,\ell,\lambda,a,c,\alpha,\beta)$, which proves the conclusion (2.1). □

Putting $m = \alpha = \beta = 0$ in Theorem 2.1, we obtain the following result.

Corollary 2.2. *Let function $f \in A$ and suppose that $g \in S^*(a,c)$. If $L(a,c)f$ is majorized by $L(a,c)g$ in U, then*

$$\left|L(a+1,c)f(z)\right| \le \left|L(a+1,c)g(z)\right| \quad (|z| \le r_2 = r(a,c)), \tag{2.22}$$

where

$$r(a,c) = \frac{3+|a|+|2-a|}{2|2-a|} - \frac{\sqrt{4+|2-a|^2 - 2|a||2-a| + 4|a| + |a|^2}}{2|2-a|}. \tag{2.23}$$

Further, putting $a = c = 1$ and $m = 0$ in Theorem 2.1, we obtain the result of Altintaş et al. [17].

Corollary 2.3. *Let function $f \in A$ and suppose that $g \in S^*((\alpha-1)e^{i\beta}) = S^*_\beta(\alpha)$, where $0 \le \alpha < 1$ and $-\pi/2 < \beta < \pi/2$. If f is majorized by g in U, then*

$$\left|f'(z)\right| \le \left|g'(z)\right| \quad \left(|z| \le r_3 = r\left(\alpha,\beta\right)\right), \tag{2.24}$$

where

$$r\left(\alpha,\beta\right) = \frac{3+\left|2(\alpha-1)e^{i\beta}-1\right|}{2\left|2(\alpha-1)e^{i\beta}-1\right|} - \frac{\sqrt{9+\left|2(\alpha-1)e^{i\beta}-1\right|^2+2|2(\alpha-1)-1|}}{2\left|2(\alpha-1)e^{i\beta}-1\right|}. \tag{2.25}$$

Putting $\beta = 0$ in Corollary 2.3, we obtain the result as follows.

Corollary 2.4. *Let function $f \in A$ and suppose that $g \in S^*(\alpha)$, where $0 \le \alpha < 1$. If f is majorized by g in U, then*

$$\left|f'(z)\right| \le \left|g'(z)\right| \quad (|z| \le r_4 = r(\alpha)), \tag{2.26}$$

where

$$r(\alpha) = \frac{3+|1-2\alpha|}{2|1-2\alpha|} - \frac{\sqrt{9+|1-2\alpha|^2+2|2(\alpha-1)-1|}}{2|1-2\alpha|}. \tag{2.27}$$

Also, putting $\alpha = \beta = 0$ in Corollary 2.3, we obtain the result of MacGregor [3].

Corollary 2.5. *Let function $f \in A$ and suppose that $g \in S^*(0)$. If f is majorized by g in U, then*

$$\left|f'(z)\right| \le \left|g'(z)\right| \quad \left(|z| \le 2-\sqrt{3}\right). \tag{2.28}$$

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